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SOLVING A NONLINEAR BIHARMONIC BOUNDARY VALUE PROBLEM*

DANG QUANG A[1], TRUONG HA HAI[2], NGUYEN THANH HUONG[3], NGO THI KIM QUY[4]

[1] *Centre for Informatics and Computing, VAST*
[2] *Thai Nguyen University of Information and Communication Technology*
[3] *College of Sciences, Thai Nguyen University*
[4] *Thai Nguyen University of Economic and Business Administration*
[3] *nguyenthanhhuong2806@gmail.com*

Abstract. In this paper we study a boundary value problem for a nonlinear biharmonic equation, which models a bending plate on nonlinear elastic foundation. We propose a new approach to existence and uniqueness and numerical solution of the problem. It is based on the reduction of the problem to finding fixed point of a nonlinear operator for the nonlinear term. The result is that under some easily verified conditions we have established the existence and uniqueness of a solution and the convergence of an iterative method for the solution. The positivity of the solution and the monotony of iterations are also considered. Some examples demonstrate the applicability of the obtained theoretical results and the efficiency of the iterative method.

Keywords. Nonlinear biharmonic boundary value problem, Existence and uniqueness of solution, Iterative method, Numerical solution.

1. INTRODUCTION

In this paper we study the following nonlinear biharmonic boundary value problem (BVP)

$$\begin{aligned} \Delta^2 u &= f(x, u, \Delta u), \quad x \in \Omega, \\ u &= 0, \quad \Delta u = 0, \quad x \in \Gamma, \end{aligned} \tag{1}$$

where Ω is a connected bounded domain in $\mathbb{R}^2$, with a smooth boundary Γ, Δ is the Laplace operator. We assume that $f(x, u, v)$ is a function continuous in a bounded domain, which will be indicated later. The problem (1) describes the static deflection of an elastic bending plate with hinged edges rested on nonlinear foundation.

For the one dimension case, the problem is of the form

$$\begin{aligned} u^{(4)}(x) &= f(x, u(x), u''(x)), \quad 0 < x < 1, \\ u(0) &= u(1), \quad u''(0) = u''(1) = 0. \end{aligned}$$

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Although there are a lot of papers concerned with this simply supported beam problem, recently it still attracts attention from some authors (see e.g. [2, 4, 9] and the references therein).

A particular case of the problem (1) in the multidimensional case, namely, the problem

$$\begin{aligned} \Delta^2 u + c\Delta u = f(x,u), \quad x \in \Omega, \\ u = 0, \quad \Delta u = 0, \quad x \in \Gamma, \end{aligned} \tag{2}$$

has been studied in many works. Below we mention some of them.

The first is an early paper of Dunninger [7], where by the maximum principle he established the uniqueness of a solution of the problem. After this work in 2007 Liu and Wang [11] by a variant version of Mountain Pass Theorem proved the existence of nontrivial solution of the problem in the case $c = 0$ provided that the function $f(x,t)$ satisfies the following hypotheses:

(H1) $f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$; $f(x,t) \equiv 0, \forall x \in \overline{\Omega}, t \le 0$; $f(x,t) \ge 0, \forall x \in \overline{\Omega}, t > 0$;

(H2) $\lim\limits_{t\to 0^+} \dfrac{f(x,t)}{t} = p(x)$, $\lim\limits_{t\to +\infty} \dfrac{f(x,t)}{t} = l$, $(0 < l \le +\infty)$ uniformly a.e. $x \in \Omega$, where $0 \le p(x) \in L^\infty(\Omega)$, $\|p(x)\|_\infty < \Lambda_1$ and Λ_1 is the first eigenvalue of $(\Delta^2, H^2(\Omega) \cap H_0^1(\Omega))$.

Under the same (H1), modified (H2) and some other conditions posed on the function $f(x,t)$ in [1] An and Liu established the existence of nontrivial solution of the problem (2) for $c < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. In a very recent work [8], Hu and Wang added to the above hypotheses (H1) and (H2), where Λ_1 now is the first eigenvalue of $(\Delta^2 + c\Delta, H^2(\Omega) \cap H_0^1(\Omega))$, the following hypothesis:

(H3) For a.e. $x \in \Omega$, $\dfrac{f(x,t)}{t}$ is nondecreasing with respect to $t > 0$.
Under the conditions (H1)-(H3) they studied the existence of a nontrivial solution to problem (2) in dependence if $0 < l < \Lambda_1$, $\Lambda_1 < l \le \infty$ or $l = \Lambda_1$. Similar results were also obtained if replacing the hypotheses (H1) and (H3) by

(H1') $f(x,t) \in C(\overline{\Omega} \times \mathbb{R})$; $f(x,0) = 0$, $\forall x \in \overline{\Omega}$; $f(x,t)t \ge 0$, $\forall (x,t) \in \overline{\Omega} \times \mathbb{R}$;

(H3') For a.e. $x \in \Omega$, $\dfrac{f(x,t)}{t}$ is nondecreasing with respect to $t > 0$ and decreasing with respect to $t < 0$.

Recently an interesting result of existence of sign-changing solutions as well as positive and negative solutions to the problem (2) was obtained in [10] under some restrictions of the hypotheses (H1) and (H2).

It should be emphasized that the existence results of solutions of the problem (2) for both cases $c = 0$ and $c \neq 0$ were established by the variational method. These results are of pure theoretical character and there are not examples of existing solutions. The question of finding solutions is not considered in the above mentioned works [1, 7, 8, 10, 11] and references therein.

Except for the variational method for studying the existence of solutions of nonlinear boundary value problems there is an another effective method for establishing the existence and uniqueness of solutions. This is the method of upper and lower solutions. The use of this method for fourth order elliptic boundary value problems was due to Pao in [12]. In this work, for the problem more general than (1), namely, for the problem

$$\begin{gathered}\Delta(a(x)\Delta u)=f(x,u,\Delta u),\quad x\in\Omega,\\ \alpha_1(x)\frac{\partial u}{\partial\nu}+\beta_1(x)u=h_1(x),\ \alpha_2(x)\frac{\partial(a\Delta u)}{\partial\nu}+\beta_2(x)(a\Delta u)=h_2(x),\ x\in\Gamma,\end{gathered}\tag{3}$$

by the method of upper and lower solutions Pao proved the existence and uniqueness of a solution of the problem under the assumptions that the function $f(x,u,v)$ satisfies the Lipschitz conditions in u,v and the monotony property in u in a sector defined by the upper and lower solutions, and some other conditions concerned with $a(x)$ and the partial derivatives of $f(x,u,v)$. In a next work [13] for solving the problem (3) numerically Pao transformed the problem into a coupled system of u and $v=-a\Delta u$ and applied the standard difference approximation to this system. Some monotone iterative schemes such as Picard, Gauss-Seidel and Jacobi iterations, which converge monotonically to a unique solution of the system of difference equations, were proposed. The further development of this monotone technique was obtained in Wang's works [17, 18, 19, 20], where some examples for illustrating effectiveness of iterative schemes, were presented.

