Tạp chí Tin học và Điều khiển học, T.23, S.1 (2007), 50–58

ABOUT CONVERGENCE RATES IN REGULARIZATION FOR ILL-POSED OPERATOR EQUATIONS OF HAMMERSTEIN TYPE

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Abstract. The aim of this paper is to study convergence rates of the regularized solutions in connection with the finite-dimensional approximations for the operator equation of Hammerstein type $x+F_2F_1(x)=f$ in reflexive Banach spaces under the perturbations for not only the operators $F_i, i=1,2$, but also f. The conditions of convergence and convergence rates given in this paper for a class of inverse-strongly monotone operators $F_i, i=1,2$, are much simpler than those in the past papers.

Tóm tắt. Mục đích của bài báo này là nghiên cứu tốc độ hội tụ của nghiệm hiệu chỉnh đã được xấp xỉ hữu hạn chiều cho phương trình toán tử loại Hammerstein $x+F_2F_1(x)=f$ trong không gian Banach phản xạ với nhiễu không chỉ có ở các toán tử $F_i,\ i=1,2$ mà cả ở f. Điều kiện hội tụ và tốc độ hội tụ trong bài báo này cho toán tử ngược đơn điệu mạnh $F_i,\ i=1,2$ là yếu hơn nhiều so với các kết quả trước.

1. INTRODUCTION

Let X be a reflexive real Banach space, and X^* be its dual which both are strictly convex. For the sake of simplicity the norms of X and X^* are denoted by the symbol $\|.\|$. We write $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$ instead of $x^*(x)$ for $x^*\in X^*$ and $x\in X$. Concerning the space X, in addition assume that it possesses the property: the weak convergence and convergence of norms for any sequence follows its strong convergence. Let $F_1 : X\to X^*$ and $F_2 : X^*\to X$ be monotone, in general nonlinear, bounded (i.e. image of any bounded subset is bounded) and continuous operators.

Our main aim of this paper is to study a stable method of finding an approximative solution for the equation of Hammerstein type

$$x+F_2F_1(x)=f,\ f\ \in\ X. \tag{1.1}$$

Usually instead of $F_i, i=1,2$, and f we know their monotone continuous approximations F_i^h and f_δ, such that

$$\|F_1^h(x)-F_1(x)\|\leqslant hg(\|x\|)\quad \forall x\in X,$$
$$\|F_2^h(x^*)-F_2(x^*)\|\leqslant hg(\|x^*\|)\quad \forall x^*\in X^*,$$
$$g(t)\leqslant Mt+N,\quad M,N\geqslant 0,$$

where $g(t)$ is a real nonegative, non-decreasing, bounded function (the image of a bounded set is bounded) with $g(0) = 0$, and $\|f_\delta - f\| \leqslant \delta$. Without additional conditions for the operators F_i such as the strongly monotone property, equation (1.1) is ill-posed (see the example at the end of the paper). To solve (1.1) we need to use stable methods. One of them is the operator equation

$$x + F^h_{2,\alpha} F^h_{1,\alpha}(x) = f_\delta \tag{1.2}$$

(see [1], [2]), where $F^h_{i,\alpha} = F^h_i + \alpha U_i$, $U_i, i = 1, 2$, are the normalized dual mappings of X and X^*, respectively (see [9]), and $\alpha > 0$ is the small parameter of regularization. For every $\alpha > 0$ equation (1.2) has a unique solution $x^{h,\delta}_\alpha$, and the sequence $\{x^{h,\delta}_\alpha\}$ converges to a solution x_0 of (1.1) as $(h+\delta)/\alpha, \alpha \to 0$. Moreover, this solution $x^{h,\delta}_\alpha$, for every fixed $\alpha > 0$, depends continuously on $F^h_i, i = 1, 2$ and f_δ, the finite-dimensional problems

$$x + F^h_{2,\alpha,n} F^h_{1,\alpha,n}(x) \;=\; f_{\delta,n},\ x \in X_n, \tag{1.3}$$

