

## NONSTANDARD FINITE DIFFERENCE SCHEMES FOR SOLVING A MODIFIED EPIDEMIOLOGICAL MODEL FOR COMPUTER VIRUSES\*

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**Abstract.** In this paper we construct two families of nonstandard finite difference (NSFD) schemes preserving the essential properties of a computer virus propagation model, such as positivity, boundedness and stability. The first family of NSFD schemes is constructed based on the nonlocal discretization and has first order of accuracy, while the second one is based on the combination of a classical Runge-Kutta method and selection of a nonstandard denominator function and it is of fourth order of accuracy. The theoretical study of these families of NSFD schemes is performed with support of numerical simulations. The numerical simulations confirm the accuracy and the efficiency of the fourth order NSFD schemes. They hint that the disease-free equilibrium point is not only locally stable but also globally stable, and then this fact is proved theoretically. The experimental results also show that the global stability of the continuous model is preserved.

**Keywords.** Computer viruses, high order NSFD schemes, Lyapunov stability theorem, NSFD schemes, numerical simulations.

### 1. INTRODUCTION

The mathematical models describing the computer virus propagation play especially important role in both theory and practice. The study of properties of these models helps us to understand the law governing the spread of computer viruses. Based on this we can make appropriate policies for controlling and preventing the spread of them. In the last two decades some authors proposed different mathematical models for computer viruses through differential equations systems, e.g., [23, 25, 26, 27, 28, 29, 30]. The idea leading to these models comes from the high similarity between computer viruses and biological viruses.

In this paper we are concerned with a modified epidemiological model for computer viruses proposed by Piqueira and Araujo [25]. The mathematical analysis there shows that the solution of the model is positive and bounded. Besides, the local stability of disease-free equilibrium points and endemic equilibrium point (if it exists) is established. The aim of our paper is to construct difference schemes preserving the essential properties of the above mentioned model. In other words, our task is to convert the continuous model to discrete

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models which are dynamically consistent with the continuous one. As is known, the establishment of stability properties of continuous models in biology, epidemiology, ecology is a problem of most interest in mathematical biology, meanwhile the conversion of these models to dynamically consistent discrete models is one of the most important problems in numerical solution of differential equations and simulation of dynamical systems. There are many methods for discretization of continuous models. Popular ones are standard finite difference methods (SFDM) such as Euler method, Runge-Kutta method and Taylor method [1]. However, for many nonlinear problems SFDM revealed a serious drawback. It is numerical instabilities [19, 20, 21, 22], when numerical methods do not preserve the properties of the differential equations. In order to overcome this phenomenon, in the 1980s, Mickens proposed a type of difference schemes, which are named as nonstandard finite difference (NSFD) schemes. These schemes can preserve essential properties of corresponding differential equations [19, 20, 21, 22].

Up to now, NSFD schemes become an important and efficient tool for solving nonlinear differential equations and simulating complicated dynamical systems [19, 20, 21, 22, 24]. They are applied to many important mathematical models in physics, chemistry, biology, medicine, and so on. Recently, we achieved some results on the construction of NSFD schemes for certain important practical models [2, 3, 4, 5, 6, 7, 8]. To our best knowledge, NSFD schemes are still not applied to computer viruses spread models although these models have great significance and many results of qualitative aspects are obtained. For this reason, in this paper we construct NSFD schemes preserving essential properties of a model for computer viruses proposed by Piqueira and Araujo [25]. It should be emphasized that in general, NSFD schemes which are dynamically consistent with differential equations, have first order of accuracy [19, 20, 21, 22]. This motivates the problem of increasing the order of accuracy of NSFD schemes. Recently, some higher order NSFD schemes were constructed [11, 17, 18]. They are based on the combination of nonstandard schemes and Richardson's extrapolation. Differently from the above way, here we construct higher order nonstandard finite methods based on Runge-Kutta methods without extrapolation. More specifically, we design NSFD schemes based on the classical Runge-Kutta methods with selection of nonstandard denominator. It is an important contribution of ours in this paper. For our schemes, the positivity is obtained from the positivity of Runge-Kutta methods, while the stability is established by Lyapunov method in combination with the stability properties of Runge-Kutta methods. It should be emphasized that the proposed method for designing NSFD schemes is applicable to some other applied models.

The results of numerical simulations reported in Section 4 confirm the validity of the obtained theoretical results. The errors and the computation time support the accuracy and efficiency of the designed high order NSFD schemes. Especially, the numerical experiments hint that the disease-free equilibrium is not only locally asymptotically stable, but is globally stable. This prediction of the global stability is proved by using a suitable Lyapunov function.

The paper is organized as follows. In Section 2 we recall a mathematical model of computer virus spread and its properties. The construction of NSFD schemes based on nonlocal discretization of the continuous model is presented in Section 3. The next section is devoted to design of high order NSFD schemes. In Section 5, the results of numerical simulations are reported. Finally, Section 6 is some conclusions.

## 2. MATHEMATICAL MODEL AND ITS PROPERTIES

First, we recall the computer virus spread model proposed by Piqueira and Araujo [25]

$$\begin{aligned}\dot{S} &= \alpha_{SA}SA - \beta SI + \sigma R, \\ \dot{I} &= \beta SI - \alpha_{IA}AI - \delta I, \\ \dot{R} &= \delta I - \sigma R, \\ \dot{A} &= \alpha_{SA}SA + \alpha_{IA}IA.\end{aligned}\tag{1}$$

This model is a modification of the original compartmental SIR model. The total population  $T$  is divided into four groups:  $S$  of non-infected computers subjected to possible infection;  $A$  of noninfected computers equipped with anti-virus;  $I$  of infected computers; and  $R$  of removed ones due to infection or not. Besides,  $\alpha_{SA}, \alpha_{IA}, \beta, \delta, \sigma$  are positive parameters. For more detail, see [25].