Differently from the above mentioned methods, in the present paper, we use the approach developed by ourselves recently in [4, 5, 6] for boundary value problems for fourth order nonlinear ordinary differential equations. This approach originates from the work [3] of the first author in 2006 when considering the problem for linear biharmonic type equation with Neumann boundary conditions. By this approach we reduce the problem (1) to an operator equation for the nonlinear term and prove that under some simple conditions, the operator is a contraction mapping. Due to this result, the existence and uniqueness of a solution and the convergence of the iterative method are established. The realization of the iterative method leads to a sequence of BVPs for the Poisson equation, which are easily solved by available efficient numerical algorithms. Some examples demonstrate the applicability of the approach and the efficiency of the iterative method, in which difference schemes of second and fourth order of accuracy are used. The advantages of the proposed method in comparison with the recent known Wang's method [19, 20] are shown.

2. EXISTENCE AND UNIQUENESS RESULTS

To investigate the problem (1) we shall reduce it to an operator equation as follows.
For functions $\varphi(x)\in C(\overline{\Omega})$ consider the nonlinear operator defined by

$$(A\varphi)(x)=f(x,u(x),\Delta u(x)),\tag{4}$$

where $u(x)$ is the solution of the problem

$$\begin{gathered}\Delta^2u=\varphi(x),\ x\in\Omega,\\ u=\Delta u=0,\ x\in\Gamma.\end{gathered}\tag{5}$$

Proposition 1. *Function $\varphi(x)$ is a solution of the operator equation*

$$A\varphi=\varphi,\tag{6}$$

i.e., $\varphi(x)$ is a fixed point of the operator A defined by (4)-(5) if and only if the function $u(x)$ being the solution of the boundary value problem (5) solves the problem (1).

Proof. Suppose $\varphi \in C(\overline{\Omega})$ is a solution of the operator equation (6) with A defined by (4). Then, it is easy to see that, the solution $u(x)$ of the problem (5) also is the solution of the problem (1).

Conversely, if a function $u(x)$ is the solution of the problem (1), then the function φ defined by

$$\varphi(x) = f(x, u(x), \Delta u(x)),$$

satisfies the operator equation (6). Thus, the proposition is proved. ■

According to the above proposition, the solution of the problem (1) is reduced to fixed point problem for the operator A defined by (4), (5).

Now we study properties of operator A. First we prove the following

Lemma 1. *For the solution of the BVP*

$$\begin{aligned} -\Delta u &= f(x),\ x \in \Omega, \\ u &= 0,\ x \in \Gamma \end{aligned}$$

there holds the estimate

$$\|u\| \le C_\Omega \|f\|, \tag{7}$$

where $\|u\| = \max_{x\in\bar{\Omega}} |u(x)|,\ C_\Omega = \dfrac{R^2}{4}$ *and R is the radius of the circle containing the domain* Ω.

If Ω is the unit square then

$$\|u\| \le \frac{1}{8}\|f\|. \tag{8}$$

Proof. In the domain Ω consider the function

$$v(x) = \frac{1}{4}\|f\|(R^2 - x_1^2 - x_2^2).$$

It is easy to calculate

$$-\Delta v = \|f\|,\ x \in \Omega.$$

Taking into account that $0 = |u(x)| \le |v(x)|$ on Γ, by the maximum principle for the Poisson equation (see e.g. [14]) we have

$$|u(x)| \le |v(x)|,\ x \in \Omega. \tag{9}$$

Since

$$0 \le v(x) \le \frac{1}{4}\|f\|R^2 = C_\Omega \|f\|$$

from (9) we obtain (7).

In the particular case if the domain Ω is the unit square we can choose $R = \sqrt{2}$, and consequently we obtain (8). The lemma is proved. ■

Now for each positive number M denote

$$D_M = \{(x, u, v)|\ x \in \Omega, |u| \le C_\Omega^2 M, |v| \le C_\Omega M\}$$

and denote by $B[O, M]$ the closed ball of the radius M in $C(\bar{\Omega})$, i.e.

$$B[O, M] = \{\varphi \in C(\bar{\Omega})|\ \|\varphi\| \le M\}.$$

Theorem 1. *Assume that there exist numbers $M, L_1, L_2 \ge 0$ such that*

1.

$$|f(x,u,v)| \le M, \quad \forall (x,u,v) \in D_M. \tag{10}$$

2.

$$|f(x,u_2,v_2)-f(x,u_1,v_1)| \le L_1|u_2-u_1|+L_2|v_2-v_1|, \quad \forall (x,u_i,v_i) \in D_M,\ i=1,2. \tag{11}$$

3.

$$q := (L_2 + C_\Omega L_1)C_\Omega < 1. \tag{12}$$

Then the BVP (1) *possesses a unique solution* $u(x) \in C(\bar{\Omega})$*, for which there holds the estimate* $\|u\| \le C_\Omega^2 M$*.*

Proof. In order to prove the theorem we shall show that the operator A defined by (4), (5) maps $B[O,M]$ into itself and is a contraction mapping. Then from the contraction mapping principle it follows the existence and uniqueness of fixed point φ of the operator A, which according to Proposition 1 generates a unique solution of the original problem.

Indeed, for any $\varphi \in B[O,M]$, we decompose the problem (5) into the couple of second order problems

$$\begin{aligned} \Delta v &= \varphi, \quad x \in \Omega, \\ v &= 0, \quad x \in \Gamma, \end{aligned}$$

$$\begin{aligned} \Delta u &= v, \quad x \in \Omega, \\ u &= 0, \quad x \in \Gamma. \end{aligned}$$

By Lemma 1 we have the following estimates for the solutions of these problems

$$\begin{aligned} \|v\| &\le C_\Omega \ \|\varphi\| \le C_\Omega M, \\ \|u\| &\le C_\Omega \|v\| \le C_\Omega^2 M. \end{aligned}$$

Consequently, when $x \in \Omega$ we have $(x,u,v) \in D_M$. Therefore, from the assumption (10) of the theorem it follows that $\|A\varphi\| \le M$, i.e., $A\varphi \in B[O,M]$.

