Journal of Computer Science and Cybernetics, V.33, N.2 (2017), 143–164
DOI 10.15625/1813-9663/33/2/10706

SOME NEW DE MORGAN PICTURE OPERATOR TRIPLES IN PICTURE FUZZY LOGIC

BUI CONG CUONG[1], ROAN THI NGAN[2], LE BA LONG[3]

[1]*Institute of Mathematics;* [1]*bccuong@gmail.com*
[2]*Basic Science Faculty, Ha Noi University of Natural Resources and Environment*
[3]*Post and Telecommunication Institute of Technology*

Abstract. A new concept of picture fuzzy sets (PFS) were introduced in 2013, which are direct extensions of the fuzzy sets and the intuitionistic fuzzy sets. Then some operations on PFS with some properties are considered in [7, 5]. Some basic operators of fuzzy logic as negation, t-norms, t-conorms for picture fuzzy sets firstly are defined and studied in [6, 9]. This paper is devoted to some classes of representable picture fuzzy t-norms and representable picture fuzzy t-conorms on PFS and a basic algebra structure of Picture Fuzzy Logic De Morgan triples of picture operators.

Keywords. Picture fuzzy sets, Picture fuzzy t-norms, Picture fuzzy t-conorm, De Morgan picture operator triple.

1. INTRODUCTION

Recently, Bui Cong Cuong and Kreinovich (2013) first defined "picture fuzzy sets" [7, 5], which are a generalization of the Zadeh's fuzzy sets [27, 28] and the Antanassov's intuitionistic fuzzy sets [2, 1]. This concept is particularly effective in approaching the practical problems in relation to the synthesis of ideas, such as decisions making problems, voting analysis, fuzzy clustering, financial forecasting. The basic definitions and basic operators in the picture fuzzy sets theory were given in [3, 4, 7, 5]. The new basic logic connectives on the PFS firstly were presented in [6, 9]. These new concepts are supporting to new computing procedures in computational intelligence problems and in other applications (see [14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26]).

In this paper we study some algebraic properties of the picture fuzzy t-norms and the picture fuzzy t-conorms on PFS, which are basic operators of the Picture Fuzzy Logics. Some classes of the representable picture fuzzy t-norms and the representable picture fuzzy t-conorms were first given in [1, 8] will be presented. Then a basic algebra structure on PFS – De Morgan picture operator triples will be considered and some new De Morgan picture operator triples will be presented.

We first recall some basic notions of the picture fuzzy sets.

Definition 1.1. [7] A picture fuzzy set A on a universe X is an object of the form

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x))\,|x \in X\},$$

where $\mu_A(x), \eta_A(x), \nu_A(x)$ are respectively called the degree of positive membership, the degree of neutral membership, the degree of negative membership of x in A, and the following conditions are satisfied:

$$0 \le \mu_A(x), \eta_A(x), \nu_A(x) \le 1 \text{ and } \mu_A(x) + \eta_A(x) + \nu_A(x) \le 1, \forall x \in X.$$

Then, $\forall x \in X:\quad 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x))$ is called the degree of refusal membership of x in A.

Consider the set $D^* = \left\{x = (x_1, x_2, x_3) | x \in [0{,}1]^3, x_1 + x_2 + x_3 \le 1\right\}$. From now on, we will assume that if $x \in D^*$, then x_1, x_2 and x_3 denote, respectively, the first, the second and the third component of x, i.e. , $x = (x_1, x_2, x_3)$.

We have a lattice $(D^*, \le_1)$, where $\le_1$ defined for $\forall x, y \in D^*$

$$(x \le_1 y) \Leftrightarrow (x_1 < y_1, x_3 \ge y_3) \vee (x_1 = y_1, x_3 > y_3) \vee (\{x_1 = y_1, x_3 = y_3, x_2 \le y_2\}),$$

$$(x = y) \Leftrightarrow (x_1 = y_1, x_2 = y_2, x_3 = y_3), \quad \forall x, y \in D^*$$

We define the first, second and third projection mapping pr_1, pr_2 and pr_3 on D^* as $pr_1(x) = x_1$ and $pr_2(x) = x_2$ and $pr_3(x) = x_3$, on all $x \in D^*$.

Note that, if for $x, y \in D^*$ that neither $x \le_1 y$ nor $y \le_1 x$, then x and y are incomparable w.r.t $\le_1$, and denoted as $x \|_{\le_1} y$.