The mathematical analysis in [25] shows that the model (1) has the following properties:

( $P_1$ ) Positivity and boundedness:

The solutions of (1) with positive initial values are always positive and the sum of solutions is constant

$$S(t) + I(t) + R(t) + A(t) \equiv T.$$

In other words, the set  $\Omega = \{(S, I, R, A) \in \mathbb{R}_+^4 : S + I + R + A = T\}$  is a positively invariant set of (1).

( $P_2$ ) Equilibrium points and locally asymptotic stability:

For all possible parameter values, the model (1) always has two disease-free equilibrium points:  $P_1 = (S, I, R, A) = (0, 0, 0, T)$  and  $P_2 = (S, I, R, A) = (T, 0, 0, 0)$ . Meanwhile, endemic equilibrium point  $P_3$  exists if and only if  $T > \delta/\beta$ .

The equilibrium point  $P_1$  is asymptotically stable and the equilibrium point  $P_2$  is unstable, while the endemic equilibrium point  $P_3$ , if existing, is unstable.

In Sections 3 and 4 we shall construct NSFD schemes preserving the properties ( $P_1$ ) and ( $P_2$ ) of the model. Notice that, due to  $S(t) + I(t) + R(t) + A(t) \equiv T$ , instead of (1) it suffices to consider the reduced system

$$\begin{aligned}\dot{S} &= -\alpha_{SA}S(T - S - I - R) - \beta SI + \sigma R, \\ \dot{I} &= \beta SI - \alpha_{IA}(T - S - I - R) - \delta I, \\ \dot{R} &= \delta I - \sigma R,\end{aligned}\tag{2}$$

with the positively invariant set  $\Omega^* = \{(S, I, R) : S + I + R \leq T\}$ .

The numerical simulations in Section 5 hint that  $P_1$  is not only locally stable but also globally stable. Therefore, below we prove this fact.

**Theorem 1.** *The equilibrium point  $P_1$  of the model (1) is globally stable.*

*Proof.* Notice that the global stability of (1) and (2) are equivalent. Therefore, it suffices to prove that the point  $(0, 0, 0)$  is globally stable equilibrium point of (2) on the positively invariant set  $\Omega^*$ . Indeed, consider the Lyapunov function

$$V(S, I, R) = S + I + R.$$

Clearly, the function  $V$  is continuous and positive definite. Moreover, the derivative of  $V$  among the trajectory of (2) is

$$\dot{V} = -\alpha_{SA}S(T - S - I - R) - \alpha_{IA}(T - S - I - R).$$

Obviously,  $\dot{V} < 0$  except for  $(S, I) = (0, 0)$ . Therefore, by the Lyapunov stability theorem [16] the global stability of the model is ensured. ■

### 3. NSFD SCHEMES FOR MODEL (1) BY NONLOCAL APPROXIMATIONS

Now we construct NSFD schemes for the model (1). Firstly, we design NSFD schemes based on the nonlocal discretization of the right-hand sides with the use of nonstandard denominators in the form

$$\begin{aligned} \frac{S_{k+1} - S_k}{\varphi(h)} &= -\alpha_{SA}S_{k+1}A_k - \beta S_{k+1}I_k + \sigma R_k, \\ \frac{I_{k+1} - I_k}{\varphi(h)} &= \beta S_{k+1}I_k - \alpha_{IA}I_{k+1}A_k - \delta I_{k+1}, \\ \frac{R_{k+1} - R_k}{\varphi(h)} &= \delta I_{k+1} - \sigma R_k, \\ \frac{A_{k+1} - A_k}{\varphi(h)} &= \alpha_{SA}S_{k+1}A_k + \alpha_{IA}I_{k+1}A_k, \end{aligned} \quad (3)$$

where  $\varphi(h) = h + \mathcal{O}(h^2)$ . For simplicity, we omit the argument  $h$  in the function  $\varphi(h)$  in some places.

**Theorem 2.** *The set  $\Omega = \{(S, I, R, A) \in \mathbb{R}_+^4 : S + I + R + A = T\}$  is a positively invariant set of the model (3) if the function  $\varphi(h)$  satisfies the condition*

$$\varphi(h) < 1/\sigma, \quad \forall h > 0. \quad (4)$$

*Proof.* Adding sides-by-sides the equations (3) we obtain

$$S_{k+1} + I_{k+1} + R_{k+1} + A_{k+1} = S_k + I_k + R_k + A_k.$$

Hence, if  $S_k + I_k + R_k + A_k = T$ , then  $S_{k+1} + I_{k+1} + R_{k+1} + A_{k+1} = T$ . On the other hands, it is easy to convert the schemes (3) to the explicit form

$$\begin{aligned} S_{k+1} &= \frac{S_k + \varphi\sigma R_k}{1 + \varphi\alpha_{SA}A_k + \varphi\beta I_k}, \\ I_{k+1} &= \frac{I_k + \varphi\beta S_{k+1}I_k}{1 + \varphi\alpha_{IA}A_k + \varphi\delta}, \\ R_{k+1} &= (1 - \varphi\sigma)R_k + \varphi\delta I_{k+1}, \\ A_{k+1} &= A_k + \varphi(\alpha_{SA}S_{k+1}A_k + \alpha_{IA}I_{k+1}A_k). \end{aligned} \quad (5)$$