Next, we show that A is a contraction operator in $B[O,M]$. Indeed, for any $\varphi_1, \varphi_2 \in B[O,M]$ denote by $v_1, u_1; v_2, u_2$ the solutions of the problems

$$\begin{aligned} \Delta v_i &= \varphi_i, \quad x \in \Omega, \\ v_i &= 0, \quad x \in \Gamma, \end{aligned} \tag{13}$$

$$\begin{aligned} \Delta u_i &= v_i, \quad x \in \Omega, \\ u_i &= 0, \quad x \in \Gamma,\ (i=1,2). \end{aligned} \tag{14}$$

As above, by Lemma 1, it follows that $(x,u_i,v_i) \in D_M$ $(i=1,2)$ for any $x \in \Omega$. From (13), (14) we have

$$\begin{aligned} \Delta(v_2 - v_1) &= \varphi_2 - \varphi_1, \quad x \in \Omega, \\ v_2 - v_1 &= 0, \quad x \in \Gamma, \end{aligned}$$

$$\Delta(u_2 - u_1) = v_2 - v_1, \quad x \in \Omega,$$
$$u_2 - u_1 = 0, \quad x \in \Gamma.$$

Therefore, using the Lipschitz condition (11) and Lemma 1 we obtain

$$\begin{aligned}|A\varphi_2 - A\varphi_1| &= |f(x, u_2, v_2) - f(x, u_1, v_1)| \\ &\le L_1|u_2 - u_1| + L_2|v_2 - v_1| \\ &\le L_1 C_\Omega^2\|\varphi_2 - \varphi_1\| + L_2 C_\Omega\|\varphi_2 - \varphi_1\| \\ &= (L_1 C_\Omega + L_2)C_\Omega\|\varphi_2 - \varphi_1\|.\end{aligned}$$

Hence,

$$\|A\varphi_2 - A\varphi_1\| \le q\|\varphi_2 - \varphi_1\|,$$

where q is defined by (12).

Therefore, if $q < 1$ then operator A is a contraction operator in $B[O, M]$. By the contraction principle the operator equation (6) has a unique solution $\varphi \in B[O, M]$. Hence, the problem (1) has a unique solution $u(x)$ satisfying the estimate $\|u\| \le C_\Omega^2 M$. The theorem is proved. ■

Now consider a particular case of Theorem 1.

Denote

$$D_M^+ = \{(x, u, v)|\ x \in \Omega, 0 \le u \le C_\Omega^2 M, -C_\Omega M \le v \le 0\}.$$

Theorem 2. *Assume that there exist numbers $M, L_1, L_2 \ge 0$ such that*

1.

$$0 \le f(x, u, v) \le M,\ \forall (x, u, v) \in D_M^+.$$

2.

$$|f(x, u_2, v_2) - f(x, u_1, v_1)| \le L_1|u_2 - u_1| + L_2|v_2 - v_1|,\ \forall (x, u_i, v_i) \in D_M^+,\ i = 1, 2.$$

3.

$$q := (L_2 + C_\Omega L_1)C_\Omega < 1.$$

Then the BVP (1) possesses a unique positive solution $u(x) \in C(\bar{\Omega})$, for which there holds the estimate $0 \le u(x) \le C_\Omega^2 M$.

Proof. The proof of the theorem is the same as of Theorem 1 with the replacement of the ball $B[O, M]$ by the strip

$$S_M = \{\varphi \in C(\bar{\Omega})|\ 0 \le \varphi(x) \le M\}.$$

■

3. ITERATIVE METHOD AND ITS NUMERICAL REALIZATIONS

Consider the following iterative process for finding fixed point φ of the operator A and simultaneously for finding the solution of the original boundary value problem u:

1. Given an initial approximation $\varphi_0 \in B[O, M]$, for example,

$$\varphi_0(x) = f(x, 0, 0),\ x \in \Omega. \tag{15}$$

2. Knowing φ_k in Ω $(k = 0, 1, ...)$ solve sequentially two Poisson problems

$$\begin{aligned} \Delta v_k &= \varphi_k, \quad x \in \Omega, \\ v_k &= 0, \quad x \in \Gamma, \end{aligned} \tag{16}$$

$$\begin{aligned} \Delta u_k &= v_k, \quad x \in \Omega, \\ u_k &= 0, \quad x \in \Gamma. \end{aligned} \tag{17}$$

3. Update the new approximation

$$\varphi_{k+1} = f(x, u_k, v_k). \tag{18}$$

Theorem 3. *Suppose that the assumptions of Theorem 1 (or Theorem 2) hold. Then the above iterative method converges and there holds the estimate*

$$||u_k - u|| \le C_\Omega^2 \frac{q^k}{(1-q)} \|\varphi_1 - \varphi_0\|, \tag{19}$$

where u is the exact solution of the problem (1) *and q is given by* (12).

Proof. Having in mind that the proposed iterative method is the successive iteration process for the fixed point of the operator A with the initial approximation from $B[O, M]$, we conclude that it converges with the rate of geometric progression and there is the estimate

$$||\varphi_k - \varphi|| \le \frac{q^k}{1-q} ||\varphi_1 - \varphi_0||.$$

In combination with the estimate $\|u_k - u\| \le C_\Omega^2 \|\varphi_k - \varphi\|$, which is easily obtained with the help of Lemma 1 we come to (19), and the theorem is proved. ■

Below we study some properties of the iterations generated by the iterative process (15)-(18). For this purpose we need a simple comparison lemma, which is a corollary from the maximum principle (see e.g. [14]).

Lemma 2. *Let $u_i(x)$ be the solutions of the problems*

$$\begin{aligned} \Delta u_i &= f_i(x), \quad x \in \Omega, \\ u_i &= 0, \quad x \in \Gamma, \ (i = 1, 2). \end{aligned}$$

If $f_1(x) \le f_2(x)$, $x \in \Omega$ then $u_1(x) \ge u_2(x)$, $x \in \Omega$.