From now on, we denote $u \wedge v = \min(u, v),\ u \vee v = \max(u, v)$ for all $u, v \in R^1$.

For each $x, y \in D^*$, we define

$$\inf(x, y) = \begin{cases} \min(x, y), & \text{if} \quad x \le_1 y \text{ or } y \le_1 x \\ (x_1 \wedge y_1, 1 - x_1 \wedge y_1 - x_3 \vee y_3, x_3 \vee y_3), & \text{else} \end{cases}$$

$$\sup(x, y) = \begin{cases} \max(x, y), & \text{if} \quad x \le_1 y \text{ or } y \le_1 x \\ (x_1 \vee y_1, 0, x_3 \wedge y_3), & \text{else} \end{cases}$$

Proposition 1.2. *With these operators $(D^*, \le_1)$ is a complete lattice.*

Proof. See [6, 9].

Using this lattice, we easily see that every picture fuzzy set

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \,| x \in X\}$$

corresponds an D^*- fuzzy set [12] mapping, i.e., we have a mapping

$$A : X \to D^* : x \to \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) \,| x \in X\}.$$

Interpreting picture fuzzy sets as D^*-fuzzy sets gives way to greater flexibility in calculating with membership degrees, since the triple of numbers formed by the three degrees is an element of D^*, and often allows to obtain more compact formulas. ■

2. PICTURE FUZZY NEGATION OPERATOR

Now we consider some basic fuzzy operators of the Picture Fuzzy Logics.

Picture fuzzy negations are an extension of the fuzzy negations [22] and the intuitionistic fuzzy negations [2]. They are defined as follows.

Definition 2.1. A mapping $N : D^* \to D^*$ satisfying conditions $N(0_{D^*}) = 1_{D^*}$ and $N(1_{D^*}) = 0_{D^*}$ and N is nonincreasing is called a picture fuzzy negation operator.

If $N(N(x)) = x$ for all $x \in D^*$, then N is called an *involutive* negation operator.

Definition 2.2. Let $f_1, f_2 : D^* \to D^*$ be mappings on D^*. We say that the mapping f_2 is greater than f_1 if $f_1(x) \leq_1 f_2(x)$, $\forall x \in D^*$, and we denote that as $f_1 \leq f_2$. We write $f_1 < f_2$, if $f_1 \leq f_2$, and $f_1 \neq f_2$.

Let $x = (x_1, x_2, x_3) \in D^*$. We first give 2 drastic picture negation operators

$$n_d(x) = \begin{cases} 0_{D^*} \text{ if } x \neq 0_{D^*} \\ 1_{D^*} \text{ if } x = 0_{D^*} \end{cases} \text{ and } n_{d2}(x) = \begin{cases} 1_{D^*} \text{ if } x \neq 1_{D^*} \\ 0_{D^*} \text{ if } x = 1_{D^*}. \end{cases}$$

Proposition 2.3. *n_d and n_{d2} are picture negation operators and for each picture negation operator $n(x)$, $n_d(x) \leq_1 n(x) \leq_1 n_{d2}(x)$, $\forall x \in D^*$.*

Definition 2.4. The mapping $n_0 : D^* \to D^*$ defined by $n_0(x) = (x_3, 0, x_1)$, for each $x \in D^*$.

Proposition 2.5. *n_0 is a picture fuzzy negation operator. It is called the simple picture negation.*

Proof. Indeed, $1_{D^*} = (1, 0, 0) \in D^*$ then $n_0(1_{D^*}) = n_0(1, 0, 0) = (0, 0, 1) = 0_{D^*}$. Analogously, $n_0(0_{D^*}) = n_0(0, 0, 1) = (1, 0, 0) = 1_{D^*} \in D^*$.

Let $x, y \in D^*$ and $x \leq_1 y$. Consider 3 subsets

$$B_1 = \{(x_1 < y_1) \wedge (x_3 \geq y_3)\},$$

$$B_2 = \{(x_1 = y_1) \wedge (x_3 > y_3)\},$$

$$B_3 = \{(x_1 = y_1) \wedge (x_3 = y_3) \wedge (x_2 \leq y_2)\}.$$

We have to consider 4 following cases

Case 1a. $x_1 < y_1$ and $x_3 = y_3$ then $(n_0(y), n_0(x)) \in B_2 \Rightarrow n_0(y) \leq_1 n_0(x)$,

Case 1b. $x_1 < y_1$ and $x_3 > y_3$ then $(n_0(y), n_0(x)) \in B_1 \Rightarrow n_0(y) \leq_1 n_0(x)$,

Case 2. $x_1 = y_1$ and $x_3 > y_3$ then $(n_0(y), n_0(x)) \in B_1 \Rightarrow n_0(y) \leq_1 n_0(x)$,

Case 3. $x_1 = y_1$ and $x_3 = y_3$ then $(n_0(y), n_0(x)) \in B_3 \Rightarrow n_0(y) \leq_1 n_0(x)$.