Therefore, if  $S_k, I_k, R_k, A_k \geq 0$  and (4) are satisfied then  $S_{k+1}, I_{k+1}, R_{k+1}, A_{k+1} \geq 0$ . The proof of the theorem is complete. ■

It is not difficult to show that the model (3) also has the equilibrium points  $P_1 = (S, I, R, A) = (0, 0, 0, T)$  and  $P_2 = (S, I, R, A) = (T, 0, 0, 0)$ , while the endemic equilibrium point  $P_3$  exists if and only if  $T > \delta/\beta$ . In a similar way as in the previous works [2, 4, 5, 6, 7, 8], it is easy to establish the local stability of  $P_2$  and  $P_3$  by the nondirect Lyapunov method. So, we have

**Proposition 1.** *Consider NSFD (3) under the assumptions (4). Then the equilibrium point  $P_2$  is unstable, while the equilibrium point  $P_3$ , if existing, also is unstable.*

As an important corollary of Theorem 2, we can establish the global stability of the model (2).

**Corollary 1.** *The equilibrium point  $P_1$  of the model (3) is globally stable.*

*Proof.* Repeating the proof of Theorem 1 with the discrete Lyapunov function

$$V(S_k, I_k, R_k) = S_k + I_k + R_k,$$

we obtain the corollary. ■

Summarizing the results in this section we have the following

**Theorem 3.** *NSFD scheme (3) preserves the properties  $(P_1)$  and  $(P_2)$  of the model (1) if the denominator function satisfies (4).*

#### 4. FOURTH ORDER NSFD SCHEMES FOR MODEL (1) BY RUNGE-KUTTA METHOD

In this section we construct NSFD schemes of fourth order accuracy based on the explicit Runge-Kutta methods. Firstly, we briefly recall the definition of explicit Runge-Kutta methods for the initial value problem (IVP) of the following form

$$y' = f(x, y), \quad y(x_0) = y_0. \quad (6)$$

**Definition 1.** (See [13], Definition 1.1). Let  $s$  be an integer (the “number of stages”) and  $a_{21}, a_{31}, a_{32}, \dots, a_{s1}, a_{s2}, \dots, a_{s,s-1}, b_1, \dots, b_s, c_2, \dots, c_s$  be real coefficients. Then the method

$$\begin{aligned} k_1 &= f(x_0, y_0), \\ k_2 &= f(x_0 + c_2h, y_0 + a_{21}hk_1), \\ k_3 &= f(x_0 + c_3h, y_0 + h(a_{31}k_1 + a_{32}k_2)), \\ &\dots \\ k_s &= f(x_0 + c_sh, y_0 + h(a_{s1}k_1 + \dots + a_{s,s-1}k_{s-1})), \\ y_1 &= y_0 + h(b_1k_1 + \dots + b_s k_s), \end{aligned}$$

is called an  $s$ -stage explicit Runge-Kutta method (ERK) for (6).

Usually,  $c_i$  satisfy the conditions

$$c_2 = a_{21}, \quad c_3 = a_{31} + a_{32}, \dots, \quad c_s = a_{s1} + \dots + a_{s,s-1}k_s,$$

or briefly,

$$c_i = \sum_{j=1}^s a_{ij}.$$

These conditions, already assumed by Kutta, express that all points where  $f$  is evaluated at are first order approximations to the solution. They greatly simplify the derivation of order conditions for high order methods. For low orders, however, these assumptions are not necessary (see [13, Chapter II]). Definition of order and order conditions for Runge-Kutta methods are presented in detail in [13].

For brevity, Runge-Kutta methods usually are denoted by  $(c, A, b^T)$  or  $RK(A, b)$  (see [13, 14, 15]), where  $c = (c_i), b = (b_i) \in \mathbb{R}^s$  and  $A = (a_{i,j}) \in \mathbb{R}^{s \times s}$ .

The problem of the positivity of Runge-Kutta methods is especially paid attention to (see [13, 14] and references therein). In this section, we shall use the result on the positivity step size threshold of Runge-Kutta methods [13]. For ease of understanding, we recall it as follows.

Consider IVPs of the form

$$U'(t) = f(t, U(t)), \quad t \geq t_0, \quad U(t_0) = u_0, \quad (7)$$

where  $t_0 \in \mathbb{R}$ ,  $n$  is a positive integer,  $u_0 \in \mathbb{R}^n$  and  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ . We assume tacitly that  $f$  is continuous and (7) has a unique uncontinuable solution for all  $t_0 \in \mathbb{R}$  and  $u_0 \in \mathbb{R}^n$  (i.e., there exists the largest  $t^* \in (t_0, \infty]$  with the property that (7) has a unique solution on  $[t_0, t^*)$ ).

We call the IVP (7) positive if  $U(t) \geq 0$  holds for all  $t \in [t_0, t^*)$  whenever  $t_0 \in \mathbb{R}$  and  $u_0 \geq 0$  are arbitrary. We denote by  $\mathcal{P}$  the set of functions  $f$  for which the corresponding IVPs of form (7) are positive. A sufficient and necessary condition for  $f$  to belong to  $\mathcal{P}$  can be found in [13, 14]. Namely  $f \in \mathcal{P}$  iff for all  $k, t$  and  $v \geq 0$  with  $v_k = 0$  we have  $f_k(t, v) \geq 0$ .