Theorem 4. *(Monotony) Assume that all the conditions of Theorem 1 (or Theorem 2) are satisfied. In addition, we assume that the function $f(x, u, v)$ is increasing in u and decreasing in v for any $(x, u, v) \in D_M$. Then, if $\varphi_0^{(1)}, \varphi_0^{(2)} \in B[O, M]$ are initial approximations and $\varphi_0^{(1)}(x) \le \varphi_0^{(2)}(x)$ for any $x \in \Omega$ then the sequences $u_k^{(1)}, u_k^{(2)}$ generated by the iterative process satisfy the relation*

$$u_k^{(1)}(x) \le u_k^{(2)}(x), \ k = 0, 1, ..., \ x \in \Omega. \tag{20}$$

Proof. First we show that (20) is valid for $k = 0$. Indeed, since

$$\begin{aligned} \Delta v_0^{(i)} &= \varphi_0^{(i)}, \quad x \in \Omega, \\ v_0^{(i)} &= 0, \quad x \in \Gamma \ (i = 1, 2) \end{aligned}$$

and $\varphi_0^{(1)}(x) \le \varphi_0^{(2)}(x)$, by Lemma 2 we have $v_0^{(1)}(x) \ge v_0^{(2)}(x)$. Consequently, once again, by the lemma we have $u_0^{(1)}(x) \le u_0^{(2)}(x)$.
Now using the assumption that the function $f(x, u, v)$ is increasing in u and decreasing in v we have

$$\varphi_1^{(1)}(x) = f(x, u_0^{(1)}, v_0^{(1)}) \le f(x, u_0^{(2)}, v_0^{(1)}) \le f(x, u_0^{(2)}, v_0^{(2)}) = \varphi_1^{(2)}(x).$$

Repeating the above argument we obtain $u_1^{(1)}(x) \le u_1^{(2)}(x)$ and in general $u_k^{(1)}(x) \le u_k^{(2)}(x)$. Thus, the theorem is proved. ∎

Corollary 1. *Denote*

$$\varphi_{\min} = \min_{(x,u,v) \in D_M} f(x, u, v),$$

$$\varphi_{\max} = \max_{(x,u,v) \in D_M} f(x, u, v).$$

Under the assumptions of Theorem 4, if starting from $\varphi_0 = \varphi_{\min}$ we obtain the increasing sequence $u_k(x)$, inversely, starting from $\varphi_0 = \varphi_{\max}$ we obtain the decreasing sequence $\widehat{u}_k(x)$, both of them converge to the exact solution $u(x)$ of the problem and $u_k(x) \le u(x) \le \widehat{u}_k(x)$.

Theorem 5. *(Lower and supper solutions) Suppose $\varphi_{\min} \le 0$, $\varphi_{\max} \ge 0$. Then, the function $\alpha(x) = u(x)$ obtained from the problems*

$$\begin{aligned} \Delta v &= \varphi_{min}, \quad x \in \Omega, \\ v &= 0, \quad x \in \Gamma, \end{aligned}$$

$$\begin{aligned} \Delta u &= v, \quad x \in \Omega, \\ u &= 0, \quad x \in \Gamma, \end{aligned}$$

is a lower solution of the problem (1), *and the function $\beta(x) = u(x)$ obtained from the problems*

$$\begin{aligned} \Delta v &= \varphi_{max}, \quad x \in \Omega, \\ v &= 0, \quad x \in \Gamma, \end{aligned}$$

$$\begin{aligned} \Delta u &= v, \quad x \in \Omega, \\ u &= 0, \quad x \in \Gamma, \end{aligned}$$

is an upper solution of the problem (1). *Here the lower and upper solutions are understood in the sense of [17].*

To numerically realize the above iterative method we denote the set of interior nodes and the set of boundary nodes of the grid covering the domain Ω by ω_h and γ_h, respectively. We shall use difference schemes of second and fourth order of accuracy for solving second order boundary value problems (16)-(17) at each iteration.

For simplicity we consider the problem in the unit square

$$\Omega = \{(x_1, x_2),\ 0 \le x_i \le 1,\ i = 1, 2\}$$

and cover it by the uniform grid

$$\overline{\omega}_h = \{(x_{1i} = ih_1,\, x_{2j} = jh_2), \quad i = 0,1,...,N; j = 0,1,...,M\},$$

where $h_1 = 1/N, h_2 = 1/M$.
On this grid the Laplace operator Δ is discretized by the central difference operator Λ, defined as follows for grid functions $z(x_1, x_2)$ (see [15])

$$\begin{aligned}
\Lambda z &= (\Lambda_1 + \Lambda_2)z,\\
\Lambda_1 z = z_{\overline{x}_1 x_1} &= \frac{z(x_1 + h_1, x_2) - 2z(x_1, x_2) + z(x_1 - h_1, x_2)}{h_1^2},\\
\Lambda_2 z = z_{\overline{x}_2 x_2} &= \frac{z(x_1, x_2 + h_2) - 2z(x_1, x_2) + z(x_1, x_2 - h_2)}{h_2^2}.
\end{aligned}$$

Below we consider two discrete versions of the iterative method (15)-(18).

The first version using the difference scheme of second order of accuracy (see [15]) has the form of Method DIM2.

Method DIM2

1. Given
$$\varphi_0^h(x) = f(x, 0, 0),\ x \in \omega_h. \tag{21}$$

2. Knowing φ_k^h in ω_h $(k = 0, 1, ...)$ solve consecutively two problems
$$\begin{aligned}\Lambda v_k^h &= \varphi_k^h, \quad x \in \omega_h,\\ v_k^h &= 0, \quad x \in \gamma_h,\end{aligned} \tag{22}$$
$$\begin{aligned}\Lambda u_k^h &= v_k^h, \quad x \in \omega_h,\\ u_k^h &= 0, \quad x \in \gamma_h.\end{aligned} \tag{23}$$

3. Update the new approximation
$$\varphi_{k+1}^h = f(x, u_k^h, v_k^h), \quad x \in \omega_h, \tag{24}$$
where the superscript h associated with continuous functions means grid approximations for the continuous functions on the grid $\bar{\omega}_h = \omega_h + \gamma_h$.

When using the difference scheme of fourth order of accuracy (see [15, Sect. 4.5]) the discrete version of the iterative method (15)-(18) has the form of Method DIM4.

