THEORETICAL ANALYSIS OF PICTURE FUZZY CLUSTERING

PHAM THI MINH PHUONG[1], PHAM HUY THONG, LE HOANG SON

Vietnam National University, Ha Noi, Viet Nam
[1]*phamthiminhphuong_t60@hus.edu.vn*

Abstract. Recently, picture fuzzy clustering (FC-PFS) has been introduced as a new computational intelligence tool for various problems in knowledge discovery and pattern recognition. However, an important question that was lacked in the related researches is examination of mathematical properties behind the picture fuzzy clustering algorithm such as the convergence, the boundary or the convergence rate, etc. In this paper, we will prove that FC-PFS converges to at least one local minimum. Analysis on the loss function is also considered.

Keywords. Convergence analysis, picture fuzzy sets, picture fuzzy clustering.

1. INTRODUCTION

One of the most efficient tools in pattern recogntion and knowledge discovery is fuzzy clustering in which the uncertainty and vagueness of data can be handled sucessfully. Fuzzy clustering, as its reminiscent names recalled, uses a membership function to assign for each data elements in the original dataset. The decision of an appropriate cluster depends on the membership values, that is to say, a greater one implies the inclusion. Fuzzy clustering sucessfully handle the problem of crisp clustering in which a data element can belong to many clusters at the same time [1, 2]. However, it was deployed on the traditional fuzzy set, which shows some limitations in dealing with practical scenarios like voting [3].

A new extension of the fuzzy set called the Picture Fuzzy Set (PFS) was presented by Cuong in [3, 4] to handle such the problem. A PFS is characterized by three membership degrees: positive, neutral, and negative degrees. In the real case of voting applications, 'positive' refers to the support for a candidate, 'negative' in reverse shows the opposition, and 'neutral' reflects the hesistant group who do not agree and disagree. There are many other cases to demonstrate the usage and practical necessity of the PFS [5].

Picture Fuzzy Set has been applied to decision making problems as in the works of Wei [6, 7, 8, 9, 10]. In these researches, the authors have applied picture aggregation opeartors and picture fuzzy entropy in multi-attribute decision problems. Some new operators based on the cosine function and their weighted variants have been utilized for recommendation of products [8]. In [9], picture Bonferroni mean operators have been given in the view of software suppliers. The 2-tuple linguistic picture operators were also examined in [10, 11]. Yang et al. [12] extended the notion of picture fuzzy soft set. Other decision making procedures in the picture fuzzy set can be retrieved in [13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. Wei [23] summarized some similarity measures in the picture fuzzy set. Indeed, Singh [24] proposed correlation coefficients for picture fuzzy sets. Zhang [25] designed Picture Fuzzy

Filters. Other researches regarding picture operators, picture fuzzy rules and database can be found in [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37].

Picture fuzzy clustering (FC-PFS) is a generalization of the traditional fuzzy clustering algorithm [38]. By adding a new membership to the fuzzy set to denote the vagueness of prototype parameters, the FC-PFS has already covered situations that require human opinions as in above. It gives precise results for clustering which has been proven through numerous researches recently [39, 40, 41, 42, 43, 44, 45]. FC-PFS showed significant roles in weather nowcasting from satellite image sequences [42], brain tumor segmentation [5], recommender systems [40], and stock prediction [39].

However, to create a solid constructed basis for the algorithm, it is necessary to perform the theoretical analysis. Proving the convergence of picture fuzzy clustering is of an important role in understanding the algorithm and how it is evolved. In this paper, the convergence of the FC-PFS algorithm is proven and some properties of its such as the boundary of the loss function are expanded. The similarities and differences between this algorithm and other clustering methods are compared. Analysis on the loss function is also considered.

Section 2 recalls the general definition of the picture fuzzy set. The convergence accompanied with some propositions is proven followed by the Zangwill theorem in Section 3. Section 4 validates the way to calculate the boundary of the loss function and describe the changing of the loss function until convergence. The final section draws the conclusion and delineates the future research directions.

2. PRELIMINARY

Definition 1. [3] A picture fuzzy set (PFS) E on the universe Y is

$$E = (y, \mu_E(y), \eta_E(y), \gamma_E(y)\|y \in Y, \tag{1}$$

where $\mu_E(y) \in [0,1]$, $\eta_E(y) \in [0,1]$, and $\gamma_E \in [0,1]$ are the positive, neutral, and negative memberships of y in Y satisfying

$$\mu_E(y) + \eta_E(y) + \gamma_E(y) \leq 1, \forall y \in Y. \tag{2}$$

Definition 2. [38] Assume Y is a dataset of N points in R dimensions and $\mu_{kj} = \mu_{kj}(y)$, $\eta_{kj} = \eta_{kj}(y)$, $\xi_{kj} = \xi_{kj}(y)$, $1 \leq j \leq C$, $1 \leq k \leq N$, C is a number of clusters, V_j is the cluster center j, $1 \leq j \leq C$, m is fuzzifier, α is exponent coefficient. The **picture fuzzy clustering** model is

$$J_m = \sum_{k=1}^{N} \sum_{j=1}^{C} (\mu_{kj}(2 - \xi_{kj}))^m \|Y_k - V_j\|^2 + \sum_{k=1}^{N} \sum_{j=1}^{C} \eta_{kj}(\ln \eta_{kj} + \xi_{kj}) \to \min, \tag{3}$$

where

$$\xi_{kj} = 1 - (\mu_{kj} + \eta_{kj} + \gamma_{kj}), \tag{4}$$

with constraints

$$\mu_{kj} + \eta_{kj} + \xi_{kj} \leq 1, \tag{5}$$

$$\mu_{kj} \in [0,1], \eta_{kj} \in [0,1], \xi_{kj} \in [0,1], \tag{6}$$

$$\sum_{j=1}^{C}(\mu_{kj}(2 - \xi_{kj})) = 1, \tag{7}$$

$$\sum_{j=1}^{C}(\eta_{kj} + \frac{\xi_{kj}}{C}) = 1. \tag{8}$$

Let us denote,

$$U_c \triangleq \{U = [\mu_{kj}] \in \mathbb{R}^{c \times n} : \mu_{kj} \text{ satisfies } (1) \ \forall i, k\},$$

$$N_c \triangleq \{N = [\eta_{kj}] \in \mathbb{R}^{c \times n} : \eta_{kj} \text{ satisfies } (1) \ \forall i, k\},$$

$$Z_c \triangleq \{Z = [\xi_{kj}] \in \mathbb{R}^{c \times n} : \xi_{kj} \text{ satisfies } (1) \ \forall i, k\}.$$