where $F^h_{2,\alpha,n} = P_n F^h_{2,\alpha} P^*_n$, $F^h_{1,\alpha,n} = P^*_n F^h_{1,\alpha} P_n$, $f_{\delta,n} = P_n f_\delta$, P_n is a linear projection from X onto its finite-dimensional subspace X_n such that $X_n \subset X_{n+1}$, $P_n x \to x$, as $n \to \infty$ for every $x \in X$, and P^*_n is the dual of P_n with $\|P_n\| \leqslant \tilde{c} =$ constant, for all n, have a unique solution $x^{h,\delta}_{\alpha,n}$, and the sequence $\{x^{h,\delta}_{\alpha,n}\}$ converges to $x^{h,\delta}_\alpha$, as $n \to \infty$, without additional conditions on $F_i,\ i = 1,\ 2$. In the case of linearity for F_2 and $f_\delta = f$ for all $\delta > 0$, the convergence rates for the sequences $\{x^{h,\delta}_\alpha\}$ and $\{x^{h,\delta}_{\alpha,n}\}$ are given in the paper [3] provided the existence of bounded inversion $(I + F_2 F'_1(x_0))^{-1}$, where I denotes the identity operator in X. It is not difficult to verify that this condition can be replaced by the bounded inversion of $(I + F'_2(x^*_0)F'_1(x_0))^{-1}$, when F_2 also is nonlinear, where $x^*_0 = F_1(x_0)$. The last requirement is equivalent to that -1 is not an eigenvalue of the operator $F'_2(x^*_0)F'_1(x_0)$ and is used in studying a method of collocation-type for nonlinear integral equations of Hammerstein type (see [6]). In general case, i.e., when both the operators $F_i, i = 1, 2$, are nonlinear, it means that $\mathcal{R}$, the range of the operator $I + F'_2(x^*_0)F'_1(x_0)$, is the whole space X. It is natural to ask if we can estimate the convergence rates for the sequences $\{x^{h,\delta}_\alpha\}$, $\{x^{h,\delta}_{\alpha,n}\}$, when $\mathcal{R}$ is not the whole space X. For this purpose, only demanding that $\mathcal{R}$ contains a necessary element of X, the convergence rates of $\{x^{h,\delta}_\alpha\}$ and $\{x^{h,\delta}_{\alpha,n}\}$ are estimated in [4], [5] on the base of the zero value of the derivatives of higher order for F_1 and F_2 at x_0 and x^*_0, respectively. This result is formulated in the following theorem.

Theorem 1.1. (see [4] or [5]). *Let the following conditions hold:*

(i) F_1 is Fréchet differentiable at some neighbourhood $\mathcal{U}_0$ of x_0 s_1-1-times if $s_1 = [s_1]$, the integer part of s_1, $[s_1]$-times if $s_1 \neq [s_1]$, and F_2 is Fréchet differentiable at some neighbourhood $\mathcal{V}_0$ of x^*_0 $s_2 - 1$-times, if $s_2 = [s_2]$, $[s_2]$-times if $s_2 \neq [s_2]$,

(ii) there exists a constant $L > 0$ such that

$$\|F^{(k)}_1(x_0) - F^{(k)}_1(y)\| \leqslant L\|x_0 - y\|,\ \forall\ y \in \mathcal{U}_0,$$
$$\|F^{(k)}_2(x^*_0) - F^{(k)}_2(y^*)\| \leqslant L\|x^*_0 - y^*\|,\ \forall\ y^* \in \mathcal{V}_0,$$

for $F^{(k)}_i$: $k = s_i - 1$ if $s_i = [s_i]$, $k = [s_i]$ if $s_i \neq [s_i]$, and if $[s_i] \geqslant 3$, then $F^{(2)}_1(x_0) = ... = F^{(k)}_1(x_0) = 0$, and $F^{(2)}_2(x^*_0) = ... = F^{(k)}_1(x^*_0) = 0$, *(iii) there exists an element* $x^1 \in X$ *such that*

$$\big(I + F_2'(x_0^*)^* F_1'(x_0)^*\big)x^1 = F_2'(x_0^*)^* U_1(x_0) - U_2(x_0^*),$$

if $s_1 = [s_1]$ *then* $L\|x^1\| < m_1 s_1!$, *and if* $s_2 = [s_2]$ *then* $L\|F_1'(x_0)^* x^1 - U_1(x_0)\| < m_2 s_2!$
Then, if α *is chosen such that* $\alpha \sim (h+\varepsilon)^\rho,\ 0 < \rho < 1$, *we have*

$$\begin{aligned} \|x_\omega - x_0\| &= O((h+\varepsilon)^\theta), \\ &\theta = \min\ \{\theta_1, \frac{1-\rho+\theta_2}{s_1 - 1}\}, \\ &\theta_i = \min\ \{\frac{1-\rho}{s_i}, \frac{\rho}{s_i}\},\ i = 1,\ 2. \end{aligned}$$

In this paper, the convergence rates of $\{x_\alpha^{h,\delta}\}$ and $\{x_{\alpha,n}^{h,\delta}\}$ are established under much weaker conditions on $F_i, i = 1, 2$. These are the assumptions that $\mathcal{R}$ contains some element of X, and $F_i, i = 1, 2$, are inverse-strongly monotone, i.e.

$$\begin{aligned} \langle F_1(x) - F_1(y), x - y\rangle &\geqslant \tilde{m}_1 \|F_1(x) - F_1(y)\|^2, \quad x, y \in X, \\ \langle F_2(x^*) - F_2(y^*), x^* - y^*\rangle &\geqslant \tilde{m}_2 \|F_2(x^*) - F_2(y^*)\|^2, \quad x^*, y^* \in X^*, \end{aligned} \tag{1.4}$$

where $\tilde{m}_i, i = 1, 2$, are some positive constants. Note that in [7] the inverse-strongly monotone property was used to estimate the convergence rates of the regularized solutions for ill-posed variational inequalities.