Method DIM4

1. Given
$$\varphi_0^h(x) = f(x, 0, 0),\ x \in \omega_h. \tag{25}$$

2. Knowing φ_k^h in ω_h $(k = 0, 1, ...)$ solve consecutively two problems
$$\begin{aligned}\Lambda^* v_k^h &= \varphi_k^{h^*}, \quad x \in \omega_h,\\ {v_k}^h &= 0, \quad x \in \gamma_h,\end{aligned} \tag{26}$$

$$\begin{aligned} \Lambda^* u_k^h &= v_k^{h*}, \quad x \in \omega_h, \\ u_k^h &= 0, \quad x \in \gamma_h, \end{aligned} \tag{27}$$

where the sign "*" associated with the difference operator Λ and grid functions are defined as follows

$$\Lambda^* w = \Lambda w + \frac{h_1^2 + h_2^2}{12} \Lambda_1 \Lambda_2 w,$$
$$\psi^* = \psi + \frac{h_1^2}{12} \Lambda_1 \psi + \frac{h_2^2}{12} \Lambda_2 \psi.$$

3. Update the new approximation

$$\varphi_{k+1}^h = f(x, u_k^h, v_k^h), \quad x \in \omega_h. \tag{28}$$

We shall consider some examples for the problem (1) in unit square. On this domain uniform grids of 33×33 and 65×65 nodes are used. The resulting systems of grid equations are solved by the method of cyclic reduction [16, Chapt. 3]. The iterative process (21)-(24) will be carried out until $e(m) = \|u_m^h - u_{m-1}^h\| \le 10^{-16}$. Here the norm of grid function w^h is defined as $\|w^h\| = \max_{x \in \bar{\omega}_h} |w^h(x)|$.

4. EXAMPLES

We shall consider some examples for demonstrating the applicability of the existence results in Section 2. and the efficiency of the iterative method in the previous section. In Examples 1 and 2 exact solutions are known, and in other examples exact solutions are not known. Examples 1 and 2 are taken from [19] and [20], where the standard second order central difference approximation for the operator Δ is used. Therefore, for them we use the methods DIM2 and DIM4 for the aim of comparing convergence of DIM2 with Wang's methods. For other examples we use the method DIM4.

Example 1. (Problem 1 in [19]) Consider the boundary value problem

$$\begin{aligned} \Delta^2 u &= \sigma u/(1+u) + p(x,y), \quad (x,y) \in \Omega, \\ u &= 0, \quad \Delta u = 0, \quad (x,y) \in \partial\Omega, \end{aligned} \tag{29}$$

where $\Omega = \{(x,y); 0 < x < 1, 0 < y < 1\}$, $k > 0$ is an arbitrary constant, σ is a positive constant and $p(x,y)$ is the function

$$p(x,y) = k\left(4\pi^4 - \frac{\sigma}{1 + k \sin \pi x \sin \pi y}\right) \sin \pi x \sin \pi y.$$

It is easy to check that the function $u(x,y) = k \sin \pi x \sin \pi y$ is an exact solution of (29).
For this problem we have

$$f = f(x,y,u,v) = \frac{\sigma u}{1+u} + p(x,y).$$

When $0 < \sigma < 4\pi^4$ we can take $M = \sigma + k(4\pi^4 - \dfrac{\sigma}{1+k})$. Then in the domain

$$D_M^+ = \{(x,y,u)|\ (x,y) \in \Omega, 0 \le u \le M/64\},$$

the conditions of Theorem 2 are satisfied with $L_1 = \sigma, L_2 = 0$. So, $q = \sigma/64$. Consequently, the problem has a unique positive solution and the iterative process (15)-(18) converges as a geometric

Table 1. The convergence of the iterative method for Example 1 on the uniform grid of 65×65 nodes

k	$\sigma = 1$				$\sigma = 5$			
	m	$error$	$m1$	$error1$	m	$error$	$m1$	$error1$
1	7	4.0205e-4	7	3.2280e-8	9	4.0350e-4	9	3.2397e-8
2	7	8.0375e-4	7	6.4533e-8	8	8.0527e-4	8	6.4655e-8
4	6	0.0016	6	1.2903e-7	7	0.0016	8	1.2914e-7
10	6	0.0040	6	3.2253e-7	6	0.0040	7	3.2260e-7
15	6	0.0060	6	4.8378e-7	6	0.0060	6	4.8383e-7
30	5	0.0121	5	9.6754e-7	6	0.0121	6	9.6757e-7

progression with ratio q if $0 < \sigma < 64$. Clearly, this convergence is independent of the constant k in the expression of $p(x, y)$. Remark that in [19] the range of σ for the convergence of Wang's iterative methods is not shown.

It should be noticed that the problem (29) is the problem (2) with $c = 0$ and the right-hand side $f(x, u) = \sigma u/(1 + u) + p(x),\ x \in \Omega$. Obviously, $f(x, u)$ does not satisfy any of the three hypotheses (H1)-(H3) in Introduction. Therefore, An and Liu [1] cannot guarantee the existence of nontrivial solution of the problem, meanwhile our Theorem 2 confirms that the problem has a unique positive solution.

We performed numerical experiments for the discrete iterative process (21)-(24) on various grids for different values of the constants k and σ. The results of the experiments are given in Table 1, where m is the number of iterations, $error = \|u - u_m^h\|$ computed by Method DIM2 (21)-(24); $m1$ is the number of iterations, $error1 = \|u - u_{m1}^h\|$ computed by Method DIM4 (25)-(28), u is the exact solution of the problem computed on grid points.

From Table 1 we see that the iterative method on discrete level (21)-(24) and (25)-(28) converges very fast, and the accuracy of the approximate solution compared with the exact solution on the grid decreases if the constant k increases and does not depend on σ. This can be explained that the magnitude of the exact solution increases with k, and this implies the decrease of the accuracy of difference approximation for the Laplacian.

In order to compare the convergence rate of our iterative method with the Wang's fastest method in [19] we show the graph of the errors $e(m)$ in Figure 1 and Figure 2. In these figures the fastest Wang's method means the Picard method, which is the fastest one among the three methods used: Picard method, Gauss-Seidel method and Jacobi method (see [19, 20]). From these figures it is seen that our method is much faster than Wang's fastest method.

