

ITERATIVE METHOD FOR SOLVING A BOUNDARY VALUE PROBLEM FOR BIHARMONIC TYPE EQUATION*

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Abstract. The solution of boundary value problems (BVP) for fourth order differential equations by their reduction to BVP for second order equations with the aim to use the achievements for the latter ones attracts attention from many researchers. In this paper, using the technique developed by ourselves in recent works, we construct iterative method for a BVP for biharmonic type equation. The performed numerical experiments show very fast convergence of the proposed algorithm.

Tóm tắt. Việc giải các bài toán biên đối với phương trình đạo hàm riêng cấp bốn bằng cách đưa chúng về các bài toán biên đối với phương trình cấp hai đã thu hút sự quan tâm của nhiều tác giả. Trong bài báo này, sử dụng kỹ thuật do chúng tôi phát triển trong nhiều công trình mới đây, một phương pháp lặp giải một bài toán biên cho phương trình song kiểu điều hòa đã được đề xuất. Sự hội tụ nhanh của phương pháp đã được chứng tỏ trên nhiều thực nghiệm tính toán.

1. INTRODUCTION

The solution of fourth order differential equations by their reduction to boundary value problems (BVP) for the second order equations, with the aim of using efficient algorithms for these, attracts attention from many researchers. Namely, for the biharmonic equation with the Dirichlet boundary condition, there is intensively developed the iterative method, which leads the problem to two problems for the Poisson equation at each iteration (see e.g. [4, 9, 11, 12]). Recently, Abramov and Ulijanova [1] proposed an iterative method for the Dirichlet problem for the biharmonic type equation, but the convergence of the method is not proved. In our previous works [6, 7] with the help of boundary or mixed boundary-domain operators appropriately introduced, we constructed iterative methods for biharmonic and biharmonic type equations associated with the Dirichlet boundary condition. Recently, in [8] for solving the Neumann BVP we introduced a purely domain operator for studying the convergence of an iterative method. It is proved that the methods are convergent with the rate of geometric progression. In this paper we develop our technique in [4] - [8] for the following problem

$$Lu \equiv \Delta^2 u - a\Delta u + bu = f \quad \text{in } \Omega, \quad (1)$$

$$u = g_0, \quad \frac{\partial \Delta u}{\partial \nu} = g_1 \quad \text{on } \Gamma, \quad (2)$$

where Δ is the Laplace operator, Ω is a bounded domain in R^n ($n \geq 2$), Γ is the sufficiently

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smooth boundary of Ω , ν is the outward normal to Γ and a, b are positive constants. The equation (1) with other boundary conditions are met, for example, in [2, 3, 8]. An iterative method reducing the problem to a sequence of Neumann and Dirichlet problems for second order equations will be proposed and investigated by experimental way.

2. CONSTRUCTION OF ITERATIVE METHOD

Set $\Delta u = v$ and

$$\varphi = -bu. \quad (3)$$

Then the problem (1)–(2) is decomposed into the two problems

$$\Delta v - av = f + \varphi \text{ in } \Omega, \quad \frac{\partial v}{\partial \nu} = g_1 \text{ on } \Gamma. \quad (4)$$

$$\Delta u = v \text{ in } \Omega, \quad u = g_0 \text{ on } \Gamma. \quad (5)$$

From here we see that the solution u of the above problems depends on the function φ , i.e., $u = u(\varphi)$. Since u and φ are related with each other by the relation (3) we get the functional equation

$$\varphi + bu(\varphi) = 0. \quad (6)$$

In order to construct iterative method for finding φ first we have to establish the concrete form of the above equation. For the purpose we introduce an operator B defined in the space $L^2(\Omega)$ by the formula

$$B\varphi = bu, \quad (7)$$

where u is found from the problems

$$\Delta v - av = \varphi \text{ in } \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \Gamma, \quad (8)$$

$$\Delta u = v \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma. \quad (9)$$

Proposition 2.1. *The problem (4), (5) is reduced to the operator equation*

$$(I + B)\varphi = F, \quad (10)$$

where

$$F = -bu_2, \quad (11)$$

u_2 being found from the problem

$$\Delta v_2 - av_2 = f \text{ in } \Omega, \quad \frac{\partial v_2}{\partial \nu} = g_1 \text{ on } \Gamma, \quad (12)$$

$$\Delta u_2 = v_2 \text{ in } \Omega, \quad u_2 = g_0 \text{ on } \Gamma. \quad (13)$$

Proof. First we notice that the functions u_2 and v_2 are completely determined by the data functions of the original problem (1), (2), namely, by f , g_0 and g_1 . Let u and v be the solutions of the problems (4), (5). Put

$$u = u_1 + u_2, \quad v = v_1 + v_2,$$

where v_1 and u_1 satisfy the problems

$$\Delta v_1 - av_1 = \varphi \quad \text{in } \Omega, \quad \frac{\partial v_1}{\partial \nu} = 0 \quad \text{on } \Gamma, \quad (14)$$

$$\Delta u_1 = v_1 \quad \text{in } \Omega, \quad u_1 = 0 \quad \text{on } \Gamma. \quad (15)$$

By the definition of B we have

$$B\varphi = bu_1.$$

Taking into account (6) from the above equality we obtain the equation (10) with F given by (11). Thus, the proposition is proved. ■

Now, consider the following two-layer iterative scheme for solving the operator equation (10)

$$\frac{\varphi^{(k+1)} - \varphi^{(k)}}{\tau} + (I + B)\varphi^{(k)} = F, \quad k = 0, 1, 2, \dots \quad (16)$$

where τ is an iterative parameter.

