Journal of Computer Science and Cybernetics, Vol.22, No.3 (2006), 235–243

FINITE-DIMENSIONAL APPROXIMATION FOR ILL-POSED VECTOR OPTIMIZATION OF CONVEX FUNCTIONALS IN BANACH SPACES

NGUYEN THI THU THUY[1], NGUYEN BUONG[2]

[1] *Faculty of Sciences, Thai Nguyen University*
[2] *Institute of Information Technology*

Abstract. In this paper we present the convergence and convergence rate for regularization solutions in connection with the finite-dimensional approximation for ill-posed vector optimization of convex functionals in reflexive Banach space. Convergence rates of its regularized solutions are obtained on the base of choosing the regularization parameter a priory as well as a posteriori by the modified generalized discrepancy principle. Finally, an application of these results for convex optimization problem with inequality constraints is shown.

Tóm tắt. Trong bài báo này chúng tôi trình bày sự hội tụ và tốc độ hội tụ của nghiệm hiệu chỉnh trong xấp xỉ hữu hạn chiều cho bài toán cực trị đa mục tiêu các phiếm hàm lồi trong không gian Banach phản xạ. Tốc độ hội tụ của nghiệm hiệu chỉnh nhận được dựa trên việc chọn tham số hiệu chỉnh trước hoặc sau bằng nguyên lý độ lệch suy rộng ở dạng cải biên. Cuối cùng là một ứng dụng của các kết quả đạt được cho bài toán cực trị lồi với ràng buộc bất đẳng thức.

1. INTRODUCTION

Let X be a real reflexive Banach space preserved a property that X and X^* are strictly convex, and weak convergence and convergence of norms of any sequence in X imply its strong convergence, where X^* denotes the dual space of X. For the sake of simplicity, the norms of X and X^* are denoted by the symbol $\|.\|$. The symbol $\langle x^*, x\rangle$ denotes the value of the linear continuous functional $x^* \in X^*$ at the point $x \in X$. Let $\varphi_j(x)$, $j = 0, 1, ..., N$, be the weakly lower semicontinuous proper convex functionals on X that are assumed to be Gâteaux differentiable with the hemicontinuous derivatives $A_j(x)$ at $x \in X$.

In [6], one of the authors has considered a problem of vector optimization: find an element $u \in X$ such that

$$\varphi_j(u) = \inf_{x\in X} \varphi_j(x), \quad \forall j = 0, 1, ..., N. \tag{1.1}$$

Set

$$Q_j = \big\{\hat{x} \in X : \varphi_j(\hat{x}) = \inf_{x\in X} \varphi_j(x)\big\},\ \ j = 0, 1, ..., N, \quad Q = \bigcap_{j=0}^{N} Q_j.$$

It is well know that Q_j coincides with the set of solutions of the following operator equation

$$A_j(x) = \theta, \tag{1.2}$$

and is a closed convex subset in X (see [11]). We suppose that $Q \neq \emptyset$, and $\theta \notin Q$, where θ is the zero element of X (or X^*).

In [6] it is showed the existence and uniqueness of the solution x_α^h of the operator equation

$$\sum_{j=0}^{N} \alpha^{\lambda_j} A_j^h(x) + \alpha U(x) = \theta, \tag{1.3}$$
$$\lambda_0 = 0 < \lambda_j < \lambda_{j+1} < 1, \;\; j = 1, 2, ..., N-1,$$

where $\alpha > 0$ is the small parameter of regularization, U is the normalized duality mapping of X, i.e., $U : X \to X^*$ satisfies the condition

$$\langle U(x), x\rangle = \|x\|^2, \quad \|U(x)\| = \|x\|,$$

A_j^h are the hemicontinuous monotone approximations for A_j in the forms

$$\|A_j(x) - A_j^h(x)\| \leqslant hg(\|x\|), \;\; \forall x \in X, \tag{1.4}$$

with level $h \to 0$, and $g(t)$ is a bounded (the image of the bounded set is bounded) nonnegative function, $t \geqslant 0$.