Below, by "a $\sim$ b" we mean "a = O(b) and b = O(a)".

2. MAIN RESULTS

Assume that the normalized dual mappings $U_i,\ i = 1,\ 2$, of the spaces X and X^* satisfy the following conditions (see [8])

$$\big\langle U_i(y_1^i) - U_i(y_2^i), y_1^i - y_2^i\big\rangle \geqslant m_i \|y_1^i - y_2^i\|^{s_i},\ \ m_i > 0,\ s_i \geqslant 2, \tag{2.1}$$

$$\|U_i(y_1^i) - U_i(y_2^i)\| \leqslant c_i(R_i)\|y_1^i - y_2^i\|^{\nu_i},\ \ 0 < \nu_i \leqslant 1, \tag{2.2}$$

where $y_1^i,\ y_2^i \in X$ or X^* on dependence of $i = 1$ or 2, respectively, and $c_i(R_i),\ R_i > 0$, are the positive increasing functions on $R_i = \max\ \{\|y_1^i\|, \|y_2^i\|\}$.

The following theorem answers the question on convergence rates for $\{x_\alpha^{h,\delta}\}$.

Theorem 2.1. *Assume that the following conditions hold:*

(i) $F_i, i = 1, 2$, *are inverse-strongly monotone and continuously Fréchet differentiable at some neighbourhoods* $\mathcal{U}$ *of* x_0 *and* $\mathcal{V}$ *of* x_0^* , *respectively, and*

$$\begin{aligned} \|F_1(x) - F_1(x_0) - F_1'(x_0)(x - x_0)\| &\leqslant \tau_1 \|F_1(x) - F_1(x_0)\|, \quad \forall x \in \mathcal{U}, \\ \|F_2(x^*) - F_2(x_0^*) - F_2'(x_0^*)(x^* - x_0^*)\| &\leqslant \tau_2 \|F_2(x^*) - F_2(x_0^*)\|, \quad \forall x^* \in \mathcal{V}, \end{aligned}$$

where $\tau_i, i = 1, 2$, *are some positive constants,* *(ii) there exists an element $x^1 \in X$ such that*

$$\big(I + F_2'(x_0^*)^* F_1'(x_0)^*\big)x^1 = F_2'(x_0^*)^* U_1(x_0) - U_2(x_0^*).$$

Then, if α is chosen such that $\alpha \sim (h+\delta)^\rho,\ 0 < \rho < 1$, we have

$$\|x_\alpha^{h,\delta} - x_0\| \ = \ O\big((h+\delta)^{\theta/s_1}\big), \quad \theta = \min\,\{\rho/2, 1-\rho\}.$$

Proof. Set

$$A = m_1\|x_\alpha^{h,\delta} - x_0\|^{s_1} + m_2\|x_\alpha^{h,\delta,*} - x_0^*\|^{s_2}, \quad x_\alpha^{h,\delta,*} = F_{1,\alpha}^h(x_\alpha^{h,\delta}).$$

It is easy to see that x_0 is a solution of (1.1) iff $z_0 = [x_0, x_0^*]$ is a solution of the system of following operator equations

$$\begin{aligned} F_1(x) - x^* &= 0,\\ F_2(x^*) + x - f &= 0. \end{aligned}$$

Similarily, $x_\alpha^{h,\delta}$ is a regularized solution of the operator equation (1.2) iff $z_\alpha^{h,\delta} = [x_\alpha^{h,\delta}, x_\alpha^{h,\delta,*}]$ is a solution of the system of following equations

$$\begin{aligned} F_1^h(x) + \alpha U_1(x) - x^* &= 0,\\ F_2^h(x^*) + \alpha U_2(x^*) + x - f_\delta &= 0. \end{aligned}$$

Consider the space $Z = X \times X^*$ with the norm $\|z\|^2 = \|x\|^2 + \|x^*\|^2, z = [x, x^*], x \in X$, and $x^* \in X^*$. Then, the two above systems of equations can be written, respectively, in form of equations

$$\begin{aligned} \mathcal{A}(z) &= \overline{f},\\ \mathcal{A}_\alpha^h(z) \equiv \mathcal{A}^h(z) + \alpha J(z) &= \overline{f}_\delta, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \mathcal{A}(z) &= [F_1(x), F_2(x^*)] + [-x^*, x],\\ \mathcal{A}^h(z) &= [F_1^h(x), F_2^h(x^*)] + [-x^*, x],\\ J(z) &= [U_1(x), U_2(x^*)],\\ \overline{f} &= [0, f], \quad \overline{f}_\delta = [0, f_\delta]. \end{aligned} \tag{2.4}$$