It shows that the mapping $n_0(x) = (x_3, 0, x_1)$ is non-increasing and the operator $n_0(x)$ is a picture negation operator. ■

Definition 2.6. Let $x = (x_1, x_2, x_3) \in D^*$. Denote $x_4 = 1 - (x_1 + x_2 + x_3)$. The mapping N_S is by $N_S(x) = (x_3, x_4, x_1)$, for each $x \in D^*$.

Proposition 2.7. *N_S is an involutive picture negation operator and is called the picture standard negation operator.*

Proof. It is analogous to the proof of the Proposition 2.5

Some other picture fuzzy negations were given in [9].

3. PICTURE FUZZY T-NORMS AND PICTURE FUZZY T-CONORMS

Fuzzy t-norms on [0, 1] and fuzzy t-conorms on [0, 1] were defined and considered in [22, 13].

In 2004 , G.Deschrijver et al. [10] introduced the notion of intuitionistic fuzzy t-norms and t-conorms and investigated under which conditions a similar representation theorem could be obtained. For further usage, we define $L^* = \{x \in D^* \,|x_2 = 0\}$.

We can consider the set L^* defined by $L^* = \{u = (u_1, u_3) \left| u \in [0,1]^2, u_1 + u_3 \leq 1\right\}$.

Consider the order relation $u \leq v$ on L^*, defined by $u \leq v \Leftrightarrow ((u_1 \leq v_1) \wedge (u_3 \geq v_3))$, for all $u, v \in L^*$.

We define the first, and the second projection mapping pr_1 and pr_3 on L^*, as $pr_1(u) = u_1$ and $pr_3(u) = u_3$, on all $u \in L^*$. The units of L^* are $1_{L^*} = (1, 0)$ and $0_{L^*} = (0, 1)$.

Definition 3.1. [10] An intuitionistic fuzzy t-norm is a commutative, associative, increasing $(L^*)^2 \to L^*$ mapping T satisfying $T(1_{L^*}, u) = u$, for all $u \in L^*$.

Definition 3.2. [10] An intuitionistic fuzzy t-conorm is a commutative, associative, increasing
$(L^*)^2 \to L^*$ mapping S satisfying $S(v, 0_{L^*}) = v$, for all $v \in L^*$.

Definition 3.3. [10] An intuitionistic fuzzy t-norm T is called t-representable iff there exist a fuzzy t-norm t_1 on [0, 1] and a fuzzy t-conorm s_3 on [0, 1] satisfying for all $u, v \in L^*$,

$$T(u, v) = (t_1(u_1, v_1), s_3(u_3, v_3)).$$

Definition 3.4. [10] An intuitionistic fuzzy t-conorm S is called t-representable iff there exist a fuzzy t-norm t_1 on [0,1] and a fuzzy t-conorm s_3 on [0,1] satisfying for all $u, v \in L^*$,

$$S(u, v) = (s_3(u_1, v_1), t_1(u_3, v_3)).$$

Now we define picture fuzzy t-norms and picture fuzzy t-conorms, which are classes of conjunction operators and classes of disjunction operators - main basic operators of the picture fuzzy logics. Picture fuzzy t-norms are direct extensions of the fuzzy t-norms in [28, 22, 13] and of the intuitionistic fuzzy t-norms in [2], and they are important operators in [11].

Let $x = (x_1, x_2, x_3) \in D^*$. Denote $I(x) = \{y \in D^* : y = (x_1, y_2, x_3), 0 \leq_1 y_2 \leq_1 x_2\}$.

Definition 3.5. A mapping $T : D^* \times D^* \to D^*$ is a *picture fuzzy t-norm* if the mapping T satisfies the following conditions

$T(x, y) = T(y, x),\ \forall x, y \in D^*$ (commutative),

$T(x, T(y, z)) = T(T(x, y), z),\ \forall x, y, z \in D^*$ (associativity),

$T(x, y) \leq_1 T(x, z),\ \forall x, y, z \in D^*, y \leq_1 z$ (monotonicity),

$T(1_{D^*}, x) \in I(x),\ \forall x \in D^*$ (boundary condition).