Clairly, the convergence and convergence rates of the sequence x_α^h to u depend on the choice of $\alpha = \alpha(h)$. In [6], one has showed that the parameter α can be chosen by the modified generalized discrepancy principle, i.e., $\alpha = \alpha(h)$ is constructed on the basis of the following equation

$$\rho(\alpha) = h^p \alpha^{-q}, \;\; p, q > 0, \tag{1.5}$$

where $\rho(\alpha) = \alpha(a_0 + t(\alpha))$, the function $t(\alpha) = \|x_\alpha^h\|$ depends continuously on $\alpha \geqslant \alpha_0 > 0$, a_0 is some positive constant.

In computation the finite-demensional approximation for (1.3) is the important problem. As usualy, it can be aproximated by the following equation

$$\sum_{j=0}^{N} \alpha^{\lambda_j} A_j^{hn}(x) + \alpha U^n(x) = \theta, \;\; x \in X_n, \tag{1.6}$$

where $A_j^{hn} = P_n^* A_j^h P_n, \; U^n = P_n^* U P_n$ and $P_n : X \longrightarrow X_n$ the linear projection from X onto X_n, X_n is the finite-dimensional subspace of X, P_n^* is the conjugate of P_n,

$$X_n \subset X_{n+1}, \;\; \forall n, \quad P_n x \longrightarrow x, \;\; \forall x \in X.$$

Without loss of generality, suppose that $\|P_n\| = 1$ (see [11]).

As for (1.3), equation (1.6) has also an unique solution $x_{\alpha,n}^h$, and for every fixed $\alpha > 0$ the sequence $\{x_{\alpha,n}^h\}$ converges to x_α^h, the solution of (1.3), as $n \to \infty$ (see [11]).

The natural problem is to ask whether the sequence $\{x_{\alpha,n}^h\}$ converges to u as $\alpha, h \to 0$ and $n \to \infty$, and how fast it converges, where u is an element in Q. The purpose of this paper is to answer these questions.

We assume, in addition, that U satisfies the condition

$$\langle U(x) - U(y), x - y\rangle \geqslant m_U \|x - y\|^s, \;\; m_U > 0, \;\; s \geqslant 2, \;\; \forall x, y \in X. \tag{1.7}$$

Set

$$\gamma_n(x) = \|(I - P_n)x\|, \;\; x \in Q,$$

where I denotes the identity operator in X.

Hereafter the symbols $\rightharpoonup$ and $\to$ indicate weak convergence and convergence in norm, respectively, while the notation $a \sim b$ is meant $a = O(b)$ and $b = O(a)$.

2. MAIN RESULT

The convergence of $\{x^h_{\alpha,n}\}$ to u is determined by the following theorem.

Theorem 1. *If h/α and $\gamma_n(x)/\alpha \to 0$, as $\alpha \to 0$ and $n \to \infty$, then the sequence $x^h_{\alpha,n}$ converges to u.*

Proof. For $x \in Q, x^n = P_n x$, it follows from (1.6) that

$$\sum_{j=0}^{N} \alpha^{\lambda_j} \langle A^{hn}_j(x^h_{\alpha,n}), x^h_{\alpha,n} - x^n\rangle + \alpha\langle U^n(x^h_{\alpha,n}) - U^n(x^n), x^h_{\alpha,n} - x^n\rangle = \alpha\langle U^n(x^n), x^n - x^h_{\alpha,n}\rangle.$$

Therefore, on the basis of (1.2), (1.7) and the monotonicity of $A^{hn}_j = P^*_n A^h_j P_n$, and $P_n P_n = P_n$ we have