It was shown in [38] that (U^*, V^*, N^*, Z^*) might be a local minimum of J_m if and only if for any $m > 1$

$$\mu_{kj}^* = \frac{1}{\sum_{i=1}^{C}(2 - \xi_{kj}^*)\left(\frac{\|Y_k - V_j^*\|}{\|Y_k - V_i^*\|}\right)^{\frac{2}{m-1}}}, \ (1 \le j \le C, \ 1 \le k \le N), \tag{9}$$

$$\eta_{kj}^* = \frac{e^{-\xi_{kj}^*}}{\sum_{i=1}^{C} e^{-\xi_{ki}^*}}\left(1 - \frac{1}{C}\sum_{i=1}^{C}\xi_{kj}^*\right), \ (1 \le j \le C, \ 1 \le k \le N), \tag{10}$$

$$\xi_{kj}^* = 1 - (\mu_{kj}^* + \eta_{kj}^*) - (1 - (\mu_{kj}^* + \eta_{kj}^*)^\alpha)^{\frac{1}{\alpha}}, \ (1 \le j \le C, \ 1 \le k \le N), \tag{11}$$

$$V_j^* = \frac{\sum_{k=1}^{N}(\mu_{kj}^*(2 - \xi_{kj}^*))^m Y_k}{\sum_{k=1}^{N}(\mu_{kj}^*(2 - \xi_{kj}^*))^m}, \ (1 \le j \le C, \ 1 \le k \le N). \tag{12}$$

The following describes the FC-PFS algorithm [38].

Picture Fuzzy Clustering algorithm
1. Input: Data Y with N elements; C is number of clusters, threshold ϵ; fuzzifier m; exponent α; maxstep ≥ 0.
2. Initialize $\mu_{kj}^0 \leftarrow$ random, $\eta_{kj}^0 \leftarrow$ random, $\xi_{kj}^0 \leftarrow$ random, $(1 \le j \le C), (1 \le k \le N)$ satisfying constraints (5-8).
3. For each iteration t, update $\mu_{kj}^t, \eta_{kj}^t, \xi_{kj}^t, V_j^t$ following equations (9-12) respectively.
4. Until $\|\mu^t - \mu^{t-1}\| + \|\eta^t - \eta^{t-1}\| + \|\xi^t - \xi^{t-1}\| < \epsilon$ or $h >$ maxstep, stop.
5. Output: matrices μ, η, ξ and centers V.

3.   CONVERGENCE OF PICTURE FUZZY CLUSTERING

In this section, we explore some propositions which ensure the convergence of FC-PFS.

Proposition 1. *Let* $\phi : U_c \to \mathbb{R}$, $\phi(U) := J_m(U, V, N, Z)$, *where* V, N, Z *are fixed. Then,* $U^* \in M_{fc}$ *is a strict local minimum of* ϕ *if and only if* U^* *is calculated as in eq. (9).*

Proof. Since μ_{kj} has two constrains in eqs. (4), (6), we consider the relaxed minimization of $\phi(U)$ via Lagrange multipliers. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ be the multipliers, and $L(U, \lambda)$ be the Lagrangian

$$L(U, \lambda) = \sum_{k=1}^{N} \sum_{j=1}^{C} (\mu_{kj}(2 - \xi_{kj}))^m \|Y_k - V_j\|^2 + \sum_{k=1}^{N} \sum_{j=1}^{C} \eta_{kj}(\ln \eta_{kj} + \xi_{kj})$$
$$- \lambda_k \left(\sum_{j=1}^{C} (\mu_{kj}(2 - \xi_{kj})) - 1 \right).$$

Then,

$$\frac{\partial L(U, \lambda)}{\partial \mu_{kj}} = m u_{kj}^{m-1}(2 - \xi_{kj})^m \|Y_k - V_j\|^2 - \lambda_k(2 - \xi_{kj}) = 0 \text{ at } U^*,$$

and calculate the second-order derivative of $L(U, \lambda)$

$$\frac{\partial}{\partial \mu_{st}} \left(\frac{\partial L(U, \lambda)}{\partial \mu_{kj}} \right) = \begin{cases} (m-1)m\mu_{kj}^{m-2}(2 - \xi_{kj})^m \|Y_k - V_j\|^2 & \text{if } s = k, t = j \\ 0 & \text{otherwise.} \end{cases}$$

Now, substitute the updated formula of μ_{kj} into the second-order derivation of $L(U, \lambda)$ to calculate the Hessian matrix $H(U^*)$. It follows that $H(U)$ is a nonzero entries matrix, whose diagonal elements are

$$\alpha_{kj,kj} = (m-1)m(\mu_{kj}^*)^{m-2}(2 - \xi_{kj})^m \|Y_k - V_j\|^2$$
$$= (m-1)m(2 - \xi_{kj})^m \|Y_k - V_j\|^2 \frac{1}{\left(\sum_{i=1}^{C} (2 - \xi_{kj}) \left(\frac{\|Y_k - V_j\|^2}{\|Y_k - V_i\|^2} \right)^{\frac{2}{m-1}} \right)^{m-2}}$$
$$> 0$$

and $\alpha_{st,kj} = 0$ for $s \neq k$ and $t \neq j$.