**Definition 2.** (See [13]) Let there be given  $\emptyset \neq \mathcal{F} \subset \mathcal{P}$ ,  $(A, b)$  a scheme of an RK method and  $H \in (0, \infty]$ . We call  $RK(A, b)$  positive on  $\mathcal{F}$  with positivity step size threshold  $H$  if the method is well-defined on (7) with step sizes less than or equal to  $H$  and  $u_m \geq 0$  for any  $m, f \in \mathcal{F}, t_0 \in \mathbb{R}, u_0 \geq 0$  and finite steps  $h_m \in [0, H]$ . If  $H$  is a positivity step size threshold and there is no greater positivity step size threshold than  $H$ , we call  $H$  the strict positivity step size threshold of  $RK(A, b)$  w.r.t.  $\mathcal{F}$ .

For any  $\alpha \in \mathbb{R}$  we define (see [13])

$$\mathcal{P}_\alpha = \left\{ f \mid f(t, v) + \alpha v \geq 0 \text{ for all } t, v \geq 0 \right\}.$$

**Theorem 4.** (See [13, Theorem 4]) Let  $(A, b)$  be the scheme of an RK method and  $\emptyset \neq \mathcal{F} \subset \mathcal{P}_\alpha$  with an  $\alpha > 0$ . Suppose that  $RK(A, b)$  is well-defined on (7) with any  $f \in \mathcal{F}, t_0 \in \mathbb{R}, u_0 \in \mathbb{R}^n$  and step sizes not larger than  $H_{def} = H_{def}((A, b), \mathcal{F})$ . Then

$$H = \min \left\{ \frac{R(A, b)}{\alpha}, H_{def} \right\},$$

is a positivity step size threshold of  $RK(A, b)$  w.r.t.  $\mathcal{F}$ .

In Theorem 4,  $R(A, b)$  is positivity radius of Runge-Kutta methods with the coefficient scheme  $(A, b)$  [13, 14, 15]. The radius  $R(A, b)$  is used by Kraaijevanger [15] in the study of contractivity of RK methods and also used in the nonlinear positivity theory for RK methods by Horvath [13, 14]. The results related to the properties of  $R(A, b)$  may be found in [15].

Next, in order to construct NSFD schemes not only preserving properties of the continuous model but also having high order accuracy, we shall use a 5-stage Runge-Kutta method  $(c, A, b^T)$  defined in [15]

$$\begin{aligned}
 b_1 &= 0.14681187608478644956; & a_{21} &= c_2; \\
 b_2 &= 0.24848290944497614757; & a_{31} &= 0.21766909626116921036; \\
 b_3 &= 0.10425883033198029567; & a_{32} &= 0.36841059305037202075; \\
 b_4 &= 0.27443890090134945681; & a_{41} &= 0.8269208665781075441; \\
 b_5 &= 0.22600748323690765039; & a_{42} &= 0.13995850219189573938; \\
 c_2 &= 0.39175222657188905833; & a_{43} &= 0.25189177427169263984; \\
 c_3 &= 0.58607968931154123111; & a_{51} &= 0.06796628363711496324; \\
 c_4 &= 0.47454236312139913362; & a_{52} &= 0.11503469850463199467; \\
 c_5 &= 0.93501063096765159845; & a_{53} &= 0.20703489859738471851; \\
 & & a_{54} &= 0.54497475022851992204.
 \end{aligned}$$

However, a difference here is that instead of the standard denominator  $h$  we use a nonstandard denominator function  $\varphi(h) = h + \mathcal{O}(h^2)$ . We call this difference scheme nonstandard Runge-Kutta (NSRK) scheme  $(c, A, b^T, \varphi)$ .

**Theorem 5.** *NSRK scheme  $(c, A, b^T, \varphi)$  preserves the positivity and boundedness (Property  $(P_1)$ ) of the model (1) if the denominator function  $\varphi(h)$  satisfies*

$$\varphi(h) < \min \left\{ \frac{r}{(\alpha_{SA} + \beta)T}, \frac{r}{(\alpha_{IA} + \delta)T}, \frac{r}{\sigma} \right\}, \quad \forall h > 0 \quad (8)$$

with

$$r = R(A, b) \approx 1.50818004918983792280.$$

*Proof.* Set  $\tau = \min\{(\alpha_{SA} + \beta)T, \alpha_{IA} + \delta)T, \sigma\}$ . It is easy to verify that the right-hand side of (1) belongs to the set  $P_\alpha$  with  $\alpha = \tau$ . Therefore, from the positivity of Runge-Kutta methods (Theorem 4) it follows the conclusion of the theorem. ■

Before the analysis of stability of NSFD we determine the eigenvalues of the Jacobian matrix of (1). We call  $J(P)$  the Jacobian matrices of (1) calculated at the equilibrium point  $P$  and  $\sigma(J(P))$  the set of the eigenvalues of  $J(P)$ . Then by [25, Section 3] we have

$$\begin{aligned}
 \sigma(J(P_1)) &= \left\{ -\alpha_{SA}T, -\alpha_{IA}T - \delta, -\sigma, 0 \right\}, \\
 \sigma(J(P_2)) &= \left\{ 0, -\beta T - \delta, -\sigma, \alpha T \right\}.
 \end{aligned}$$

On the other hand, if the equilibrium point  $P_3$  exists ( $T > \delta/\beta$ ) then  $J(P_3)$  has a positive eigenvalue defined by

$$\lambda^* = \alpha_{SA}\delta\sigma + \alpha_{SA}\delta^2 + \alpha_{IA}\beta\sigma T - \alpha_{IA}\delta\sigma.$$

**Theorem 6.** Consider NSRK scheme  $(c, A, b^T, \varphi)$  under the assumption (8). Then

1. The equilibrium point  $P_2$  is unstable.
2. The equilibrium point  $P_3$  if existing, is unstable.
3. There exists a number  $\tau^* > 0$  such that if the denominator function  $\varphi(h)$  satisfies  $\varphi(h) < \tau^*$  then  $P_1$  is locally asymptotically stable.