Proposition 2.2. *The iterative scheme (16) can be realized by the following iterative process*

1. Given $\varphi^{(0)} \in L_2(\Omega)$.
2. Knowing $\varphi^{(k)}(x)$ in Ω ($k = 0, 1, \dots$) solve consecutively two problems

$$\begin{aligned} \Delta v^{(k)} - av^{(k)} &= f + \varphi^{(k)} \quad \text{in } \Omega, \\ \frac{\partial v^{(k)}}{\partial \nu} &= g_1 \quad \text{on } \Gamma, \end{aligned} \quad (17)$$

$$\begin{aligned} \Delta u^{(k)} &= v^{(k)} \quad \text{in } \Omega, \\ u^{(k)} &= g_0 \quad \text{on } \Gamma. \end{aligned} \quad (18)$$

3. Compute the new approximation

$$\varphi^{(k+1)} = (1 - \tau)\varphi^{(k)} - \tau bu^{(k)}. \quad (19)$$

Proof. It is easy to verify that the solution $u^{(k)}$ and $v^{(k)}$ of the problems (15), (16) can be represented in the form

$$u^{(k)} = u_1^{(k)} + u_2, \quad v^{(k)} = v_1^{(k)} + v_2,$$

where u_2, v_2 are solutions of (12), (13) and $u^{(k)}, v^{(k)}$ satisfy (14), (15) with upper indexes k . Therefore, according to the definition of B we have

$$B\varphi^{(k)} = bu_1^{(k)},$$

and consequently,

$$bu^{(k)} = B\varphi^{(k)} - F.$$

In view of this, rewriting (19) in the form

$$\frac{\varphi^{(k+1)} - \varphi^{(k)}}{\tau} + \varphi^{(k)} + bu^{(k)} = 0 \quad (20)$$

we obtain the formula (16). Thus, the proposition is proved.

3. NUMERICAL RESULTS

As we see from the iterative process described in the Proposition 2.2 at each iteration we have to solve two BVP for second order equations for $v^{(k)}$ and $u^{(k)}$. For this purpose we can use many available efficient algorithms. Here for definiteness we consider the original problem in unit square and construct difference schemes of second order approximation. Then for solving the obtained systems of grid equations we use the method of complete reduction [13] with computational complexity of $O(MN\log_2 N)$.

We perform some limited experiments in MATLAB for testing the convergence of the iterative process (17)–(19) in dependence on iterative parameter τ and values of a and b . The stopping criterion for the iterative process is $\|\varphi^{(k+1)} - \varphi^{(k)}\|_\infty < \varepsilon = h_1 h_2$, where h_1 and h_2 are grid stepsizes. In all examples below we take exact solutions u and the functions f, g_0, g_1 are calculated in accordance with the given exact solutions. The results of experiments for $\tau = 1$ are presented in tables, where K is the number of iterations, $Error = \|u - u_{app}\|_\infty$, u_{app} is the computed approximate solution. The computations are carried out on PC Pentium 4 CPU 1.80 Ghz .

Table 1. $u = 0.25x^4 + 0.25y^4 + x^2 + y^2$, $a = 1$, $b = 1.5$

Grid	$h_1 * h_2$	K	Error	Time (secs)
32×32	$9.76 e - 4$	4	$2.80 e - 4$	0.79
64×64	$2.44 e - 4$	4	$8086 e - 5$	2.04
128×128	$6.10 e - 5$	5	$1.38 e - 5$	9.70
256×256	$1.52 e - 5$	6	$4.18 e - 6$	68.20

Table 2. $u = \sin(\pi x) \sin(\pi y)$, $a = 1$, $b = 1$

Grid	$h_1 * h_2$	K	Error	Time (secs)
32×32	$9.76 e - 4$	2	0.0014	0.66
64×64	$2.44 e - 4$	2	$3.73 e - 4$	1.62
128×128	$6.10 e - 5$	2	$9.30 e - 5$	6.12
256×256	$1.52 e - 5$	3	$8.62 e - 7$	43.07

Table 3. $u = (x^2 - 1)^2 e^y + (y^2 - 1)e^x$, $a = 1$, $b = 1$

Grid	$h_1 * h_2$	K	Error	Time (secs)
32×32	$9.76 e - 4$	3	$9.50 e - 4$	0.73
64×64	$2.44 e - 4$	3	$2.32 e - 4$	1.79
128×128	$6.10 e - 5$	4	$5.98 e - 5$	8.81
256×256	$1.52 e - 5$	4	$1.51 e - 5$	55.59

From Tables 1–3 we see that the proposed iterative process (17)–(19) with the chosen $\tau = 1$ converges very fast. Some experiments with other values of τ require more iterations.

For the fixed $\tau = 1$ we also perform experiments for number of iterations in dependence on b with fixed a and in dependence on a with fixed b . The results of the experiments in the case of the exact solution $u = (x^2 - 1)^2 e^y + (y^2 - 1)e^x$, grid 128×128 are given in Tables 4, 5.

Table 4. Number of iterations
in dependence on b for fixed $a = 1$

b	1	3	5	7	9	11	13	15	16
K	4	7	10	14	20	29	46	95	177

Table 5. Number of iterations
in dependence on a for fixed $b = 2$

a	2	1	0.8	0.6	0.4	0.3	0.15	0.13	0.12
K	4	5	6	7	9	12	41	88	236

From Tables 4, 5 we see that the proposed iterative process (17)–(19) shows very good convergence for the case $b/a \leq 5$ that is met in real plate bending problems.

4. CONCLUDING REMARK

In the paper an iterative method for solving the boundary value problem (1)–(2) for biharmonic type equation was proposed. Its idea is to reduce the problem to a sequence of Neumann and Dirichlet problems for second order equations. The fast convergence of the method was shown on various experiments, where the value of iterative parameter $\tau = 1$ seems to be optimal. Theoretical proof of this convergence remains an open problem to be studied.

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