$$\begin{aligned}
\alpha m_U \|x^h_{\alpha,n} - x^n\|^s &\leqslant \alpha\langle U(x^h_{\alpha,n}) - U(x^n), x^h_{\alpha,n} - x^n\rangle = \alpha\langle U^n(x^h_{\alpha,n}) - U^n(x^n), x^h_{\alpha,n} - x^n\rangle \\
&= \sum_{j=0}^{N} \alpha^{\lambda_j} \langle A^{hn}_j(x^h_{\alpha,n}), x^n - x^h_{\alpha,n}\rangle + \alpha\langle U^n(x^n), x^n - x^h_{\alpha,n}\rangle \\
&\leqslant \sum_{j=0}^{N} \alpha^{\lambda_j} \langle A^{hn}_j(x^n), x^n - x^h_{\alpha,n}\rangle + \alpha\langle U^n(x^n), x^n - x^h_{\alpha,n}\rangle \\
&= \sum_{j=0}^{N} \alpha^{\lambda_j} \langle A^h_j(x^n) - A_j(x^n) + A_j(x^n) - A_j(x), x^n - x^h_{\alpha,n}\rangle + \alpha\langle U(x^n), x^n - x^h_{\alpha,n}\rangle. \qquad (2.1)
\end{aligned}$$

On the other hand, by using (1.4) and

$$\|A_j(x^n) - A_j(x)\| \leqslant K\gamma_n(x),$$

where K is some positive constant depending only on x, it follows from (2.1) that

$$m_U \|x^h_{\alpha,n} - x^n\|^s \leqslant \frac{1}{\alpha}\Big[(N+1)\big(hg(\|x^n\|) + K\gamma_n(x)\big)\Big]\|x^n - x^h_{\alpha,n}\| + \langle U(x^n), x^n - x^h_{\alpha,n}\rangle. \quad (2.2)$$

Because of h/α, $\gamma_n(x)/\alpha \to 0$ as $\alpha \to 0$, $n \to \infty$ and $s \geqslant 2$, this inequality gives us the boundedness of the sequence $\{x^h_{\alpha,n}\}$. Then, there exists a subsequence of the sequence $\{x^h_{\alpha,n}\}$ converging weakly to $\hat{x}$ in X. Without loss of generality, we assume that $x^h_{\alpha,n} \rightharpoonup \hat{x}$ as $h, h/\alpha \to 0$ and $n \to \infty$. First, we prove that $\hat{x} \in Q_0$. Indeed, by virtue of the monotonicity of $A^{hn}_j = P^*_n A^h_j P_n$, $U^n = P^*_n U P_n$ and (1.6) we have

$$\begin{aligned}
&\langle A^{hn}_0(P_n x), P_n x - x^h_{\alpha,n}\rangle \geqslant \langle A^{hn}_0(x^h_{\alpha,n}), P_n x - x^h_{\alpha,n}\rangle \\
&= \sum_{j=1}^{N} \alpha^{\lambda_j} \langle A^{hn}_j(x^h_{\alpha,n}), x^h_{\alpha,n} - P_n x\rangle + \alpha\langle U^n(x^h_{\alpha,n}), x^h_{\alpha,n} - P_n x\rangle \\
&\geqslant \sum_{j=1}^{N} \alpha^{\lambda_j} \langle A^{hn}_j(P_n x), x^h_{\alpha,n} - P_n x\rangle + \alpha\langle U^n(P_n x), x^h_{\alpha,n} - P_n x\rangle, \ \forall x \in X.
\end{aligned}$$

Because of $P_nP_n = P_n$, so the last inequality has form

$$\langle A_0^h(P_nx), P_nx - x_{\alpha,n}^h\rangle \geqslant \sum_{j=1}^{N} \alpha^{\lambda_j}\langle A_j^h(P_nx), x_{\alpha,n}^h - P_nx\rangle + \alpha\langle U(P_nx), x_{\alpha,n}^h - P_nx\rangle, \ \ \forall x \in X.$$

By letting $h, \alpha \to 0$ and $n \to \infty$ in this inequality we obtain

$$\langle A_0(x), x - \hat{x}\rangle \geqslant 0, \quad \forall x \in X.$$

Consequently, $\hat{x} \in Q_0$ (see [11]). Now, we shall prove that $\hat{x} \in Q_j$, $j = 1, 2, ..., N$. Indeed, by (1.6) and making use of the monotonicity of A_j^{hn} and U^n, it follows that