*Proof.* Call  $J$  and  $J^*$  the Jacobian matrices of (1) and NSRK  $(c, A, b^T, \varphi)$  calculated at a certain equilibrium point  $E^*$ , respectively. Then, according to the results of the stability function of Runge-Kutta methods [1] we obtain: if  $\lambda$  is an eigenvalue of  $J$  then  $\mu = \mu(\lambda)$  is a corresponding eigenvalue of  $J^*$ , where  $\mu$  is defined by

$$\mu = a_5 z^5 + \frac{1}{24} z^4 + \frac{1}{6} z^3 + \frac{1}{2} z^2 + z + 1,$$

with  $a_5 = b^T A^4 \mathbf{1} > 0$  and  $z = \lambda\varphi$ .

For the equilibrium point  $P_2$ , corresponding to the eigenvalue  $\lambda = \alpha T$  of  $J$  there is the eigenvalue  $\mu = \mu(\lambda)$  of  $J^*$ . Since  $\lambda > 0$  then  $\mu > 1$ . Therefore, the matrix  $J^*(P_2)$  has an eigenvalue lying outside of the unit circle. By Lyapunov theorem [10, 16] the point  $P_2$  is unstable.

In a similar way, as is known, if the equilibrium point  $P_3$  exists ( $T > \delta/\beta$ ) then  $J(P_3)$  has a positive eigenvalue  $\lambda^*$ . Corresponding to the eigenvalue  $\lambda^*$  of  $J$  we have the eigenvalue  $\mu^*$  of  $J^*$ . Obviously,  $\mu^* > 1$  because  $\lambda^* > 0$ . Hence, the point  $P_3$  is unstable.

Now we consider the stability of the equilibrium point  $P_1$ . Recall that  $\sigma(J(P_1)) = \{-\alpha_{SA}T, -\alpha_{IA}T - \delta, -\sigma, 0\}$ . Corresponding to the eigenvalue  $\lambda = 0$  of  $J$  we have the eigenvalue  $\mu = 1$  of  $J^*$ . Nevertheless, as in the continuous case, this eigenvalue does not imply bifurcation or central manifold for the model [25], representing only the fact that one equation can be expressed as a linear combination of the other three. Corresponding to the eigenvalues  $\lambda_1 = -\alpha_{SA}T, \lambda_2 = -\alpha_{IA}T - \delta, \lambda_3 = -\sigma$  of  $J$  we have the following eigenvalues of  $J^*$

$$\begin{aligned} \mathcal{P}(z) &= a_5 z^5 + \frac{1}{24} z^4 + \frac{1}{6} z^3 + \frac{1}{2} z^2 + z + 1, & z &= \varphi\lambda_1, \\ \mathcal{P}(z) &= a_5 z^5 + \frac{1}{24} z^4 + \frac{1}{6} z^3 + \frac{1}{2} z^2 + z + 1, & z &= \varphi\lambda_2, \\ \mathcal{P}(z) &= a_5 z^5 + \frac{1}{24} z^4 + \frac{1}{6} z^3 + \frac{1}{2} z^2 + z + 1, & z &= \varphi\lambda_3. \end{aligned}$$

By Lyapunov theorem [10, 16], the necessary and sufficient condition for  $P_1$  to be locally stable is  $|\mathcal{P}(\lambda_i)| < 1, i = 1, 2, 3$ . This is equivalent to the system

$$a_5 \varphi^5 (\lambda_i)^5 + \frac{1}{24} \varphi^4 (\lambda_i)^4 + \frac{1}{6} \varphi^3 (\lambda_i)^3 + \frac{1}{2} \varphi^2 (\lambda_i)^2 + \varphi \lambda_i < 0, \tag{9}$$

$$\mathcal{P}_i(\varphi) := a_5 \varphi^5 (\lambda_i)^5 + \frac{1}{24} \varphi^4 (\lambda_i)^4 + \frac{1}{6} \varphi^3 (\lambda_i)^3 + \frac{1}{2} \varphi^2 (\lambda_i)^2 + \varphi \lambda_i > -2, \tag{10}$$

for  $i = 1, 2, 3$ . We also see that (9) is equivalent to

$$\mathcal{Q}_i := a_5 \varphi^4 (\lambda_i)^5 + \frac{1}{24} \varphi^3 (\lambda_i)^4 + \frac{1}{6} \varphi^2 (\lambda_i)^3 + \frac{1}{2} \varphi (\lambda_i)^2 + \lambda_i < 0.$$



It is easy to see that  $\mathcal{P}_i(\varphi) \rightarrow 0$  as  $\varphi \rightarrow 0$  and  $\mathcal{Q}_i(\varphi) \rightarrow \lambda_i < 0$  as  $\varphi \rightarrow 0$ . Therefore, from the definition of limit of a function it follows that there exists a number  $\tau^* > 0$  such that  $\mathcal{P}_i(\varphi) > -2$  and  $\mathcal{Q}_i(\varphi) < 0$  for any  $\varphi < \tau^*$ , or in other words, (9) and (10) are satisfied if  $\varphi < \tau^*$ . Thus, the theorem is proved. ■

**Remark 1.** In Theorem 6, the number  $\tau^*$  can be determined as  $\tau^* = \min_{i=1,2,3}\{p_i, q_i\}$ , where  $p_i$  and  $q_i$  are minimal root of the polynomials  $\mathcal{P}_i(\varphi)$  and  $\mathcal{Q}_i(\varphi)$ , respectively.