It is easy to verify that $\mathcal{A}$ and $\mathcal{A}^h$ are the monotone operators from Z to $Z^* = X^* \times X$, and the operator J is the normalized duality mapping of the space Z. Hence, from (2.1), (2.3), (2.4) and the monotone property of $\mathcal{A}^h$ it implies that

$$\begin{aligned} A \leqslant &\langle J(z_\alpha^{h,\delta}) - J(z_0), z_\alpha^{h,\delta} - z_0\rangle \leqslant \langle J(z_0), z_0 - z_\alpha^{h,\delta}\rangle\\ &+ \frac{1}{\alpha}[\langle \overline{f}_\delta - \overline{f}, z_\alpha^{h,\delta} - z_0\rangle + \langle \mathcal{A}(z_0) - \mathcal{A}^h(z_0), z_\alpha^{h,\delta} - z_0\rangle]. \end{aligned} \tag{2.5}$$

It is not difficult to verify that

$$\|\mathcal{A}^h(z) - \mathcal{A}(z)\| \leqslant \sqrt{2}hg(\|z\|). \tag{2.6}$$

Further, from (1.4) it follows

$$\begin{aligned}
&\langle \mathcal{A}(z_\alpha^{h,\delta}) - \mathcal{A}(z_0), z_\alpha^{h,\delta} - z_0\rangle = \langle F_1(x_\alpha^{h,\delta}) - x_\alpha^{h,\delta,*} - (F_1(x_0) - x_0^*), x_\alpha^{h,\delta} - x_0\rangle \\
&+ \langle F_2(x_\alpha^{h,\delta,*}) + x_\alpha^{h,\delta} - (F_2(x_0^*) + x_0), x_\alpha^{h,\delta,*} - x_0^*\rangle \\
&= \langle F_1(x_\alpha^{h,\delta}) - F_1(x_0), x_\alpha^{h,\delta} - x_0\rangle + \langle F_2(x_\alpha^{h,\delta,*}) - F_2(x_0^*), x_\alpha^{h,\delta,*} - x_0^*\rangle \\
&\geqslant \tilde{m}_1\|F_1(x_\alpha^{h,\delta}) - F_1(x_0)\|^2 + \tilde{m}_2\|F_2(x_\alpha^{h,\delta,*}) - F_2(x_0^*)\|^2 \\
&\geqslant \min\{\tilde{m}_1, \tilde{m}_2\}C^2, \quad C^2 = \|F_1(x_\alpha^{h,\delta}) - F_1(x_0)\|^2 + \|F_2(x_\alpha^{h,\delta,*}) - F_2(x_0^*)\|^2.
\end{aligned}$$

On the other hand, from (2.3), (2.4)-(2.6) and the properties of $\mathcal{A}, \mathcal{A}^h, J, g$ we have

$$\begin{aligned}
\langle \mathcal{A}(z_\alpha^{h,\delta}) - \mathcal{A}(z_0), z_\alpha^{h,\delta} - z_0\rangle &\leqslant \langle \overline{f}_\delta - \overline{f}, z_\alpha^{h,\delta} - z_0\rangle \\
&+ \alpha\langle J(z_0), z_0 - z_\alpha^{h,\delta}\rangle + \langle \mathcal{A}(z_\alpha^{h,\delta}) - \mathcal{A}^h(z_\alpha^{h,\delta}), z_\alpha^{h,\delta} - z_0\rangle,
\end{aligned}$$

and $\{z_\alpha^{h,\delta}\}$ is bounded, as $(h+\delta)/\alpha \to 0$. Therefore,

$$C^2 \leqslant \frac{1}{\min\{\tilde{m}_1, \tilde{m}_2\}}[\delta + \alpha\|J(z_0)\| + \sqrt{2}hg(\|z_\alpha^{h,\delta}\|)]\|z_\alpha^{h,\delta} - z_0\|.$$

Consequently, $C \leqslant O(\sqrt{h+\delta+\alpha})$. Hence,

$$\begin{aligned}
\|F_1(x_\alpha^{h,\delta}) - F_1(x_0)\| &\leqslant O(\sqrt{h+\delta+\alpha}), \\
\|F_2(x_\alpha^{h,\delta,*}) - F_2(x_0^*)\| &\leqslant O(\sqrt{h+\delta+\alpha}).
\end{aligned} \tag{2.7}$$

Now, we shall estimate the value $\langle J(z_0), z_0 - z_\alpha^{h,\delta}\rangle$. For this purpose, set $x^2 = U_1(x_0) - F_1'(x_0)^*x^1$. From condition (ii) of the theorem it follows that x^1 and x^2 ($\in X^*$) satisfy the system of following equalities

$$\begin{aligned}
F_1'(x_0)^*x^1 + x^2 &= U_1(x_0), \\
F_2'(x_0^*)^*x^2 - x^1 &= U_2(x_0^*).
\end{aligned}$$