Example 2. (Problem 2 in [20]) Consider the boundary value problem

$$\begin{aligned} \Delta^2 u = \sigma(x, y)\Delta u/(1 + u) + p(x, y), \quad (x, y) \in \Omega, \\ u = 0, \quad \Delta u = 0, \quad (x, y) \in \Gamma, \end{aligned} \tag{30}$$

where $k > 0$ is arbitrary, $\sigma(x, y)$ is a sign-changing continuous function and $p(x, y)$ is a nonnegative continuous function

$$p(x, y) = 2\pi^2 k \left(2\pi^2 + \frac{\sigma(x, y)}{1 + k \sin \pi x \sin \pi y}\right) \sin \pi x \sin \pi y.$$

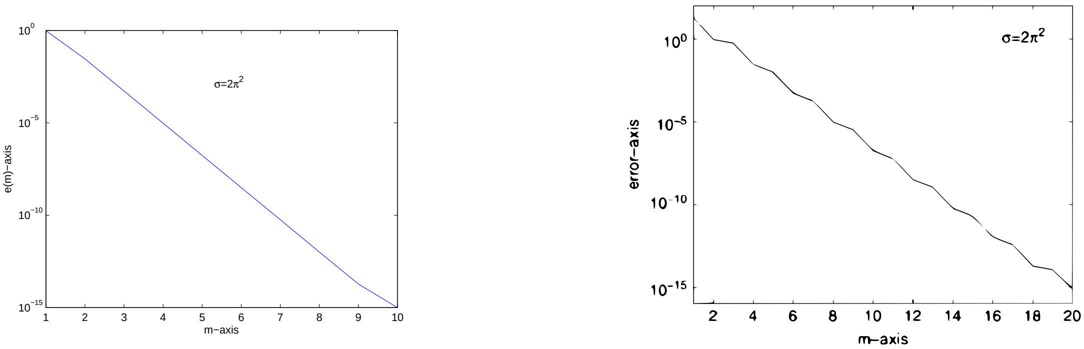

Figure 1. The errors e(m) of our Method DIM2 (left) vs. the Wang's fastest method (right) in Example 1 for $k = 1, \sigma = 2\pi^2$

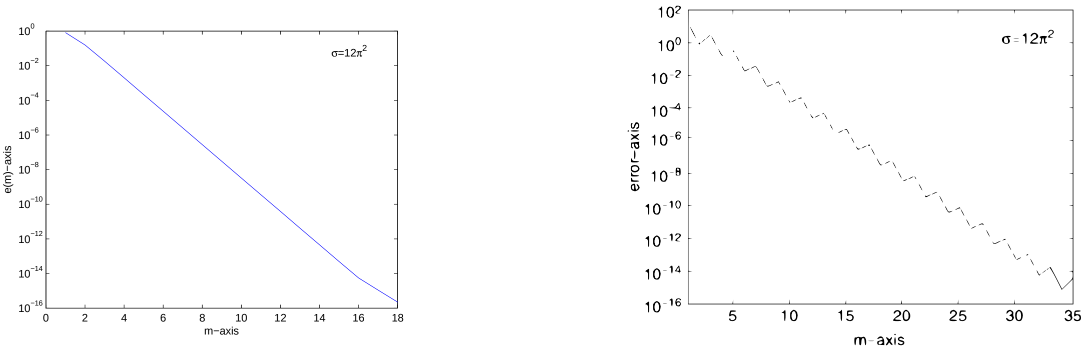

Figure 2. The errors e(m) of our Method DIM2 (left) vs. the Wang's fastest method (right) in Example 1 for $k = 1, \sigma = 12\pi^2$

For this problem the function $u(x, y) = k \sin \pi x \sin \pi y$ is the exact solution and

$$f(x, y, u, v) = \frac{\sigma(x, y)v}{1 + u} + p(x, y).$$

Let $\sigma(x, y) = \cos \pi x \cos \pi y$. Then for each fixed k it is possible to choose M, such that $0 \leq f(x, y, u, v) \leq M$ in the domain

$$D_M^+ = \{(x, y, u, v)|\ 0 \leq x, y \leq 1, 0 \leq u \leq \frac{M}{64}, -\frac{M}{8} \leq v \leq 0\}.$$

Namely, $\frac{16}{7}\pi^2 k(2\pi^2 + 1) \leq M \leq 16\pi^2 k(2\pi^2 - 1)$. Additionally, if $M < 447$ then by Theorem 2 the problem (30) has a unique positive solution and the iterative method (15)-(18) converges as the geometric progression with ratio $q = \frac{M}{512} + \frac{1}{8}$. This is guaranteed for $0 < k < 0.95$.

We performed numerical experiments for the discrete iterative method (21)-(24) on various grids for different values of k. The results of the experiments are presented in Table 2.

From Table 2 it is seen that the convergence of the iterative method (21)-(24) for Example 2 is fast and almost independent of the grid sizes and the values of k, even for large k. This phenomenon is not surprise because Theorem 3 gives only sufficient conditions for convergence. As in Example

Table 2. The convergence of the iterative method for Example 2

k	Grid 33×33				Grid 65×65			
	m	$error$	$m1$	$error1$	m	$error$	$m1$	$error1$
0.1	8	1.6078e-4	8	5.1580e-8	8	4.0169e-4	8	3.2252e-9
0.4	9	6.4313e-4	9	2.0632e-7	9	1.6068e-4	10	1.2901e-8
1	8	0.0016	9	5.1580e-7	9	4.0169e-4	9	3.2251e-8
2	9	0.0032	9	1.0316e-6	8	8.0337e-4	9	6.4502e-8
3	8	0.0048	9	1.5474e-6	8	0.0012	9	9.6753e-8

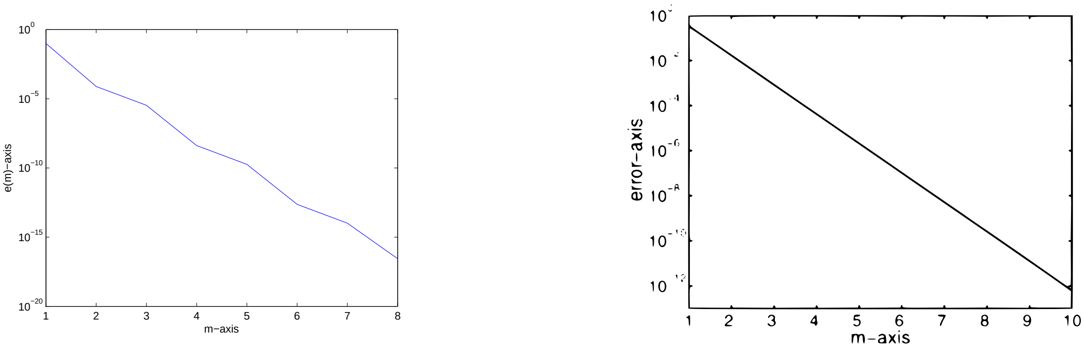

Figure 3. The errors e(m) of our Method DIM2 (left) vs. the Wang's fastest method (right) in Example 2 for $k = 0.1$.