Summarizing the results in this section we obtain

**Theorem 7.** NSRK (5) preserves the properties  $(P_1)$  and  $(P_2)$  of (1) if the denominator function satisfies the condition

$$\varphi(h) < \min \left\{ \frac{r}{(\alpha_{SA} + \beta)T}, \frac{r}{(\alpha_{IA} + \delta)T}, \frac{r}{\sigma}, \tau^* \right\}, \quad \forall h > 0. \tag{11}$$

Clearly, the denominator function  $\varphi(h) = h$  does not satisfy (11). Therefore, we should select the denominator function satisfying (11) and not influencing on the accuracy order of the original Runge-Kutta methods. For doing this we need the following

**Corollary 2.** NSRK  $(c, A, b^T, \varphi)$  is of fourth order of accuracy if the denominator function satisfies the condition

$$\varphi(h) = h + \mathcal{O}(h^5). \tag{12}$$

Thus, we have to select denominator functions satisfying simultaneously (11) and (12). The selection of such functions is an interesting and important problem. Analogously as in the recent work [6], we select denominator functions of the form

$$\varphi(h) = (1 - \theta(h)) \frac{1 - e^{-\tau h}}{\tau} + \theta(h) h e^{-\mu h^m}, \quad \theta(h) = 1 + \mathcal{O}(h^p).$$

### 5. NUMERICAL SIMULATIONS

In this section we report the results of some numerical simulations in order to confirm the validity of obtained theoretical results and to demonstrate the efficiency of designed NSFD schemes. It should be emphasized that all numerical simulations in [2, 4, 5, 6, 7, 8, 19, 20, 21, 22] showed that standard difference schemes do not preserve essential properties of the corresponding continuous models.

**Example 1.** Consider the model (1) with the parameters

$$\beta = 0.1, \delta = 20, \sigma = 0.8, \alpha_{SA} = 0.25, \alpha_{IA} = 0.25.$$

In this case we take  $T = 100$ . The numerical solutions obtained by NSFD schemes (3) for the model (2) is depicted in Figure 1. From the figure it is seen that  $P_1$  is globally asymptotically stable,  $P_2$  is unstable and  $P_3$  does not exist. Moreover, the properties of the continuous model are preserved.

**Example 2.** Consider the model (1) with the parameters

$$\beta = 0.1, \delta = 9, \sigma = 0.8, \alpha_{SA} = 0.25, \alpha_{IA} = 0.25.$$

For this case we take  $T = 100$ . The numerical solutions of the model (2) obtained by the NSFD schemes (3) are depicted in Figure 2. Obviously,  $P_1$  is globally asymptotically stable,  $P_2$  and  $P_3$  are unstable. Moreover, the properties of the continuous model are preserved.

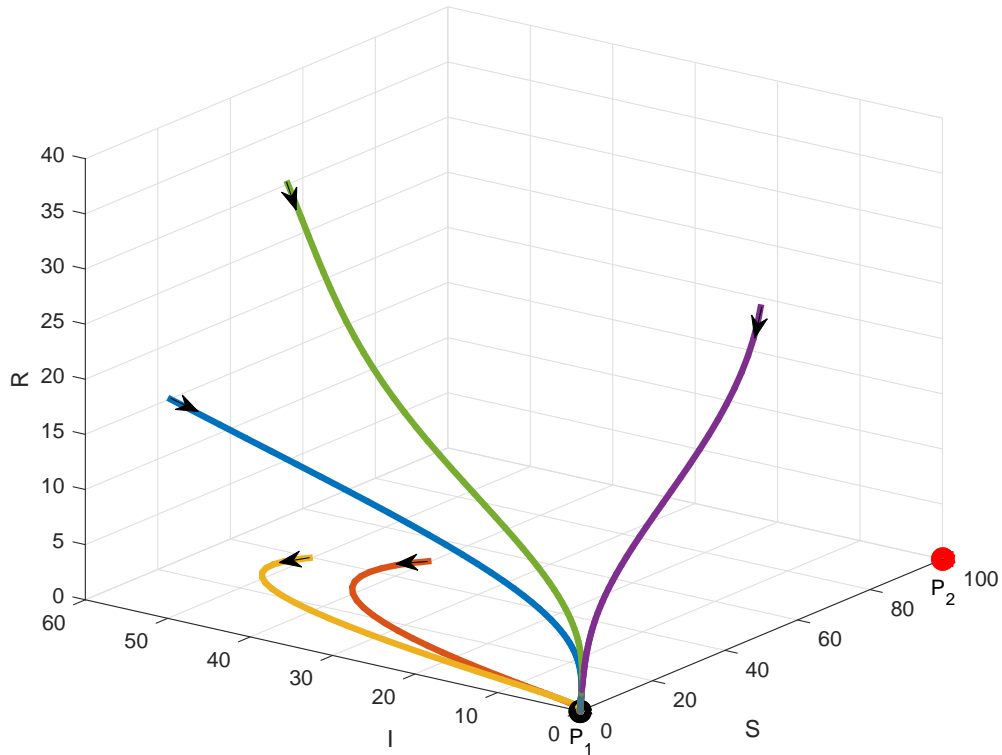


Figure 1. Numerical solutions obtained by NSFD schemes (3) with  $\varphi(h) = 1 - e^{-h}$  and  $h = 1$