By virtue of

$$\begin{aligned}
\langle J(z_0), z_0 - z_\alpha^{h,\delta}\rangle &= \langle U_1(x_0), x_0 - x_\alpha^{h,\delta}\rangle + \langle U_2(x_0^*), x_0^* - x_\alpha^{h,\delta,*}\rangle \\
&=\langle F_1'(x_0)^*x^1 + x^2, x_0 - x_\alpha^{h,\delta}\rangle + \langle F_2'(x_0^*)^*x^2 - x^1, x_0^* - x_\alpha^{h,\delta,*}\rangle \\
&=\langle x_\alpha^{h,\delta,*} - x_0^* - F_1'(x_0)(x_\alpha^{h,\delta} - x_0), x^1\rangle \\
&\quad + \langle x_0 - x_\alpha^{h,\delta} - F_2'(x_0^*)(x_\alpha^{h,\delta,*} - x_0^*), x^2\rangle \\
&=\langle F_1(x_\alpha^{h,\delta}) - F_1(x_0) - F_1'(x_0)(x_\alpha^{h,\delta} - x_0), x^1\rangle \\
&\quad + \alpha\langle U_1(x_\alpha^{h,\delta}), x^1\rangle + \langle F_1^h(x_\alpha^{h,\delta}) - F_1(x_\alpha^{h,\delta}), x^1\rangle \\
&\quad + \langle F_2(x_\alpha^{h,\delta,*}) - F_2(x_0^*) - F_2'(x_0^*)(x_\alpha^{h,\delta,*} - x_0^*), x^2\rangle \\
&\quad + \langle \alpha U_2(x_\alpha^{h,\delta,*}) + f - f_\delta, x^2\rangle + \langle F_2^h(x_\alpha^{h,\delta,*}) - F_(x_\alpha^{h,\delta,*}), x^2\rangle,
\end{aligned}$$

we have

$$\begin{aligned}
\langle J(z_0), z_0 - z_\alpha^{h,\delta}\rangle &\leqslant \max\{\tau_1\|x^1\|, \tau_2\|x^2\|\} \times \\
&(\|F_1(x_\alpha^{h,\delta}) - F_1(x_0)\| + \|F_2(x_\alpha^{h,\delta,*}) - F_2(x_0^*)\|) + O(h+\delta+\alpha).
\end{aligned}$$

Thus, for sufficiently small h, δ, α $(h+\delta+\alpha<1)$ from (2.5)-(2.7) we have got

$$A \leqslant O((h+\delta)^{1-\rho}) + O(\sqrt{h+\delta+\alpha}),$$

It means that

$$\|x_\alpha^{h,\delta} - x_0\| \; = \; O\big((h+\delta)^{\theta/s_1}\big).$$

Theorem is proved. ■

Theorem 2.2. *Assume that the conditions of Theorem 2.1 hold, and* α *is chosen such that* $\alpha \sim (h+\delta+\gamma_n)^\rho,\ 0<\rho<1,$ *where*

$$\gamma_n = \max\{\|(I-P_n)x_0\|, \|(I-P_n)f\|, \|(I-P_n)x^1\|, \|(I^*-P_n^*)x_0^*\|, \|(I^*-P_n^*)x^2\|\},$$

and I^* *denotes the identity operator in* X^*. *Then,*

$$\begin{aligned} \|x_{\alpha,n}^{h,\delta} - x_0\| &= O\big((h+\delta)^\eta + \gamma_n^\mu\big),\\ \eta &= \min\,\{\frac{1-\rho}{s_1}, \frac{\rho}{2s_1}\},\\ \mu &= \min\,\{\eta, \frac{\nu_1}{s_1}, \frac{\nu_2}{s_1}\}. \end{aligned}$$

Proof. Set

$$B = m_1\|x_{\alpha,n}^{h,\delta} - x_{0,n}\|^{s_1} + m_2\|x_{\alpha,n}^{h,\delta,*} - x_{0,n}^*\|^{s_2},$$

with $x_{0,n} = P_n x_0,\ x_{\alpha,n}^{h,\delta,*} = F_{1,\alpha,n}^h(x_{\alpha,n}^{h,\delta})$, and $x_{0,n}^* = P_n^* x_0^*$. It is easy to see that $x_{\alpha,n}^{h,\delta}$ is a solution of (1.3) iff $x_{\alpha,n}^{h,\delta}$ and $x_{\alpha,n}^{h,\delta,*}$ are the solutions of the system of following equations