The Hessian of ϕ at U^* has all positive eigenvalues which are $\alpha_{kj,kj}, 1 \leq k \leq N, 1 \leq j \leq C$. It is sufficient to show that U^* is a strict local minimum of ϕ. ∎

Next, we fix $U \in M_c, N \in N_c, Z \in Z_c$ and consider the minimization of J_m in variables $V = \{V_i\}$.

Proposition 2. *Let $\psi : \mathbb{R}^{CS} \to \mathbb{R}$, $\psi(V) \doteq J_m(U, V, N, Z)$, where U, N, Z are fixed. Then, V^* is a strict local minimum of ψ if and only if $V_i^*, 1 \leq i \leq C$ is calculated via the updated formula in eq. (12).*

Proof. Since there is no constrains for V, in order to minimize ψ over $\mathbb{R}^{CS}$, it is necessary to require $\bigtriangledown_{V_i}\psi(V^*)$ to vanish for every i,

$$\frac{\partial \psi(V)}{\partial V_j} = \sum_{k=1}^{N}(u_{kj}(2 - \xi_{kj}))^m(-2Y_k + 2V_j) = 0 \text{ at } V^*,$$

then take the second-order derivative,

$$\frac{\partial}{\partial V_s}\left(\frac{\partial \psi(V)}{\partial V_j}\right) = \begin{cases} \sum_{k=1}^{N}(u_{kj}(2 - \xi_{kj}))^m > 0 & \text{if } s = j \\ 0 & \text{otherwise.} \end{cases}$$

The Hessian matrix of $\psi(V)$ at V^* has all positive eigenvalues. Therefore, V^* is sufficient to be minimum point of $\psi(V)$. ∎

A similar way to prove that η^* is sufficient to minimize J_m when U, V, Z are fixed in their spaces.

Proposition 3. *Let $f : N \to \mathbb{R} \doteq J_m(U, V, N, Z)$ where U, V, Z are fixed. Then, N^* is a strict local minimum of f if and only if $\eta_{kj}, 1 \leq k \leq n, 1 \leq j \leq C$ is calculated via eq. (10).*

Proof. Since each η_{kj} has it own constrains in eq.(7), we consider the minimization of $f(N)$ via Lagrange multipliers obtained constrains. Let $\beta = (\beta_1, \beta_2, ..., \beta_n)$ be the multipliers, and $L(N, \beta)$ be the Lagrangian

$$L(N, \beta) = \sum_{k=1}^{N}\sum_{j=1}^{C}(\mu_{kj}(2 - \xi_{kj}))^m\|Y_k - V_j\|^2 + \sum_{k=1}^{N}\sum_{j=1}^{C}\eta_{kj}(\ln \eta_{kj} + \xi_{kj})$$

$$- \beta_k\left(\sum_{j=1}^{C}\left(\eta_{kj} + \frac{\xi_{kj}}{C}\right) - 1\right).$$

Since N^* is the root of equation system,

$$\frac{\partial L(N, \beta)}{\partial \eta_{kj}} = \ln \eta_{kj} + 1 - \beta_k + \xi_{kj} = 0,$$

and

$$\frac{\partial}{\partial \eta_{st}}\left(\frac{\partial L(N, \beta)}{\partial \eta_{kj}}\right) = \begin{cases} \dfrac{1}{\eta_{kj}} > 0 & \text{if } s = k; t = j \\ 0 & \text{otherwise.} \end{cases}$$

The Hessian matrix of $L(N^*, \beta)$ at N^* has all positive eigenvalues and N^* also minimizes $f(N)$. ∎

The updated formula for the neutral degree is $h(\zeta_{kj}) = 1 - (\mu_{kj} + \eta_{kj}) - (1 - (\mu_{kj} + \eta_{kj})^\alpha)^{\frac{1}{\alpha}}$, where $h : Z \to \mathbb{R}$ which is based on the Yager's operator. When the neutral and refusal degrees of each elements increase, the entropy decreases. When the update of centroids change in minor variation, J_m may slowly increase to the convergence point.

Now, to show that the algorithm makes J_m converge, we use the Zangwill theorem below.

Proposition 4. *[1] Let $f : D_f \in \mathbb{R}^m \to \mathbb{R}$ and $S = \{y^* \in D_f : f(y^*) < f(y) \,\forall\, y \in B^o(y^*, r)\}$. Let $A : D_f \to D_f$ be an iterative algorithm and $y_{k+1} = E(y_k); k = 0, 1, ...$*

If E is continuous on $D_f \setminus S$, g is a descent function for A, S and the iterative sequences $E(Y_k) : k = 0, 1, 2, ...; y_0 \in D_f \subset K$ are contained in a compact set $K \subseteq D_f$ for arbitrary $y_0 \in D_f$, then for each iterative sequence Y_k generated by E, we have either Y_k terminates at a solution $y^* \in S$ or there exists a subsequence $y_{kj} \subset y_k$ so that $y_{kj} \to y^* \in S$.

To apply the Zangwill theorem, we need to show that J_m is a descent function and the algorithm is continuous on $[0, 1]^4 \setminus S$. Then, we only need to show that J_m is a descent function. Now, let P_m be the algorithm to update the parameters in eqs. (9-12).

Proposition 5. *Let*
$$S = \{(U*, V*, N*, Z*) : J_m(U*, V*, N*, Z*) <$$
$$J_m(U, V, N, Z), \forall (U, V, N, Z) \in B^o((U*, V*, N*, Z*), r)\}.$$
Then J_m is descent function for P_m, S.