**Example 3.** Accuracy and computation time of schemes  
 Consider the model (1) with the parameters

$$\beta = 0.01, \delta = 0.02, \sigma = 0.5, \alpha_{SA} = 0.025, \alpha_{IA} = 0.02$$

and the initial values (20, 30, 20, 30). We report the errors of the NSFD and NSRK ( $c, A, b^T, \varphi$ ). For comparison we also consider the method proposed by Wood and Koruharov in [27], which preserves the positivity and stability of dynamical systems based on nonlocal discretization. The denominator functions used for NSFD scheme (3), NSRK scheme and Wood and Koruharov’s scheme, respectively are

$$\varphi(h) = \frac{1 - e^{-0.6h}}{0.6}, \quad \varphi(h) = e^{-h^8} h e^{-0.0001h^6} + \frac{(1 - e^{-h^8})(1 - e^{-1.1h})}{1.1}, \quad \varphi(h) = \frac{1 - e^{-0.6h}}{0.6h}.$$

Since it is impossible to find the exact solution of the model, as a benchmark we use the numerical solution obtained by 11-stage Runge-Kutta method [9]. The benchmark solution is depicted in Figure 3. From the figure it is seen that the components of the solution change very quickly in a short time from the starting points, after that they become stable. The differential problem in this case is stiff. Table 1 provides the errors and the rates of the NSFD

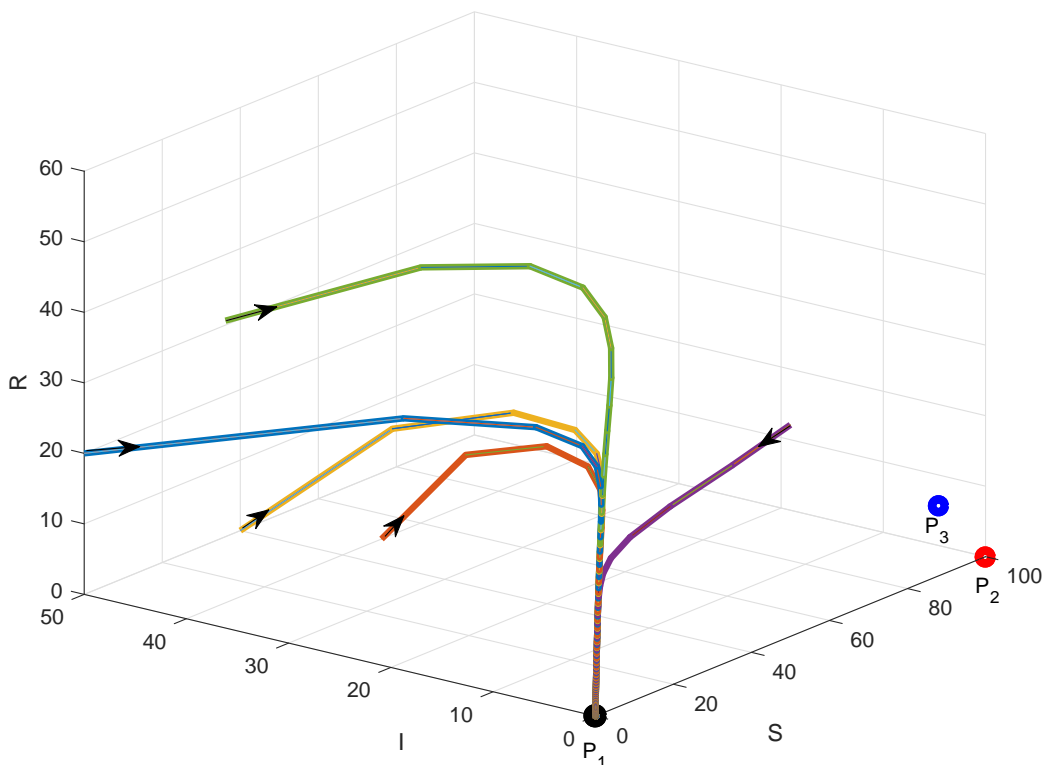


Figure 2. Numerical solutions obtained by NSFD schemes (3) with  $\varphi(h) = e^{-h^8} h e^{-0.0001h^6} + (1 - e^{-h^8})(1 - e^{-1.1h})/1.1$  and  $h = 1$

scheme, Wood and Koruharov’s (W-K’s) scheme and NSRK scheme for different stepsizes. There errors are computed by the formula

$$err = \max_k \{|S_k - S(t_k)| + |I_k - I(t_k)| + |R_k - R(t_k)| + |A_k - A(t_k)|\},$$

where  $U_k$  and  $U(t_k)$  are the solutions obtained by a scheme and the benchmark solution, respectively. Besides,  $rate := \log_{\frac{h_1}{h_2}}(err1/err2)$  (see [1, Example 4.1]) is an approximation for accuracy order of the schemes. In the last column of Table 1 (rate of NSRK scheme), we see an unexpected phenomenon, when  $h$  is small the rates decrease. A similar phenomenon also was indicated in [1, Example 4.1]) when studying explicit standard Runge-Kutta methods. The reason of this is that the rounding errors generally increase as  $h$  decreases.