$$\begin{aligned} F_{1,n}^h(x) + \alpha U_1^n(x) - x^* &= 0,\\ F_{2n}^h(x^*) + \alpha U_2^n(x^*) + x - f_{\delta,n} &= 0, \end{aligned}$$

with $U_1^n = P_n^* U_1 P_n, U_2^n = P_n U_2 P_n^*, F_{1,n}^h = P_n^* F_1^h P_n, F_{2,n}^h = P_n F_2^h P_n^*$, and $f_{\delta,n} = P_n f_\delta$. As in the proof of theorem 2.1, $z_{\alpha,n}^{h,\delta} := [x_{\alpha,n}^{h,\delta}, x_{\alpha,n}^{h,\delta,*}]$ is the solution of the following operator equation

$$\mathcal{A}_{\alpha,n}^h(z) \equiv \mathcal{A}_n^h(z) + \alpha J^n(z) = \overline{f}_{\delta,n}, \tag{2.8}$$

where

$$\begin{aligned} \mathcal{A}_n^h(z) &= [F_{1,n}^h(x), F_{2,n}^h(x^*)] + [-x^*, x],\\ J^n(z) &= [U_1^n(x), U_2^n(x^*)], \quad \overline{f}_{\delta,n} = [0, f_{\delta,n}]. \end{aligned} \tag{2.9}$$

The operators $\mathcal{A}_n^h$ and $\mathcal{A}_n$, defined by $\mathcal{A}_n(z) = [F_{1,n}(x), F_{2,n}(x^*)]+[-x^*, x]$, $F_{1,n} = P_n^* F_1 P_n, F_{2,n} = P_n F_2 P_n^*$, are the monotone operators, and act from $Z_n := X_n \times X_n^*$ into Z_n^*, and J^n is the normalized duality mapping of the space Z_n.

From (2.8) we obtain

$$\begin{aligned} \mathcal{A}_n(z_{\alpha,n}^{h,\delta}) - \mathcal{A}_n(z_{0,n}) + \alpha[J^n(z_{\alpha,n}^{h,\delta}) - J^n(z_{0,n})] = \overline{f}_{\delta,n} +&\\ \mathcal{A}_n(z_{\alpha,n}^{h,\delta}) - \mathcal{A}_n^h(z_{\alpha,n}^{h,\delta}) - \mathcal{A}_n(z_{0,n}) - \alpha J^n(z_{0,n}).& \end{aligned} \tag{2.10}$$

Therefore, from (1.4) and the properties of the projections P_n, P_n^* it implies that

$$\begin{aligned}
&\langle \mathcal{A}_n(z_{\alpha,n}^{h,\delta}) - \mathcal{A}_n(z_{0,n}), z_{\alpha,n}^{h,\delta} - z_{0,n}\rangle = \langle F_1(x_{\alpha,n}^{h,\delta}) - F_1(x_{0,n}), x_{\alpha,n}^{h,\delta} - x_{0,n}\rangle \\
&\quad + \langle F_2(x_{\alpha,n}^{h,\delta,*}) - F_2(x_{0,n}^*), x_{\alpha,n}^{h,\delta,*} - x_{0,n}^*\rangle \\
&\quad \geqslant \tilde{m}_1\|F_1(x_{\alpha,n}^{h,\delta}) - F_1(x_{0,n})\|^2 + \tilde{m}_2\|F_2(x_{\alpha,n}^{h,\delta,*}) - F_2(x_{0,n}^*)\|^2 \\
&\quad \geqslant \min\{\tilde{m}_1, \tilde{m}_2\}C_n^2, \quad C_n^2 = \|F_1(x_{\alpha,n}^{h,\delta}) - F_1(x_{0,n})\|^2 + \|F_2(x_{\alpha,n}^{h,\delta,*}) - F_2(x_{0,n}^*)\|^2.
\end{aligned}$$

On the other hand, from (2.8), (2.9) we also obtain

$$\begin{aligned}
\mathcal{A}_n^h(z_{\alpha,n}^{h,\delta}) - \mathcal{A}_n^h(z_{0,n}) + \alpha[J^n(z_{\alpha,n}^{h,\delta}) - J^n(z_{0,n})] = \overline{f}_{\delta,n} \\
- \mathcal{A}_n^h(z_{0,n}) - \alpha J^n(z_{0,n}).
\end{aligned} \tag{2.11}$$

Hence, on the base of the property of J and (2.11) we can write

$$\begin{aligned}
B &\leqslant \frac{1}{\alpha}\langle \overline{f}_\delta - \overline{f} - \alpha J(z_{0,n}), z_{\alpha,n}^{h,\delta} - z_{0,n}\rangle \\
&+ \frac{1}{\alpha}\langle \mathcal{A}(z_0) - \mathcal{A}^h(z_{0,n}), z_{\alpha,n}^{h,\delta} - z_{0,n}\rangle \\
&\leqslant \frac{1}{\alpha}[\delta + \|\mathcal{A}(z_0) - \mathcal{A}(z_{0,n})\| + hg(\|z_{0,n}\|)]\|z_{\alpha,n}^{h,\delta} - z_{0,n}\| \\
&+ \langle J^n(z_{0,n}), z_{0,n} - z_{\alpha,n}^{h,\delta}\rangle.
\end{aligned} \tag{2.12}$$