Proof. Since the norm function and the exponent function are continuous, we call the sum of products of such functions as J_m. Obviously, J_m is also continuous on $M_{fc} \times \mathbb{R}^{CS}$. Suppose $(U, V, N, Z) \notin S$ then

$$
\begin{aligned}
J_m(P_m(U, V, N, Z)) &= J_m(P_1 \circ P_2 \circ P_3 \circ P_4(U, V, N, Z)) \\
&= J_m(P_1 \circ P_2 \circ P_3(U, V, N, h(Z))) \\
&< J_m(P_1 \circ P_2 \circ P_3(U, V, N, Z)) \\
&= J_m(P_1 \circ P_2(U, V, f(N), Z)) \\
&< J_m(P_1 \circ P_2(U, V, N, Z)) \\
&= J_m(P_1(U, \psi(V), N, Z)) \\
&< J_m(P_1(U, V, N, Z)) \\
&= J_m(\phi(U), V, N, Z) \\
&< J_m(U, V, N, Z).
\end{aligned}
$$

Hence J_m is a descent function.   ∎

However, in some cases, J_m will slowly increase because of updating of η. Because the difference between period centroids and the next centroids changes very small, it still guarantees that J_m converges.

4. SOME PROPERTIES

4.1. Property of the loss function

We consider the loss function

$$J_m = \sum_{k=1}^{N}\sum_{j=1}^{C}(\mu_{kj}(2-\xi_{kj}))^m\|Y_k-V_j\|^2 + \sum_{k=1}^{N}\sum_{j=1}^{C}\eta_{kj}(\ln\eta_{kj}+\xi_{kj}).$$

We also know that the first part $J_1 = \sum_{k=1}^{N}\sum_{j=1}^{C}(\mu_{kj}(2-\xi_{kj}))^m\|Y_k-V_j\|^2$ converges to a value called M. Now, we find the upper bound and lower bound of the second part. Let

$$J_2 = \sum_{k=1}^{N}\sum_{j=1}^{C}\eta_{kj}(\ln\eta_{kj}+\xi_{kj}).$$

We see that

$$\eta_{kj}(\ln\eta_{kj}+\xi_{kj}) \le \eta_{kj}(\ln\eta_{kj}+1-\eta_{kj}) \le 0.$$

Indeed, consider $f(y) = y(\ln y + 1 - y)$ with $y \in [0,1]$. Therefore,

$$f'(y) = 2 + \ln y - 2y.$$

$$f'(y) = 0 \leftrightarrow y = 1 \text{ or } y = -\frac{1}{2}W\left(-\frac{2}{e^2}\right).$$

From $f(1) = 0$ and $f\left(-\frac{1}{2}W\left(-\frac{2}{e^2}\right)\right) < 0$ we get $f(y) \le 0, \forall y \in [0,1]$.

Therefore, $\eta_{kj}(\ln\eta_{kj}+\xi_{kj}) \le 0$ and it leads to $J_2 = \sum_{k=1}^{N}\sum_{j=1}^{C}\eta_{kj}(\ln\eta_{kj}+\xi_{kj}) \le 0$.

On the other hand,

$$\eta_{kj}\xi_{kj} \ge 0.$$

Let us consider

$$g(y) = y.\ln y \text{ where } y \in [0,1],$$

we have

$$g'(y) = 1 + \ln y,$$

$$g'(y) = 0 \leftrightarrow y = \frac{1}{e},$$

and $f\left(\frac{1}{e}\right) = -\frac{1}{e}$ is the minimal value of this function.
We have