Fisrt we present some picture fuzzy t-norms on picture fuzzy sets.

Definition 3.6. A picture fuzzy t-norm T is called *representable* iff there exist two fuzzy t-norms t_1, t_2 on [0,1] and a fuzzy t-conorm s_3 on [0,1] satisfying

$$T\left(x, y\right) = \left(t_1\left(x_1, y_1\right), t_2\left(x_2, y_2\right), s_3\left(x_3, y_3\right)\right), \forall x, y \in D^*$$

.

We give some representable picture fuzzy t-norms, for all $x, y \in D^*$

1. $T_{\min}(x,y) = (\min(x_1,y_1), \min(x_2,y_2), \max(x_3,y_3))$.
2. $T_{02}(x,y) = (\min(x_1,y_1), x_2y_2, \max(x_3,y_3))$.
3. $T_{03}(x,y) = (x_1y_1, x_2y_2, \max(x_3,y_3))$.
4. $T_{04}(x,y) = (x_1y_1, x_2y_2, x_3+y_3-x_3y_3)$.
5. $T_{05}(x,y) =$ $\left(\begin{cases} x_1 \wedge y_1 \text{ if } x_1 \vee y_1 = 1 \\ 0 if x_1 \vee y_1 < 1 \end{cases}, \begin{cases} x_2 \wedge y_2 \; if x_2 \vee y_2 = 1 \\ 0 if x_2 \vee y_2 < 1 \end{cases}, \begin{cases} x_3 \vee y_3 \; if x_3 \wedge y_3 = 0 \\ 1 if x_3 \wedge y_3 \neq 0 \end{cases}\right)$.
6. $T_{06}(x,y) = (\max(0, x_1+y_1-1), \max(0, x_2+y_2-1), \min(1, x_3+y_3))$.
7. $T_{07}(x,y) = (\max(0, x_1+y_1-1), \max(0, x_2+y_2-1), x_3+y_3-x_3y_3)$.
8. $T_{08}(x,y) =$ $\left(\max\left\{\frac{1}{2}(x_1+y_1-1+x_1y_1), 0\right\}, \max\left\{\frac{1}{2}(x_2+y_2-1+x_2y_2), 0\right\}, x_3+y_3-x_3y_3\right)$.
9. $T_{09}(x,y) = (x_1y_1, \max(0, x_2+y_2-1), x_3+y_3-x_3y_3)$.
10. $T_{010}(x,y) = (\max(0, x_1+y_1-1), x_2y_2, x_3+y_3-x_3y_3)$.

In this part we give some detailed proofs of picture t - norms.

Proposition 3.7. *Let $x, y \in D^*$, $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$.*

The mapping $T_{\min}$ is a picture fuzzy t-norm.

Proof. Let

$$x, y \in D^*, \text{ then } x_1 + x_2 \leq 1 - x_3, \text{ and } y_1 + y_2 \leq 1 - y_3,$$

$$(x_1 \wedge y) + (x_2 \wedge y_2) \leq \min(1-x_3, 1-y_3) = 1 - \max(x_3, y_3),$$

$$(x_1 \wedge y) + (x_2 \wedge y_2) + \max(x_3, y_3) \leq 1,$$

$$T_{\min}(x,y) = ((x_1 \wedge y), (x_2 \wedge y_2), \max(x_3, y_3)) \in D^*.$$

The mapping $T_{\min}$ is a picture fuzzy t-norm, since other conditions easily are verified. ∎
Let $x, y \in D^*$, $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$.

Proposition 3.8. *The mapping $T_{02}(x,y) = (\min(x_1,y_1),\ x_2y_2,\ \max(x_3,y_3))$ is a picture fuzzy t-norm.*

Proof. We remark that $x_2.y_2 \leq x_2 \wedge y_2 \ \Rightarrow ((x_1 \wedge y_1) + x_2.y_2 + \max(x_3, y_3)) \leq T(x,y) \in D^*$. It implies that the mapping $T_{02}(x,y) = (\min(x_1,y_1),\ x_2y_2,\ \max(x_3,y_3))$ is a picture fuzzy t-norm. ∎

Proposition 3.9. *The mapping $T_{04}(x,y) = (x_1y_1, x_2y_2,\ x_3+y_3-x_3y_3)$ is a picture fuzzy t-norm.*