$$\begin{aligned}
\alpha^{\lambda_1}\langle A_1^{hn}(x_{\alpha,n}^h), &x_{\alpha,n}^h - P_nx\rangle + \sum_{j=2}^{N} \alpha^{\lambda_j}\langle A_j^{hn}(x_{\alpha,n}^h), x_{\alpha,n}^h - P_nx\rangle + \alpha\langle U^n(x_{\alpha,n}^h), x_{\alpha,n}^h - P_nx\rangle \\
&= \alpha^{\lambda_0}\langle A_0^{hn}(x_{\alpha,n}^h), P_nx - x_{\alpha,n}^h\rangle \leqslant \langle A_0^{hn}(P_nx), P_nx - x_{\alpha,n}^h\rangle \\
&= \langle A_0^h(P_nx) - A_0(P_nx) + A_0(P_nx) - A_0(x), P_nx - x_{\alpha,n}^h\rangle, \ \ \forall x \in Q_0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle A_1^h(P_nx), x_{\alpha,n}^h - P_nx\rangle &+ \sum_{j=2}^{N} \alpha^{\lambda_j-\lambda_1}\langle A_j^h(P_nx), x_{\alpha,n}^h - P_nx\rangle + \alpha^{1-\lambda_1}\langle U(P_nx), x_{\alpha,n}^h - P_nx\rangle \\
&\leqslant \frac{1}{\alpha}\Big[h\alpha^{1-\lambda_1}g(\|P_nx\|) + K\gamma_n(x)\Big]\|P_nx - x_{\alpha,n}^h\|, \ \ \forall x \in Q_0.
\end{aligned}$$

After passing $h, \alpha \to 0$ and $n \to \infty$, we obtain

$$\langle A_1(x), \hat{x} - x\rangle \leqslant 0, \ \ \forall x \in Q_0.$$

Thus, $\hat{x}$ is a local minimizer for φ_1 on S_0 (see [9]). Since $S_0 \cap S_1 \neq \emptyset$, then $\hat{x}$ is also a global minimizer for φ_1, i.e., $\hat{x} \in S_1$.

Set $\tilde{Q}_i = \cap_{k=0}^{i} Q_k$. Then, $\tilde{Q}_i$ is also closed convex, and $\tilde{Q}_i \neq \emptyset$.

Now, suppose that we have proved $\hat{x} \in \tilde{Q}_i$ and we need to show that $\hat{x}$ belongs to Q_{i+1}. Again, by virtue of (1.6) for $x \in \tilde{Q}_i$, we can write

$$\begin{aligned}
&\langle A_{i+1}^{hn}(x_{\alpha,n}^h), x_{\alpha,n}^h - P_nx\rangle + \sum_{j=i+2}^{N} \alpha^{\lambda_j-\lambda_{i+1}}\langle A_j^{hn}(x_{\alpha,n}^h), x_{\alpha,n}^h - P_nx\rangle \\
&\qquad + \alpha^{1-\lambda_{i+1}}\langle U^n(x_{\alpha,n}^h), x_{\alpha,n}^h - P_nx\rangle = \sum_{k=0}^{i} \alpha^{\lambda_k-\lambda_{i+1}}\langle A_k^{hn}(x_{\alpha,n}^h), P_nx - x_{\alpha,n}^h\rangle \\
&\leqslant \frac{1}{\alpha}\sum_{k=0}^{i} \alpha^{\lambda_k+1-\lambda_{i+1}}\langle A_k^h(P_nx) - A_k(P_nx) + A_k(P_nx) - A_k(x), P_nx - x_{\alpha,n}^h\rangle \\
&\leqslant \frac{1}{\alpha}(i+1)\big(hg(\|P_nx\|) + K\gamma_n(x)\big)\|P_n(x) - x_{\alpha,n}^h\|.
\end{aligned}$$

Therefore,

$$\langle A_{i+1}^h(P_nx), x_{\alpha,n}^h - P_nx\rangle + \sum_{j=i+2}^{N} \alpha^{\lambda_j-\lambda_{i+1}}\langle A_j^h(P_nx), x_{\alpha,n}^h - P_nx\rangle$$
$$+\alpha^{1-\lambda_{i+1}}\langle U(P_nx), x_{\alpha,n}^h - P_nx\rangle \leqslant \frac{hg(\|P_nx\|) + K\gamma_n(x)}{\alpha}(N+1)\|P_nx - x_{\alpha,n}^h\|.$$

By letting $h, \alpha \to 0$ and $n \to \infty$, we have

$$\langle A_{i+1}(x), \hat{x} - x\rangle \leqslant 0,\ \ \forall x \in \tilde{Q}_i.$$

As a result, $\hat{x} \in Q_{i+1}$.