The computation time is given in Table 2. From the tables we see that NSFD (3) has better accuracy and faster than the Wood and Koruharov’s scheme. The reason is that at each step Wood and Koruharov’s scheme needs to consider the sign of the right-hand sides for choosing appropriate discretization. NSRK scheme has the best accuracy but the computation time is largest because at each step it requires computation of values of stages  $K_i$ . However, to obtain the solution of high order of accuracy the use of NSRK is more efficient than the use of extrapolation, which combines the solutions of first order of

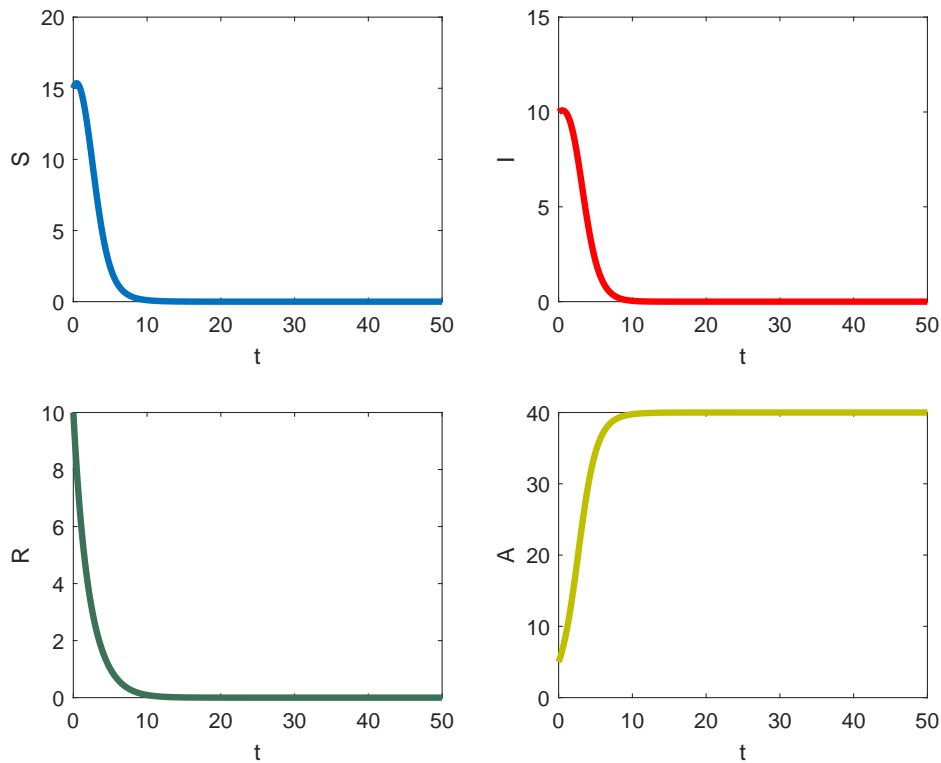


Figure 3. Benchmark solution obtained by RK8 with  $h = 10^{-5}$

Table 1. Errors and rates of the schemes

Stepsize	NSFD scheme	Rate	W-K's scheme	Rate	NSRK scheme	Rate
0.25	6.463466701595602		7.713254298410573		1.2780e-04	
0.2	5.243459924068896	0.9374	6.056566836376636	1.0836	2.7012e-05	6.9650
0.15	3.989310828795733	0.9502	4.459865753797996	1.0638	5.5149e-06	5.5229
0.1	2.698912044130053	0.9638	2.920085782158711	1.0445	9.2956e-07	4.3913
0.05	1.369838435912756	0.9784	1.434358023863455	1.0256	5.6487e-08	4.0405
0.001	0.027798934145437	0.9963	0.028200224634866	1.0044	1.7708e-12	2.6509
0.00001	0.000278071734776	0.9999	0.000281905383956	1.0001	9.3258e-15	1.1392

accuracy. The advantage of NSRK is that it has high order of accuracy for small stepsizes and preserves the properties of the model for large stepsizes. Moreover, it is explicit, easy to be programmed.

## 6. CONCLUSIONS

In this paper we have constructed two families of NSFD schemes preserving the essential properties of a computer virus spread model. They are positivity, boundedness and local stability. Besides, the first NSFD schemes are globally stable, the second NSFD schemes are

Table 2. Computation times of the schemes in seconds

Stepsize	NSFD scheme (3)	W-Ks scheme	NSRK scheme
0.2	0.001897	0.027662	0.048200
0.1	0.001942	0.012523	0.043975
0.05	0.002224	0.016025	0.064037
0.01	0.006202	0.066508	0.306687
0.005	0.006793	0.118272	0.632497
0.001	0.028386	0.616064	2.859388
0.0001	0.210997	5.866315	27.817605
0.00001	2.069937	57.343785	273.413824

of fourth order of accuracy. The numerical simulations confirm the validity of obtained theoretical results. Among these constructed schemes, NSFD schemes have advantage in order of accuracy for small stepsizes and preserves the properties of the corresponding continuous model for large stepsizes. The method for designing high order, dynamically consistent schemes are applicable to some other applied models. This is the subject of our research in the future.

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