Proof. We have

$$\begin{aligned}
x_1y_1 + x_2y_2 &\le (1 - x_2 - x_3)(1 - y_2 - y_3) + x_2y_2 \\
&= (1 - y_2 - y_3 - x_2 + x_2y_2 + x_2y_3 - x_3 + x_3y_2 + x_3y_3) + x_2y_2 \\
&= (1 - x_3 - y_3 + x_3y_3) + (x_2y_2 + x_2y_3 + x_2y_2 + x_3y_2 - x_2 - y_2) \\
&= (1 - x_3 - y_3 + x_3y_3) + (x_2(y_2 + y_3 - 1) + y_2(x_2 + x_3 - 1)) \\
&\le 1 - x_3 - y_3 + x_3y_3 \\
\Rightarrow x_1y_1 + x_2y_2 + x_3 + y_3 - x_3y_3 \le 1.
\end{aligned}$$

∎

Proposition 3.10. *Let mapping t_2 is a fuzzy t-norm on $[0,1]$, then the mapping*

$$T_{0t_2}(x,y) = (\min(x_1,y_1), t_2(x_2,y_2), \max(x_3,y_3)),$$

is a picture fuzzy t - norm.

Proof. See the proof of the Proposition 3.8.

Definition 3.11. A mapping $S : D^* \times D^* \to D^*$ is a *picture fuzzy t-conorm* if S satisfies all of the following conditions

1. $S(x,y) = S(y,x), \forall x,y \in D^*$ (commutative),
2. $S(x,S(y,z)) = S(S(x,y),z), \forall x,y,z \in D^*$ (associativity),
3. $S(x,y) \le_1 S(x,z), \forall x,y,z \in D^*, y \le_1 z$ (monotonicity),
4. $S(0_{D^*},x) \in I(x), \forall x \in D^*$ (boundary condition).

Definition 3.12. A picture fuzzy t-conorm S is called *representable* iff there exist two fuzzy t-norms t_1, t_2 on [0,1] and a fuzzy t-conorm s_3 on [0,1] satisfying.

Some examples of representable picture fuzzy t-conorms, for all $x,y \in D^*$.

1. $S_{\max}(x,y) = (\max(x_1,y_1), \min(x_2,y_2), \min(x_3,y_3))$.
2. $S_{02}(x,y) = (\max(x_1,y_1), x_2y_2, \min(x_3,y_3))$.
3. $S_{03}(x,y) = (\max(x_1,y_1), x_2y_2, x_3y_3)$.
4. $S_{04}(x,y) = (x_1 + y_1 - x_1y_1, x_2y_2, x_3y_3)$.
5. $S_{05}(x,y) = \left(x_1 \vee y_1, \begin{cases} x_2 \wedge y_2 & \text{if } x_2 \vee y_2 = 1 \\ 0 & \text{if } x_2 \vee y_2 < 1 \end{cases}, x_3 \wedge y_3\right)$.
6. $S_{06}(x,y) = \left(\begin{cases} x_1 \vee y_1 & \text{if } x_1 \wedge y_1 = 0 \\ 1 & \text{if } x_1 \wedge y_1 \ne 0 \end{cases}, x_2 \wedge y_2, \begin{cases} x_3 \wedge y_3 & \text{if } x_3 \vee y_3 = 1 \\ 0 & \text{if } x_3 \vee y_3 < 1 \end{cases}\right)$.

Proposition 3.13. *For any representable picture fuzzy t-norm T we have*

$$T_{05}(x,y) \leq_1 T(x,y) \leq_1 T_{\min}(x,y), \forall x,y \in D^*.$$

Proposition 3.14. *For any representable picture fuzzy t-conorm S we have*

$$S_{05}(x,y) \leq_1 S(x,y) \leq_1 S_{06}(x,y), \forall x,y \in D^*.$$

Proposition 3.15. *Assume $T(u,v)$ is a t- representable intuitionistic fuzzy t-norm*

$$T(u,v) = (t_1(u_1,v_1), s_3(u_3,v_3)), \forall u = (u_1,u_3), v = (v_1,v_3) \in L^*$$

where, t_1 is a fuzzy t-norm on $[0,1]$, s_3 is a fuzzy t-conorm on $[0,1]$. Assume t_2 is a t-norm on $[0,1]$ satisfying.

$$0 \leq t_1(x_1,y_1) + t_2(x_2,y_2) + s_3(x_3,y_3) \leq 1, \forall x,y \in D^*$$

then

$$T(x,y) = (t_1(x_1,y_1), t_2(x_2,y_2), s_3(x_3,y_3)), \forall x,y \in D^*$$

is a representable picture fuzzy t-norm.