$$J_2 = \sum_{k=1}^{N}\sum_{j=1}^{C}\eta_{kj}(\ln\eta_{kj}+\xi_{kj}) \ge \sum_{k=1}^{N}\sum_{j=1}^{C}\left(-\frac{1}{e}+0\right) = \frac{-1}{e}\times N \times C.$$
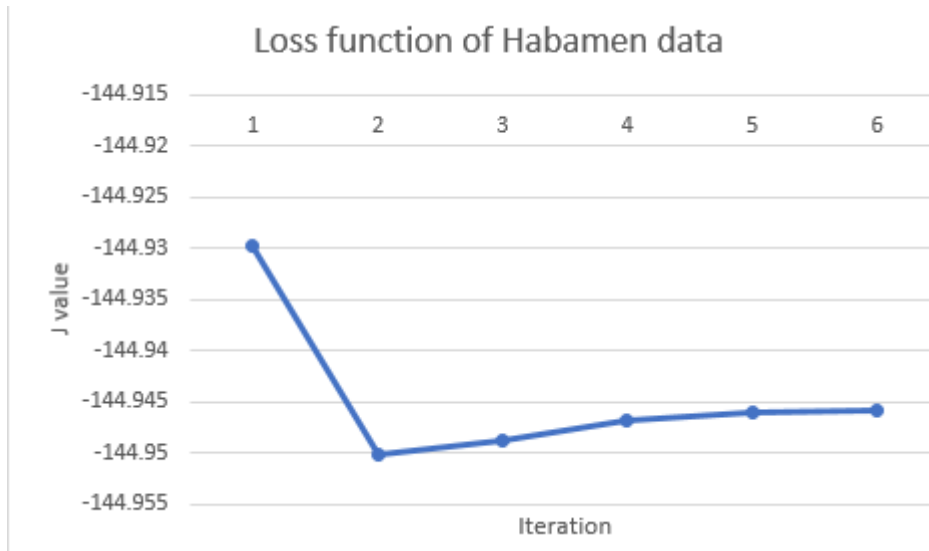
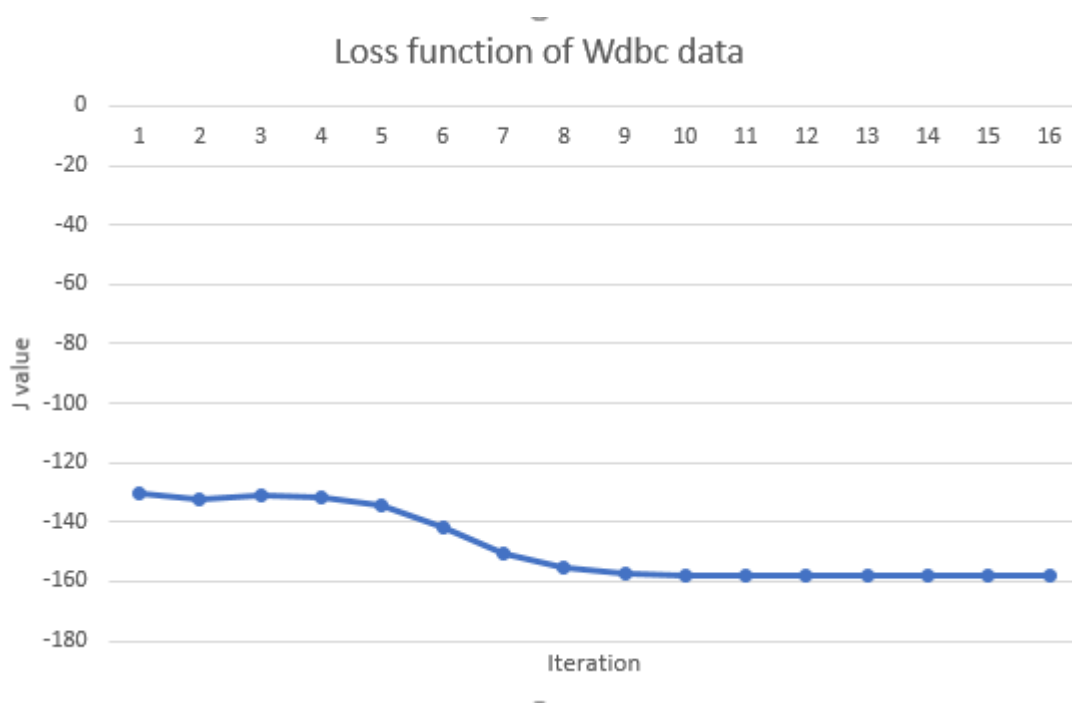*Figure 1.* Loss function of Haberman dataset [46]
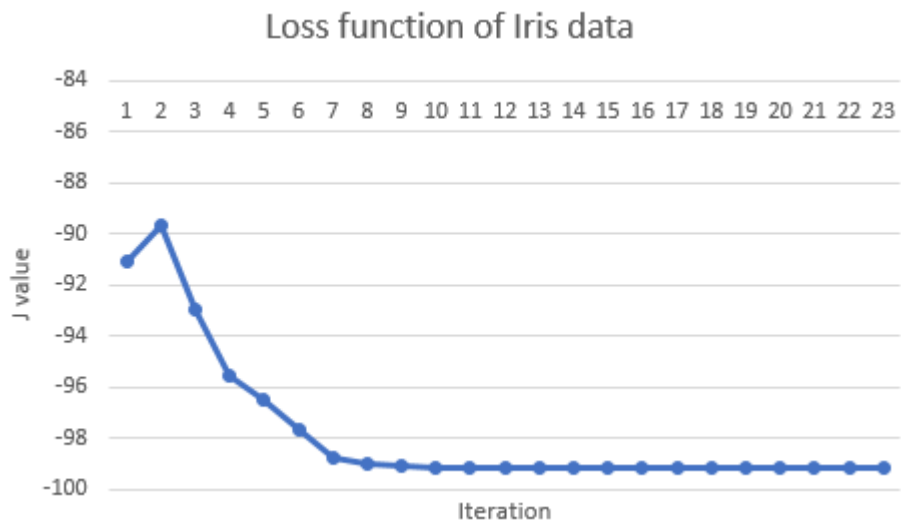

Figure 2. Loss function of Wdbc dataset [46]

Loss function of Iris data

Figure 3. Loss function of Iris dataset [46]
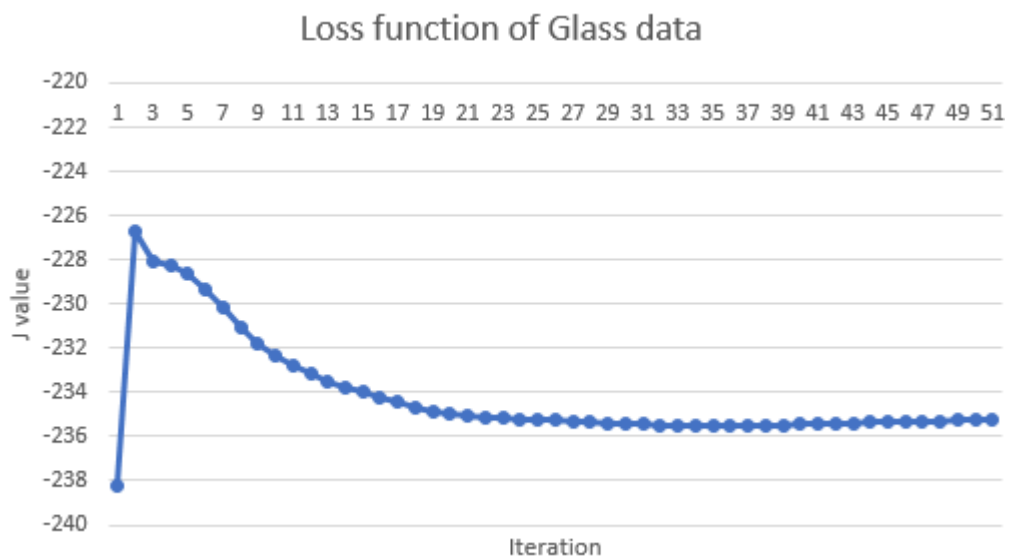
Loss function of Glass data

Figure 4. Loss function of Glass dataset [46]

Therefore, the upper bound of J_m is M, the lower bound is $M - \dfrac{1}{e} \times N \times C$.