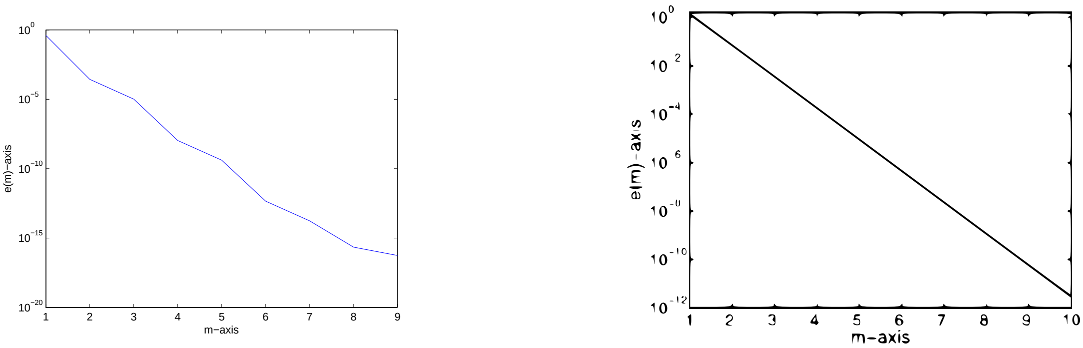

Figure 4. The errors e(m) of our Method DIM2 (left) vs. the Wang's fastest method (right) in Example 2 for $k = 0.4$.

1 the accuracy of the approximate solution compared with the exact solution on the grids decreases when the constant k increases.

In order to compare the convergence rate of our iterative method with the Wang's fastest method in [20] we present the graph of the errors $e(m)$ in Figure 3 and Figure 4. From this figure it is seen that our method is much faster than Wang's method.

Example 3. (The example in [17]) Consider the boundary value problem

$$\begin{aligned} \Delta^2 u &= f(x, y, u, \Delta u), \quad (x, y) \in \Omega, \\ u &= 0, \quad \Delta u = 0, \quad (x, y) \in \Gamma, \end{aligned}$$

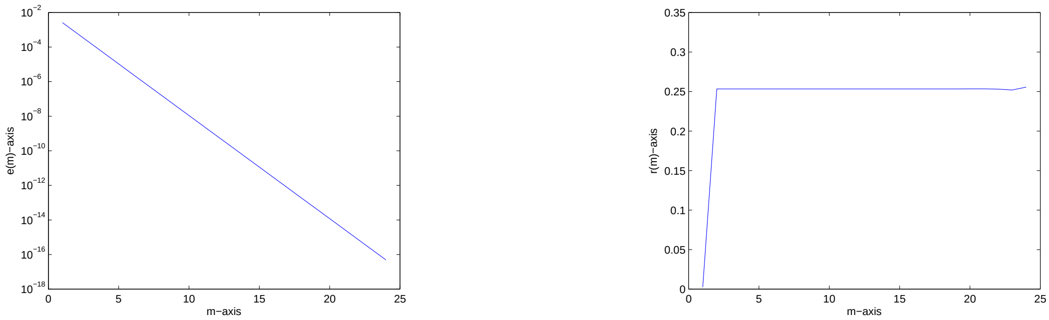

Figure 5. The errors $e(m)$ and the ratios $r(m)$ in Example 3

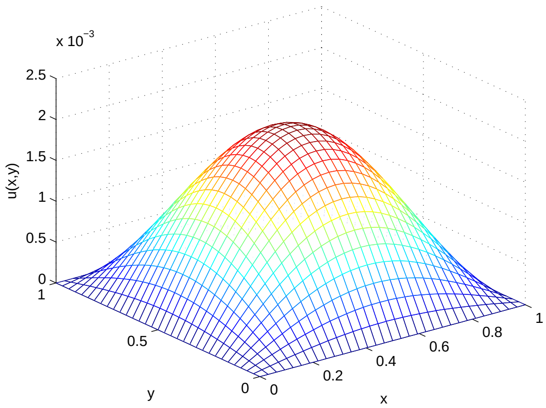

Figure 6. The approximate solution in Example 3

where

$$f(x, y, u, v) = 5v + \sigma(x, y)u^4 + \sin \pi x \sin \pi y.$$

In this example the exact solution is not known. If the function $\sigma(x, y)$ oscillates in the interval $[-1; 1]$ we can take $M = 3$, so that in the domain D_M all the conditions of Theorem 1 are satisfied with $L_1 = 0.0005,\ L_2 = 5,\ q = 0.6250$. Therefore, the problem has a unique solution, meanwhile, it is easy to verify that Hu and Wang [8] cannot guarantee the existence of nontrivial solution of the problem. Further, by Theorem 2 the iterative process (15)-(18) converges as the geometric progression with ratio q. The results of convergence for the case $\sigma(x, y) = \cos \pi x \cos \pi y$ are given in Figure 5, where $r(m) = e(m)/e(m-1)$ and the graph of the obtained approximate solution is depicted in Figure 6.

From Figure 5 we see that the actual ratio $r(m) \approx 0.25$, which is much less than the theoretical estimated ratio $q = 0.6250$ of the geometric progression.

Example 4. Consider the boundary value problem (1) with the exponential nonlinearity

$$f(x, y, u, \Delta u) = e^u.$$

For this example we take $M = 1.2$. It is easy to verify that in D_M^+ all the conditions of Theorem 2 are satisfied with $L_1 = 1.0189,\ L_2 = 0,\ q = 0.0159$. Therefore, the problem has a unique positive