On the other hand, it follows from (2.2) that

$$\langle U(x), x - \hat{x}\rangle \geqslant 0,\ \ \forall x \in Q.$$

Since Q_j is closed convex, Q is also closed convex. Replacing x by $t\hat{x} + (1-t)x$, $t \in (0,1)$ in the last inequality, and dividing by $(1-t)$ and letting t to 1, we obtain

$$\langle U(\hat{x}), x - \hat{x}\rangle \geqslant 0,\ \ \forall x \in Q.$$

Hence $\|\hat{x}\| \leqslant \|x\|$, $\forall x \in Q$. Because of the convexity and the closedness of Q, and the strictly convexity of X we deduce that $\hat{x} = u$. So, all sequence $\{x_{\alpha,n}^h\}$ converges weakly to u. Set $x^n = u^n = P_nu$ in (2.2) we deduce that the sequence $\{x_{\alpha,n}^h\}$ converges strongly to u as $h \to 0$ and $n \to \infty$. The proof is complete. ■

In the following, we consider the finite-dimensional variant of the generalized discrepancy principle for the choice $\tilde{\alpha} = \alpha(h,n)$ so that $x_{\tilde{\alpha},n}^h$ converges to u, as $h, \alpha \to 0$ and $n \to \infty$.

Note that, the generalized discrepancy principle for parameter choice is presented first in [8] for the linear ill-posed problems. For the nonlinear ill-posed equation involving a monotone operator in Banach space the use of a discrepancy principle to estimate the rate of convergence of the regularized solutions was considered in [5]. In [4] the convergence rates of regularized solutions of ill-posed variational inequalities under arbitrary perturbative operators were investigated when the regularization parameter was chosen arbitrarily such that $\alpha \sim (\delta + \varepsilon)^p$, $0 < p < 1$. In this paper, we consider the modified generalized discrepancy principle for selecting $\tilde{\alpha}$ in connection with the finite-dimensional and obtain the rates of convergence for the regularized solutions in this case.

The parameter $\alpha(h,n)$ can be chosen by

$$\alpha(a_0 + \|x_{\alpha,n}^h\|) = h^p\alpha^{-q},\ \ p, q > 0 \tag{2.3}$$

for each $h > 0$ and n. It is not difficult to verify that $\rho_n(\alpha) = \alpha(a_0 + \|x_{\alpha,n}^h\|)$ possesses all properties as well as $\rho(\alpha)$ does, and

$$\lim_{\alpha\to+\infty} \alpha^q\rho_n(\alpha) = +\infty, \qquad \lim_{\alpha\to+0} \alpha^q\rho_n(\alpha) = 0.$$

To find α by (2.3) is very complex. So, we consider the following rule.

The rule. Choose $\tilde{\alpha} = \alpha(h,n) \geqslant \alpha_0 := (c_1h + c_2\gamma_n)^p$, $c_i > 1, i = 1,2,$ $0 < p < 1$ such that the following inequalities

$$\tilde{\alpha}^{1+q}(a_0 + \|x_{\tilde{\alpha},n}^h\|) \geqslant d_1h^p,$$
$$\tilde{\alpha}^{1+q}(a_0 + \|x_{\tilde{\alpha},n}^h\|) \leqslant d_2h^p,\ \ d_2 \geqslant d_1 > 1,$$

hold.

In addition, assume that U satisfies the following condition

$$\|U(x) - U(y)\| \leqslant C(R)\|x - y\|^{\nu}, \ \ 0 < \nu \leqslant 1, \tag{2.4}$$

where $C(R), R > 0$, is a positive increasing function on $R = max\{\|x\|, \|y\|\}$ (see [10]).