In Fig.1, the loss function of Habamen data takes 6 iterations to converge to the local minimum. Because of the update of η, it increases gradually from the second iteration but still reaches the convergence.

The loss function of the Wdbc dataset in Fig.2 decreases slowly in each iteration and converges at the 16th iteration. Because the initialization elements are random, the value of the first iteration is also random. However, from the second step the value of J decreases significantly and from the 8th step, the stability appears and the loss function slowly converges.

From the 2nd iteration in Figs.3 and 4, the loss functions of Iris and Glass datasets decrease but sometimes they increase slightly and converge to a stable point.

4.2. Property of centroid

We can see the updating of centroids in each iteration similar to the update of parameters in Gradient Descent. From eq. (11), we have

$$
\begin{aligned}
V_j^{t+1} &= \frac{\sum\limits_{k=1}^{N} (\mu_{kj}^t (2 - \xi_{kj}^t))^m Y_k}{\sum\limits_{k=1}^{N} (\mu_{kj}^t (2 - \xi_{kj}^t))^m} \\
&= V_j^t - \frac{1}{\sum\limits_{k=1}^{N} (\mu_{kj}^t (2 - \xi_{kj}^t))^m} \cdot \left(\sum\limits_{k=1}^{N} (\mu_{kj}^t (2 - \xi_{kj}^t))^m (-2Y_k + 2V_j^t) \right) \\
&= V_j^t - \alpha_j^t . \nabla_{V_j} J_m^t (V_j^t),
\end{aligned}
$$

where $\alpha_j^t = \dfrac{1}{\sum\limits_{k=1}^{N} (\mu_{kj}^t (2 - \xi_{kj}^t))^m}.$

5. CONCLUSIONS

This paper presented some theoretical properties of FC-PFS and proved the convergence of this algorithm. We have pointed out that this algorithm converges to at least local minimum which guaranties to archive acceptable solutions. Specifically, Propositions 1 to 5 stated that the membership matrices and cluster centers converge if and only if their values are computed by updated equations. Moreover, the objective function is descent in the domain. Some properties of PF-PFS were also considered such as the boundary of the loss function. This is significant in understanding the mechanism of the picture fuzzy clustering.

In the future, we will assess the maximum and minimum changes of the objective function through interval steps and others. Relationship between picture fuzzy set and neutrosophic set [47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60] in terms of clustering algorithms will also be our target.

ACKNOWLEDGMENT

REFERENCES

[1] J.C. Bezdek, R. Ehrlich, W. Full, "FCM: the fuzzy c-means clustering algorithm", *Comput Geosci*, vol. 10, no. 2, pp. 191–203, 1984.

[2] K.T. Atanassov, "Intuitionistic fuzzy sets", *Fuzzy Sets Syst*, vol. 20, pp. 87–96, 1986.

[3] B.C. Cuong, V. Kreinovich, "Picture Fuzzy Sets-a new concept for computational intelligence problems", *Information and Communication Technologies (WICT)*, Third World Congress on (pp. 1-6). IEEE, 2013, Ha Noi, Viet Nam (pp. 1-6).

[4] B.C. Cuong, "Picture fuzzy sets", *J Comput Sci Cybern*, vol. 30, no. 4, pp. 409–420, 2014.

[5] S. A. Kumar, B. S. Harish, V. M. Aradhya, "A picture fuzzy clustering approach for brain tumor segmentation", *Cognitive Computing and Information Processing (CCIP)*, Second International Conference on (pp. 1-6). IEEE, Noida, India, 2016 (pp. 1-6).

[6] G. Wei, "Picture fuzzy aggregation operators and their application to multiple attribute decision making", *Journal of Intelligent Fuzzy Systems*, vol. 33, no. 2, pp. 713–724, 2017.

[7] G. Wei, "Picture fuzzy cross-entropy for multiple attribute decision making problems", *Journal of Business Economics and Management*, vol. 17, no. 4, pp. 491–502, 2016.

[8] G. Wei, "Some cosine similarity measures for picture fuzzy sets and their applications to strategic decision making", *Informatica*, vol. 28, no. 3, pp. 547–564, 2017.

[9] G. Wei, "Picture 2-tuple linguistic Bonferroni mean operators and their application to multiple attribute decision making", *International Journal of Fuzzy Systems*, vol. 19, no. 4, pp. 997–1010, 2017.

[10] G. Wei, , M. Lu, F.E. Alsaadi, T. Hayat, A. Alsaedi, "Pythagorean 2-tuple linguistic aggregation operators in multiple attribute decision making", *Journal of Intelligent Fuzzy Systems*, vol. 33, no. 2, pp. 1129–1142, 2017.

[11] R. X. Nie, J. Q. Wang, L. Li, "A shareholder voting method for proxy advisory firm selection based on 2-tuple linguistic picture preference relation", *Applied Soft Computing*, vol. 60, pp. 520–539, 2017.

[12] Y. Yang, C. Liang, S. Ji, T. Liu, "Adjustable soft discernibility matrix based on picture fuzzy soft sets and its applications in decision making", *Journal of Intelligent Fuzzy Systems*, vol. 29, no. 4, pp. 1711–1722, 2015.

[13] C.Y. Wang, X.Q. Zhou, H.N. Tu, S.D. Tao, "Some geometric aggregation operators based on picture fuzzy sets and their application in multiple attribute decision making", *Ital. J. Pure Appl. Math,* vol. 37, pp. 477–492, 2017.

[14] X. Peng, J. Dai, "Algorithm for picture fuzzy multiple attribute decision-making based on new distance measure", *International Journal for Uncertainty Quantification*, vol. 7, no. 2, 2017.