Moreover, using the continously Fréchet differentiable property of F_1, F_2 and the definition of γ_n we can also write

$$\begin{aligned}
\|\mathcal{A}(z_{0,n}) - \mathcal{A}(z_0)\| &\leqslant (\|F_1(x_{0,n}) - F_1(x_0)\|^2 \\
&\quad + \|F_2(x_{0,n}^*) - F_2(x_0^*)\|^2)^{1/2} + \sqrt{2}\gamma_n \\
&\leqslant (\max\{\tilde{c}_1, \tilde{c}_2\} + \sqrt{2})\gamma_n,
\end{aligned}$$

where $\tilde{c}_1 = \max_{0\leqslant t\leqslant 1}\|F_1'(x_0 + t(x_{0,n} - x_0))\|$ and $\tilde{c}_2 = \max_{0\leqslant t\leqslant 1}\|F_2'(x_0^* + t(x_{0,n}^* - x_0^*))\|$. Consequently, $\{z_{\alpha,n}^{h,\delta}\}$ is bounded, when $(h+\delta+\gamma_n)/\alpha \to 0$. By virtue of (2.10) we have

$$\begin{aligned}
&\langle \mathcal{A}_n(z_{\alpha,n}^{h,\delta}) - \mathcal{A}_n(z_{0,n}), z_{\alpha,n}^{h,\delta} - z_{0,n}\rangle \leqslant \langle \overline{f}_{\delta,n} - \mathcal{A}_n(z_{0,n}), z_{\alpha,n}^{h,\delta} - z_{0,n}\rangle \\
&\quad + \langle \mathcal{A}_n(z_{\alpha,n}^{h,\delta}) - \mathcal{A}_n^h(z_{\alpha,n}^{h,\delta}) - \alpha J^n(z_{0,n}), z_{\alpha,n}^{h,\delta} - z_{0,n}\rangle \\
&\quad \leqslant \langle \overline{f}_\delta - \overline{f} + \mathcal{A}(z_0) - \mathcal{A}(z_{0,n}), z_{\alpha,n}^{h,\delta} - z_{0,n}\rangle \\
&\quad + \langle \mathcal{A}_n(z_{\alpha,n}^{h,\delta}) - \mathcal{A}_n^h(z_{\alpha,n}^{h,\delta}) - \alpha J(z_{0,n}), z_{\alpha,n}^{h,\delta} - z_{0,n}\rangle \\
&\quad \leqslant O(h+\delta+\alpha+\gamma_n)\|z_{0,n} - z_{\alpha,n}^{h,\delta}\rangle\|.
\end{aligned}$$

Therefore, $\tilde{C}_n \leqslant O(\sqrt{h+\delta+\alpha+\gamma_n})$. Hence,

$$\begin{aligned}
\|F_1(x_{\alpha,n}^{h,\delta}) - F_1(x_{0,n})\| &\leqslant O(\sqrt{h+\delta+\alpha+\gamma_n}), \\
\|F_2(x_{\alpha,n}^{h,\delta,*}) - F_2(x_{0,n}^*)\| &\leqslant O(\sqrt{h+\delta+\alpha+\gamma_n}).
\end{aligned}$$

Now, we obtain the esimation for $\langle J^n(z_{0,n}), z_{0,n} - z^{h,\delta}_{\alpha,n}\rangle$. From (2.2), (2.8) and the condition of the theorem we have got

$$\begin{aligned}
\langle J^n(z_{0,n}), z_{0,n} - z^{h,\delta}_{\alpha,n}\rangle &= \langle J(z_{0,n}), z_{0,n} - z^{h,\delta}_{\alpha,n}\rangle \\
&= \langle J(z_{0,n}) - J(z_0), z_{0,n} - z^{h,\delta}_{\alpha,n}\rangle + \langle J(z_0), z_{0,n} - z^{h,\delta}_{\alpha,n}\rangle \\
&\leqslant \overline{C}\gamma_n^{\nu}\|z^{h,\delta}_{\alpha,n} - z_{0,n}\| + \langle F_1'(x_0)^* x^1 + x^2, x_{0,n} - x^{h,\delta}_{\alpha,n}\rangle \\
&\quad + \langle F_2'(x_0^*)^* x^2 - x^1, x^*_{0,n} - x^{h,\delta,*}_{\alpha,n}\rangle \\
&\leqslant \overline{C}\gamma_n^{\nu}\|z^{h,\delta}_{\alpha,n} - z_{0,n}\| + \langle x^1, x^{h,\delta,*}_{\alpha,n} - x^*_{0,n} - F_1'(x_0)(x^{h,\delta}_{\alpha,n} - x_{0,n})\rangle \\
&\quad + \langle x^2, x_{0,n} - x^{h,\delta}_{\alpha,n} - F_2'(x_0^*)(x^{h,\delta,*}_{\alpha,n} - x^*_{0,n})\rangle,
\end{aligned}$$