Set

$$\gamma_n = \max_{x \in Q}\{\gamma_n(x)\}.$$

Lemma 1.

$$\lim_{\substack{h\to 0\\ n\to\infty}} \alpha(h, n) = 0.$$

Proof. Obviously, it follows from the rule that

$$\begin{aligned}\alpha(h,n) &\leqslant d_2^{1/(1+q)}\big(a_0 + \|x^h_{\alpha(h,n),n}\|\big)^{-1/(1+q)} h^{p/(1+q)}\\ &\leqslant d_2^{1/(q+1)} a_0^{-1/(1+q)} h^{p/(1+q)}.\end{aligned}$$

■

Lemma 2. *If* $0 < p < 1$ *then*

$$\lim_{\substack{h\to 0\\ n\to\infty}} \frac{h + \gamma_n}{\alpha(h,n)} = 0.$$

Proof. Obviously using the rule we get

$$\frac{h+\gamma_n}{\alpha(h,n)} \leqslant \frac{c_1 h + c_2\gamma_n}{(c_1 h + c_2 \gamma_n)^p} = (c_1 h + c_2\gamma_n)^{1-p} \to 0$$

as $h \to 0$ and $n \to \infty$. ■

Now, let $x^h_{\tilde{\alpha},n}$ be the solution of (1.6) with $\alpha = \tilde{\alpha}$. By the argument in the proof of Theorem 1, we obtain the following result.

Theorem 2. *The sequence* $x^h_{\tilde{\alpha},n}$ *converges to* u *as* $h \to 0$ *and* $n \to \infty$.

The next theorem shows the convergence rates of $\{x^h_{\tilde{\alpha},n}\}$ to u as $h \to 0$ and $n \to \infty$.

Theorem 3. *Assume that the following conditions hold:*

(i) A_0 *is continuously Frchet differentiable, and satifies the condition*

$$\|A_0(x) - A_0'(u)(x-u)\| \leqslant \tau\|A_0(x)\|, \ \forall u \in Q,$$

where τ *is a positive constant, and* x *belongs to some neighbourhood of* Q*;*

(ii) $A_h(X_n)$ *are contained in* X_n^* *for sufficiently large* n *and small* h*;*

(iii) there exists an element $z \in X$ *such that* $A_0'(u)^* z = U(u)$*;*

(vi) the parameter $\tilde{\alpha} = \alpha(h,n)$ *is chosen by the rule.*

Then, we have

$$\|x^h_{\tilde{\alpha},n} - u\| = O\big((h+\gamma_n)^{\eta_1} + \gamma_n^{\eta_2}\big),$$

$$\eta_1 = \min\left\{\frac{1-p}{s-1}, \frac{\mu_1 p}{s(1+q)}\right\}, \quad \eta_2 = \min\left\{\frac{1}{s}, \frac{\nu}{s-1}\right\}.$$

Proof. Replacing x^n by $u^n = P_n u$ in (2.2) we obtain

$$m_U \|x^h_{\tilde\alpha,n} - u^n\|^s \leqslant \frac{1}{\tilde\alpha}\Bigg[(N+1)hg(\|u^n\|) + K\gamma_n\Bigg]\|u^n - x^h_{\tilde\alpha,n}\| \\ +\langle U(u^n) + U(u) - U(u), u^n - x^h_{\tilde\alpha,n}\rangle. \tag{2.5}$$

By (2.4) it follows that

$$\langle U(u^n) - U(u), u^n - x^h_{\tilde\alpha,n}\rangle \leqslant C(\tilde R)\|u^n - u\|^\nu \|u^n - x^h_{\tilde\alpha,n}\| \leqslant C(\tilde R)\gamma_n^\nu \|u^n - x^h_{\tilde\alpha,n}\|, \tag{2.6}$$

where $\tilde R > \|u\|$.