[15] S.J. Wu, G.W. Wei, "Picture uncertain linguistic aggregation operators and their application to multiple attribute decision making", *International Journal of Knowledge-based and Intelligent Engineering Systems*, vol. 21, no. 4, pp. 243–256, 2017.

[16] P. Liu, X. Zhang, "A novel picture fuzzy linguistic aggregation operator and its application to group decision-making", *Cognitive Computation*, vol. 10, no. 2, pp. 242–259.

[17] S.M. Peng, "Study on enterprise risk management assessment based on picture fuzzy multiple attribute decision-making method", *Journal of Intelligent and Fuzzy Systems*, vol. 33, no. 6, pp. 3451–3458, 2017.

[18] D.X. Li, H. Dong, X. Jin, "Model for evaluating the enterprise marketing capability with picture fuzzy information", *Journal of Intelligent and Fuzzy Systems*, vol. 33, no. 6, pp. 3255–3263, 2017.

[19] G. Wei, , F.E. Alsaadi, T. Hayat, A. Alsaedi, "Projection models for multiple attribute decision making with picture fuzzy information", *International Journal of Machine Learning and Cybernetics*, vol. 9, no. 4, pp. 713–719, 2018.

[20] C. Bo, X. Zhang, "New operations of picture fuzzy relations and fuzzy comprehensive evaluation", *Symmetry*, vol. 9, no. 11, pp. 268–285, 2017.

[21] H. Garg, "Some Picture Fuzzy Aggregation Operators and Their Applications to Multicriteria Decision-Making", *Arabian Journal for Science and Engineering*, vol. 42, no. 12, pp. 5275–5290, 2017.

[22] G. Wei, , "Picture uncertain linguistic Bonferroni mean operators and their application to multiple attribute decision making", *Kybernetes*, vol. 46, no. 10, pp. 1777–1800, 2017.

[23] G.W. Wei, "Some similarity measures for picture fuzzy sets and their applications", *Iranian Journal of Fuzzy Systems*, vol. 15, no. 1, pp. 77–89, 2018.

[24] P. Singh, "Correlation coefficients for picture fuzzy sets", *Journal of Intelligent and Fuzzy Systems*, vol. 28, no. 2, pp. 591–604, 2015.

[25] X. Zhang, C. Boc, C. Park, "Picture fuzzy filters of pseudo-BCI algebras", *Fuzzy Systems and Data Mining III: Proceedings of FSDM*, Hualien, Taiwan, 2017 (299, 254).

[26] P. Dutta, S. Ganju, "Some aspects of picture fuzzy set", *Transactions of A. Razmadze Mathematical Institute*, vol. 172, no. 2, pp. 164–175, 2018.

[27] B.C. Cuong, P.V. Hai, "Some fuzzy logic operators for picture fuzzy sets", *Knowledge and Systems Engineering (KSE), 2015 Seventh International Conference on. IEEE.*, Ho Chi Minh city, Viet Nam, October 8-10, 2015, (pp. 132-137).

[28] B.C. Cuong, R.T. Ngan, B.D. Hai, "An involutive picture fuzzy negator on picture fuzzy sets and some De Morgan triples", *Knowledge and Systems Engineering (KSE), 2015 Seventh International Conference on. IEEE.*, Ho Chi Minh city, Viet Nam, October 8-10, 2015, (pp. 126-131).

[29] B.C. Cuong, V. Kreinovitch, R.T. Ngan, "A classification of representable t-norm operators for picture fuzzy sets", *Knowledge and Systems Engineering (KSE), 2016 Eighth International Conference on. IEEE,* Ha Noi, Viet Nam, October 6-8, 2016 (pp. 19-24).

[30] P.H. Phong, D. T. Hieu, R.T. Ngan, P. T. Them, "Some compositions of picture fuzzy relations", *Proceedings of the 7th National Conference on Fundamental and Applied Information Technology Research (FAIR'7)*, Thai Nguyen, 2014 (pp. 19–20).

[31] P.H. Phong, B.C. Cuong, "Multi-criteria Group Decision Making with Picture Linguistic Numbers", *VVNU Journal of Science: Comp. Science & Com. Eng.*, vol. 32, no. 3, pp. 39–53, 2017.

[32] L.H. Son, P.V. Viet, P.V. Hai, "Picture inference system: a new fuzzy inference system on picture fuzzy set", *Applied Intelligence*, vol. 46, no. 3, pp. 652–669, 2017.

[33] P.V. Viet, HTM. Chau, L.H. Son, P.V. Hai, ''Some extensions of membership graphs for picture inference systems", *Knowledge and Systems Engineering (KSE), 2015 Seventh International Conference on. IEEE.*, Ho Chi Minh city, Viet Nam, October 8-10, 2015 (pp. 192–197).

[34] P. H. Thong, L.H. Son, H. Fujita, "Interpolative picture fuzzy rules: A novel forecast method for weather nowcasting", *Fuzzy Systems (FUZZ-IEEE), 2016 IEEE International Conference on IEEE.*, Vancouver, Canada, July 24-29, 2016 (pp. 86-93).

[35] N.X. Thao, N.V. Dinh, "Rough picture fuzzy set and picture fuzzy topologies", *Journal of Computer Science and Cybernetics*, vol. 31, no. 3, pp. 245, 2015.

[36] N. Van Dinh, N.X. Thao, "Some measures of picture fuzzy sets and their application", *The University of Danang, Journal of Science and Technology: Issue on Information and Communications Technology*, vol. 3, no. 2, pp. 35–40, 2017.

[37] N.V. Dinh, N.X. Thao, N.M. Chau, "On the picture fuzzy database: theories and application", *Vietnam National University of Agriculture, J. Sci. & Devel.*, vol. 13, no. 6, pp. 1028–1035, 2015.