where $\overline{C}$ is some positive constant, and $\nu = \min\{\nu_1, \nu_2\}$. Obviously,

$$\begin{aligned}
\langle x^1, x^{h,\delta,*}_{\alpha,n} - x^*_{0,n} - F_1'(x_0)(x^{h,\delta}_{\alpha,n} - x_{0,n})\rangle &= \langle x^1, F^h_{1,n}(x^{h,\delta}_{\alpha,n}) + \alpha U_1^n(x^{h,\delta}_{\alpha,n}) - x^*_{0,n}\rangle \\
&\quad + \langle x^1, -F_1'(x_0)(x^{h,\delta}_{\alpha,n} - x_0) + F_1'(x_0)(x_{0,n} - x_0)\rangle \\
= \langle x^1_n, F_1(x^{h,\delta}_{\alpha,n}) - F_1(x_0) - F_1'(x_0)(x^{h,\delta}_{\alpha,n} - x_0)\rangle & \\
\quad + \alpha\langle x^1, U_1^n(x^{h,\delta}_{\alpha,n})\rangle + \langle x^1, F_1'(x_0)(x_{0,n} - x_0)\rangle & \\
\quad + \langle (I - P_n)x^1, -F_1'(x_0)(x^{h,\delta}_{\alpha,n} - x_0)\rangle + \langle x^1_n, F_1^h(x^{h,\delta}_{\alpha,n}) - F_1(x^{h,\delta}_{\alpha,n})\rangle & \\
\leqslant \tau_1\|x^1_n\|\|F_1(x^{h,\delta}_{\alpha,n}) - F_1(x_0)\| + O(h + \alpha + \gamma_n), &
\end{aligned}$$

where $x^1_n = P_n x^1$. By the argument, we also obtain the estimate

$$\begin{aligned}
\langle x^2, x_{0,n} - x^{h,\delta}_{\alpha,n} - F_2'(x_0^*)(x^{h,\delta,*}_{\alpha,n} - x^*_{0,n})\rangle &\leqslant \tau_2\|x^2_n\|\|F_2(x^{h,\delta,*}_{\alpha,n}) - F_2(x_0^*)\| \\
&\quad + O(h + \delta + \alpha + \gamma_n).
\end{aligned}$$

Therefore,

$$\langle J^n(z_0), z_0 - z^{h,\delta}_{\alpha}\rangle \leqslant O(\gamma_n^{\nu}) + O(\sqrt{h + \delta + \gamma_n + \alpha}).$$

Thus, from (2.12) and the properties of $\mathcal{A}^h, J$ it follows

$$B \leqslant O((h + \delta + \gamma_n)^{1-\rho} + \gamma_n^{\nu} + O((h + \delta + \gamma_n)^{\rho/2}).$$

Consequently,

$$\|x^{h,\delta}_{\alpha,n} - x_0\| \;=\; O\big((h + \delta)^{\eta} + \gamma_n^{\mu}).$$

Theorem is proved. ■

Example 1. Consider the simple example, when $X \equiv X^* = \mathbf{E}^2$, the Euclid space, and

$$F_1 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, F_2 = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, x = (x_1, x_2).$$

It is easy to verify that $\langle F_1 x, x\rangle = x_1^2 \geqslant 0$, and $\langle F_2 x, x\rangle = x_2^2 \geqslant 0 \forall x \in \mathbf{E}^2$. It means that $F_i, i = 1, 2$, are monotone. Equation (1.1) has the form $0x_1 = f_1, \quad 2x_1 = f_2$ with $f = (f_1, f_2)$. Obviously, this system of equations has a unique solution when $f = (0, f_2)$ for arbitrary f_2. When $f_\delta = (f_1^\delta, f_2)$ with $f_1^\delta \neq 0$ equation (1.1) in this case there isn't a solution. So, equation

(1.1) with the monotone operators $F_1, i = 1, 2$, in general is ill-posed. On the other hand, equation $\mathcal{A}(z) = \overline{f}$ for $z = (x_1, x_2, x_1^*, x_2^*)$ is the system of 4 linear equations with the matrix

$$\mathcal{A} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

having det $\mathcal{A} = 0$. Consequently, the system of equations is also ill-posed.

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Received on April 6 - 2006
Revised on November 14 - 2006