On the other hand, using conditions $(i), (ii), (iii)$ of the theorem we can write

$$\langle U(u), u^n - x^h_{\tilde\alpha,n}\rangle = \langle U(u), u^n - u\rangle + \langle z, A_0'(u)(u - x^h_{\tilde\alpha,n})\rangle \leqslant \tilde R\gamma_n + \|z\|(\tau+1)\|A_0(x^h_{\tilde\alpha,n})\|$$

$$\leqslant \tilde R\gamma_n + \|z\|(\tau+1)\Big[hg(\|x^h_{\tilde\alpha,n}\|) + \|A_0^h(x^h_{\tilde\alpha,n})\|\Big]$$

$$\leqslant \tilde R\gamma_n + \|z\|(\tau+1)\Bigg[\sum_{j=1}^N \tilde\alpha^{\lambda_j}\|A_j^h(x^h_{\tilde\alpha,n})\| + \tilde\alpha\|x^h_{\tilde\alpha,n}\| + hg(\|x^h_{\tilde\alpha,n}\|)\Bigg]. \tag{2.7}$$

Combining (2.6) and (2.7) inequality (2.5) has form

$$m_U\|x^h_{\tilde\alpha,n} - u^n\|^s \leqslant \frac{1}{\tilde\alpha}\Big[(N+1)hg(\|u^n\|) + K\gamma_n\Big]\|u^n - x^h_{\tilde\alpha,n}\| + C(\tilde R)\gamma_n^\nu\|u^n - x^h_{\tilde\alpha,n}\|$$

$$+\tilde R\gamma_n + \|z\|(\tau+1)\Bigg[\sum_{j=1}^N \tilde\alpha^{\lambda_j}\|A_j^h(x^h_{\tilde\alpha,n})\| + \tilde\alpha\|x^h_{\tilde\alpha,n}\| + hg(\|x^h_{\tilde\alpha,n}\|)\Bigg]. \tag{2.8}$$

On the other hand, making use of the rule and the boundedness of $\{x^h_{\tilde\alpha,n}\}$ it implies that

$$\tilde\alpha = \alpha(h,n) \geqslant (c_1 h + c_2\gamma_n)^p,$$
$$\tilde\alpha = \alpha(h,n) \leqslant C_1 h^{p/(1+q)},\ C_1 > 0,$$
$$\tilde\alpha = \alpha(h,n) \leqslant 1,$$

for sufficiently small h and large n.

Consequently, in view of (2.8) it follows that

$$m_U\|x^h_{\tilde\alpha,n} - u^n\| \leqslant \Bigg[\frac{(N+1)hg(\|u^n\|) + K\gamma_n}{(c_1h + c_2\gamma_n)^p} + C(\tilde R)\gamma_n^\nu\Bigg]\|u^n - x^h_{\tilde\alpha,n}\| \\ + \tilde R\gamma_n + C_2(h+\gamma_n)^{\lambda_1 p/(1+q)} \\ \leqslant \tilde C_1\big[(h+\gamma_n)^{1-p} + \gamma_n^\nu\big]\|u^n - x^h_{\tilde\alpha,n}\| + \tilde C_2\gamma_n + \tilde C_3(h+\gamma_n)^{\lambda_1 p/(1+q)},$$

C_2 and $\tilde C_i$, $i = 1, 2, 3$ are the positive constants.

Using the implication

$$a, b, c \geqslant 0, \quad p_1 > q_1, \quad a^{p_1} \leqslant b a^{q_1} + c \Rightarrow a^{p_1} = O\big(b^{p_1/(p_1-q_1)} + c\big)$$

we obtain

$$\|x^h_{\tilde{\alpha},n} - u^n\| = O\big((h+\gamma_n)^{\eta_1} + \gamma_n^{\eta_2}\big).$$

Thus,

$$\|x^h_{\tilde{\alpha},n} - u\| = O\big((h+\gamma_n)^{\eta_1} + \gamma_n^{\eta_2}\big),$$

which completes the proof. ■

Remarks. If $\tilde{\alpha} = \alpha(h,n)$ is chosen a priori such that $\tilde{\alpha} \sim (h+\gamma_n)^{\eta}$, $0 < \eta < 1$, then inequality (2.8) has the form

$$m_U\|x^h_{\tilde{\alpha},n} - u^n\| \leqslant \overline{C_1}\Bigg[(h+\gamma_n)^{1-\eta} + \gamma_n^{\nu}\Bigg]\|u^n - x^h_{\tilde{\alpha},n}\| + \overline{C_2}\gamma_n + \overline{C_3}(h+\gamma_n)^{\lambda_1\eta},$$