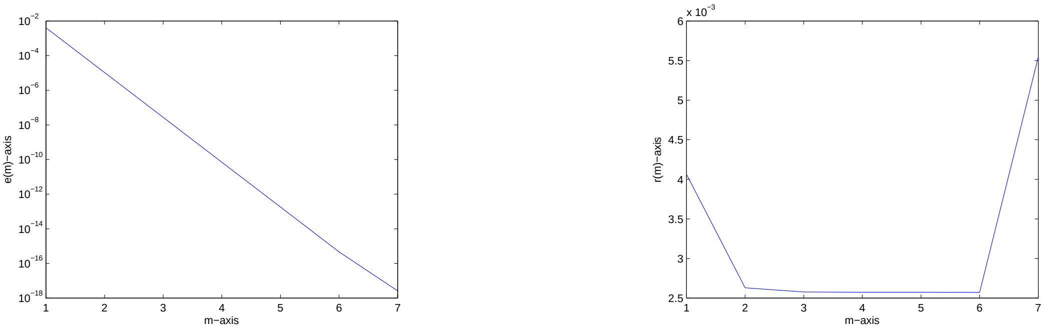

Figure 7. The errors $e(m)$ and the ratios $r(m)$ in Example 4

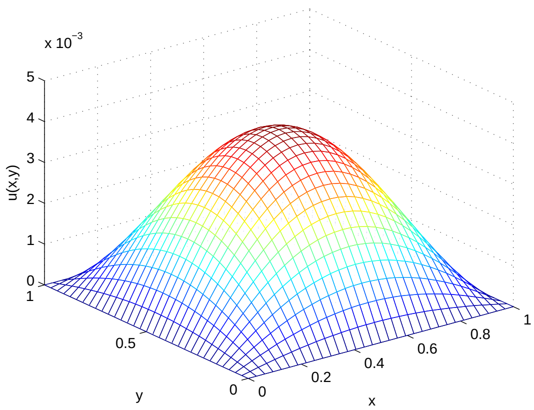

Figure 8. The approximate solution in Example 4

solution, meanwhile An and Liu [1] cannot guarantee the existence of nontrivial solution since it is easy to see that the hypotheses (H1)-(H3) are not satisfied. By Theorem 3 the iterative process (15)-(18) converges as the geometric progression with ratio q. The results of convergence are given in Figure 7 and the graph of the obtained approximate solution is depicted in Figure 8.

From Figure 7 we see that the actual ratio $r(m) \approx 0.0026$, which is much less than the theoretical estimated ratio $q = 0.0159$ of the geometric progression.

Notice that in the case if $f(x, y, u, \Delta u) = e^{\Delta u}$ taking $M = 1.5$ and using Theorem 2 we also conclude that the problem (1) has a unique positive solution.

5. CONCLUSIONS

In this paper, we have proposed a novel approach to a nonlinear biharmonic boundary value problem. The essence of it is the reduction of the problem to a nonlinear operator equation for the right-hand side function. Under some simple conditions we have proved that this operator is a contraction operator. As a result, the existence and uniqueness of a solution of the problem is established and the convergence of an iterative method is proved. Some properties of the solution and the iterations are also studied. The applicability of the theoretical results and the efficiency of the iterative method are demonstrated on some examples. The advantages of our method over the Wang's methods in convergence rate are shown on these examples.

In the future we shall develop the method to other boundary value problems for fourth order ordinary and partial differential equations including the problems with nonlinear and periodic boundary conditions.

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REFERENCES

[1] Y. An and R. Liu, "Existence of nontrivial solutions of an asymptotically linear fourth-order elliptic equation," *Nonlinear Analysis*, vol. 68, pp. 3325–3331, 2008.

[2] Z. Bai, W. Ge, and Y. Wang, "The method of lower and upper solutions for some fourth-order equations," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 5, 2004.

[3] Q. A. Dang, "Iterative method for solving the neumann boundary value problem for biharmonic type equation, journal of computational and applied mathematics," *Journal of Computational and Applied Mathematics*, vol. 96, pp. 634 –643, 2006.

[4] Q. A. Dang, Q. L. Dang, and T. K. Q. Ngo, "A novel efficient method for nonlinear boundary value problems," *Numerical Algorithms*, vol. 76, pp. 427–439, 2017.

[5] Q. A. Dang and T. K. Q. Ngo, "Existence results and iterative method for solving the cantilever beam equation with fully nonlinear term," *Nonlinear Analysis: Real World Applications*, vol. 36, pp. 56–68, 2017.

[6] ——, "New fixed point approach for a fully nonlinear fourth order boundary value problem," *Boletim da Sociedade Paranaense de Matematica*, vol. 36, no. 4, pp. 209–223, 2018.

[7] Q. R. Dunninger, "Maximum principles for solutions of some fourth-order elliptic equations," *Journal of Mathematical Analysis and Applications*, vol. 37, pp. 655–658, 1972.

[8] S. Hu and L. Wang, "Existence of nontrivial solutions for fourth-order asymptically linear elliptic equations," *Nonlinear Analysis*, vol. 94, pp. 120–132, 2014.

[9] Y. Li, "A monotone iterative technique for solving the bending elastic beam equations," *Applied Mathematics and Computation*, vol. 217, pp. 2200–2208, 2010.

[10] X. Liu and Y. Huang, "On sign-changing solution for a fourth-order asymptotically linear elliptic problem," *Nonlinear Analysis*, vol. 72, pp. 2271–2276, 2010.

[11] Y. Liu and Z. P. Wang, "Biharmonic equations with asymptotically linear nonlinearities," *Acta Mathematica Scientia*, vol. 27B, no. 3, pp. 549–560, 2007.

[12] C. V. Pao, "On fourth-order elliptic boundary value problems," *Proceedings of the American Mathematical Society*, vol. 128, pp. 1023–1030, 2000.

[13] ——, "Numerical methods for fourth order nonlinear elliptic boundary value problems," *Numerical Methods for Partial Differential Equations*, vol. 17, pp. 347–368, 2001.

[14] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations.* Springer, 1984.

[15] A. Samarskii, *The theory of difference schemes.* Marcel Dekker, Inc., 2001.

[16] A. Samarskii and E. Nikolaev, *Numerical methods for grid equations.* Birkhäuser, Basel, 1989.

[17] Y. M. Wang, "On fourth-order elliptic boundary value problems with nonmonotone nonlinear function," *Journal of Mathematical Analysis and Applications*, vol. 307, pp. 1–11, 2005.

[18] ——, "Convergence analysis of a monotone method for fourth-order semilinear elliptic boundary value problems," *Applied Mathematics Letters*, vol. 19, pp. 332–339, 2006.

[19] ——, "Error and stability of monotone method for numerical solutions of fourth-order semilinear elliptic boundary value problems," *Journal of Computational and Applied Mathematics*, vol. 200, pp. 503–519, 2007.

[20] ——, "Monotone iterative technique for numerical solutions of fourth-order nonlinear elliptic boundary value problems," *Applied Numerical Mathematics*, vol. 57, pp. 1081–1096, 2007.

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