[38] P. H. Thong, L.H. Son, "Picture fuzzy clustering: a new computational intelligence method", *Soft Computing*, vol. 20, no. 9, pp. 3549–3562, 2016.

[39] P. H. Thong, L.H. Son, "A new approach to multi-variable fuzzy forecasting using picture fuzzy clustering and picture fuzzy rule interpolation method", *Knowledge and Systems Engineering*, Springer, Cham., 2015 (pp. 679-690).

[40] N. T. Thong, L.H. Son, "HIFCF: An effective hybrid model between picture fuzzy clustering and intuitionistic fuzzy recommender systems for medical diagnosis", *Expert Systems with Applications*, vol. 42, no. 7, pp. 3682–3701, 2015.

[41] L.H. Son, "Generalized picture distance measure and applications to picture fuzzy clustering", *Applied Soft Computing*, vol. 46(C), pp. 284–295, 2016.

[42] L.H. Son, P. H. Thong, "Some novel hybrid forecast methods based on picture fuzzy clustering for weather nowcasting from satellite image sequences", *Applied Intelligence*, vol. 46, no. 1, pp. 1–15, 2017.

[43] P.H. Thong, L.H. Son, "A novel automatic picture fuzzy clustering method based on particle swarm optimization and picture composite cardinality", *Knowledge-Based Systems*, vol. 109, pp. 48–60, 2016.

[44] L.H. Son, "Measuring analogousness in picture fuzzy sets: from picture distance measures to picture association measures", *Fuzzy Optimization and Decision Making*, vol. 16, pp. 359–378, 2017.

[45] P.H. Thong, L.H. Son, "Picture fuzzy clustering for complex data", *Engineering Applications of Artificial Intelligence*, vol. 56, pp. 121–130, 2016.

[46] UC Irvine Machine Learning Repository, https://archive.ics.uci.edu/ml/index.php.

[47] M. Khan, L.H. Son, M. Ali, HTM. Chau, NTN. Na, F. Smarandache, "Systematic review of decision making algorithms in extended neutrosophic sets", *Symmetry-Basel* vol. 10, pp. 314-342, 2018.

[48] S. Jha, R. Kumar, L.H. Son, J.M. Chatterjee, M. Khari, N. Yadav, F. Smarandache, "Neutrosophic soft set decision making for stock trending analysis", *Evolving Systems*, pp. 1–7, 2018, DOI: 10.1007/s12530-018-9247-7.

[49] A. Dey, S. Broumi, L.H. Son, A. Bakali, M. Talea, F. Smarandache, "A new algorithm for finding minimum spanning trees with undirected neutrosophic graphs", *Granular Computing*, pp. 1-7, 2018, DOI: 10.1007/s41066-018-0084-7.

[50] M. Ali, L.H. Son, N.D. Thanh, N.V. Minh, "A neutrosophic recommender system for medical diagnosis based on algebraic neutrosophic measures", *Applied Soft Computing*, pp. 1-18, 2018, DOI: 10.1016/j.asoc.2017.10.012.

[51] M. Ali, L.H. Son, M. Khan, N.T. Tung, "Segmentation of dental X-ray images in medical imaging using neutrosophic orthogonal matrices", *Expert Systems with Applications*, vol. 91, pp. 434–441, 2018.

[52] G. N. Nhu, L.H. Son, A. S. Ashour, N. Dey, "A survey of the state-of-the-arts on neutrosophic sets in biomedical diagnoses", *International Journal of Machine Learning and Cybernetics*, pp. 1–13, 2018. DOI: 10.1007/s13042-017-0691-7.

[53] S. Broumi, L.H. Son, A. Bakali, M. Talea, F. Smarandache, G. Selvachandran, "Computing Operational Matrices in Neutrosophic Environments: A Matlab Toolbox", *Neutrosophic Sets & Systems,* vol. 18, pp. 58 - 66, 2017.

[54] S. Broumi, A. Dey, A. Bakali, M. Talea, F. Smarandache, L.H. Son, D. Koley, *Uniform Single Valued Neutrosophic Graphs. New Trends in Neutrosophic Theory and Applications*, Florentin Smarandache, Surapati Pramanik (Eds.), Infinite Study. 2017, 424 pages.

[55] M. Ali, L.H. Son, I. Deli, N.D. Tien, "Bipolar neutrosophic soft sets and applications in decision making", *Journal of Intelligent and Fuzzy Systems*, vol. 33, no. 6, pp. 4077–4087, 2017.

[56] N. X. Thao, B.C. Cuong, M. Ali, L.H. Lan, "Fuzzy equivalence on standard and rough neutrosophic sets and applications to clustering analysis", *Information Systems Design and Intelligent Applications*, Springer, Singapore, 2018 (pp. 834-842).

[57] N. D. Thanh, M. Ali, L.H. Son, "A novel clustering algorithm in a neutrosophic recommender system for medical diagnosis", *Cognitive Computation*, vol. 9, no. 4, pp. 526–544, 2017.

[58] N. D. Thanh, L.H. Son, M. Ali, "Neutrosophic recommender system for medical diagnosis based on algebraic similarity measure and clustering", *In Fuzzy Systems (FUZZ-IEEE), 2017 IEEE International Conference on*, Naples, Italy, 2017 (pp. 1–6).

[59] L.H. Son, P.V. Hai, "A novel multiple fuzzy clustering method based on internal clustering validation measures with gradient descent", *International Journal of Fuzzy Systems*, vol. 18, no. 5, pp. 894–903, 2016.

[60] M. Ali, L.Q. Dat, L.H. Son, F. Smarandache, "Interval complex neutrosophic set: formulation and applications in decision-making", *International Journal of Fuzzy Systems*, vol. 20, no. 3, pp. 986–999, 2018.