where $\overline{C_i}, i = 1, 2, 3$ are the positive constants.
Therefore,

$$\|x^h_{\tilde{\alpha},n} - u^n\| = O\big((h+\gamma_n)^{\theta_1} + \gamma_n^{\theta_2}\big),$$

whence,

$$\|x^h_{\tilde{\alpha},n} - u\| = O\big((h+\gamma_n)^{\theta_1} + \gamma_n^{\theta_2}\big),$$

$$\theta_1 = \min\left\{\frac{1-\eta}{s-1}, \frac{\lambda_1\eta}{s}\right\}, \quad \theta_2 = \min\left\{\frac{1}{s}, \frac{\nu}{s-1}\right\}.$$

3. AN APPLICATION

In this section we consider a constrained optimization problem:

$$\inf_{x\in X} f_N(x) \tag{3.1}$$

subject to

$$f_j(x) \leqslant 0, \ \ j = 0, ..., N-1, \tag{3.2}$$

where $f_0, f_1, ..., f_N$ are weakly lower semicontinuous and properly convex functionals on X that are assumed to be *Gteaux* differentiable at $x \in X$.

Set

$$Q_j = \{x \in X : f_j(x) \leqslant 0\}, \ \ j = 0, ..., N-1. \tag{3.3}$$

Obviously, Q_j is the closed convex subset of X, $j = 0, ..., N-1$.
Define

$$\varphi_N(x) = f_N(x), \quad \varphi_j(x) = \max\{0, f_j(x)\}, \ \ j = 0, ..., N-1. \tag{3.4}$$

Evidently, φ_j are also convex functionals on X and

$$Q_j = \{\bar{x} \in X : \varphi_j(\bar{x}) = \inf_{x\in X} \varphi_j(x)\}, \ \ 0, 1, ..., N.$$

So, $\bar{x}$ is a solution of the problem:

$$\varphi_j(\bar{x}) = \inf_{x\in X} \varphi_j(x), \ \ \forall j = 0, 1, ..., N.$$

REFERENCES

[1] Ya. I. Alber, On solving nonlinear equations involving monotone operators in banach spaces, *Sib. Mat. Zh.* **26** (1975) 3–11.

[2] Ya.I. Alber and I. P. Ryazantseva, On solutions of nonlinear problems involving monotone discontinuous operators, *Differ. Uravn.* **25** (1979) 331–342.

[3] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff Int. Publ. Leyden (Ed. Acad. Bucuresti, Romania, Netherlands) 1976.

[4] Ng. Buong, Convergence rates and finite-dimensional approximation for a class of ill-posed variational inequalities, *Ukrainian Math. J.* **49** (1997) 629–637.

[5] Ng. Buong, On a monotone ill-posed problem, *Acta Mathematica Sinica, English Series* **21** (2005) 1001–1004.

[6] Ng. Buong, Regularization for unconstrained vector optimization of convex functionals in Banach spaces, *Comp. Mat. and Mat. Phy.* **46** (2006) 354–360.

[7] I. Ekeland and R. Temam, *Convex analysis and variational problems*, North-Holland Publ. Company, Amsterdam, Holland, 1970.

[8] H. W. Engl, Discrepancy principle for tikhonov regularization of ill-posed problems leading to optimal convergence rates, *J. of Optimization Theory and Appl.* **52** (1987) 209–215.

[9] I. P. Ryazantseva, Operator method of ragularization for problems of optimal programming with monotone maps, *Sib. Mat. Zh.* **24** (1983) 214.

[10] I. P. Ryazantseva, An algorithm for solving nonlinear monotone equations with unknown input data error bound, *USSR Comput. Mat. and Mat. Phys.* **29** (1989) 225–229.

[11] M. M. Vainberg, *Variational Method and Method of Monotone Operators in the Theory of Nonlinear Equations*, New York, John Wiley, 1973.

Received on May 29, 2006
Revised on August 2, 2006