Proposition 3.16. *Assume $S(u,v)$ is a t-representable intuitionistic fuzzy t-conorm*

$$S(u,v) = (s_3(u_1,v_1), t_1(u_3,v_3)), \quad \forall\, u = (u_1,u_3), v = (v_1,v_3) \in L^*$$

where, t_1 is a fuzzy t-norm on $[0,1]$, s_3 is a fuzzy t-conorm on $[0,1]$. Assume t_2 is a t-norm on $[0,1]$ satisfies $0 \leq t_1(x_1,y_1) + t_2(x_2,y_2) + s_3(x_3,y_3) \leq 1, \forall x,y \in D^*$ then

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x,y \in D^*$$

is a representable picture fuzzy t-conorm.

Now we define some new concepts for the picture fuzzy logic.

Definition 3.17. A picture fuzzy t-norm T is called *Achimerdean* iff

$$\forall x \in D^* \backslash \{0_{D^*}, 1_{D^*}\},\ T(x,x) <_1 x.$$

Definition 3.18. A picture fuzzy t-norm T is called

1. Nilpotent iff: $\exists x,\ y \in D^* \backslash \{0_{D^*}\},\ T(x,y) = 0_{D^*}$.
2. Strict iff: $\forall x,\ y \in D^* \backslash \{0_{D^*}\},\ T(x,y) \neq 0_{D^*}$.

With these defitions we have the following propositions.

Proposition 3.19. *Let*

$$T^* = \{Nilpotent\ picture\ t-norms\},$$
$$T^{**} = \{strict\ picture\ t-norms\}$$

then $T^* \cap T^{**} = \emptyset$.

Definition 3.20. A picture fuzzy t-conorm S is called *Achimerdean* iff

$$\forall x \in D^* \backslash \{0_{D^*}, 1_{D^*}\}, S(x,x) >_1 x.$$

Definition 3.21. A picture fuzzy t-conorm S is called

1. Nilpotent iff: $\exists x,\ y \in D^*\setminus\{1_{D^*}\},\ S(x,y) = 1_{D^*}$.

2. Strict iff: $\forall x,\ y \in D^*\setminus\{1_{D^*}\},\ S(x,y) \neq 1_{D^*}$.

Proposition 3.22. *Let*

$$S^* = \{nilpotent\ picture\ fuzzy\ t-conorms\},$$
$$S^{**} = \{strict\ picture\ t-conorms\}$$

then $S^* \cap S^{**} = \emptyset$.

Proposition 3.23. *Assume T is a representable picture fuzzy t-norm*

$$T(x,y) = (t_1(x_1,y_1), t_2(x_2,y_2), s_3(x_3,y_3)), \forall x,y \in D^*,$$

and t_1, t_2, s_3 are Archimedean on $[0,1]$, then T is Archimedean.

Proof. For all $x \in D^*\setminus\{0_{D^*}, 1_{D^*}\}$, we have

$$T(x,x) = (t_1(x_1,x_1), t_2(x_2,x_2), s_3(x_3,x_3)).$$

Since t_1, t_2, s_3 are Archimedean on [0,1]. It follows that $t_1(x_1,x_1) < x_1,\ s_3(x_3,x_3) > x_3$, so $T(x,x) <_1 x$. Thus T is Archimedean. ■

Proposition 3.24. *Assume S is a representable picture fuzzy t-conorm*

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x,y \in D^*.$$

and t_1, t_2, s_3 are Archimedean on $[0,1]$, then S is Archimedean.

4. SOME SUBCLASSES OF REPRESENTABLE PICTURE FUZZY T-NORMS

We can give some subclasses of representable picture fuzzy t-norms.

4.1. Strict-strict-strict t-norms subclass, denoted by Δ_{sss}

Definition 4.1. A picture fuzzy t-norm T is called *strict-strict-strict* iff

$$T(x,y) = (t_1(x_1,y_1), t_2(x_2,y_2), s_3(x_3,y_3)), \forall x,y \in D^*,$$

where t_1, t_2 are strict fuzzy t-norms on [0,1] and s_3 is a strict fuzzy t-conorm on [0,1].

Example 4.1. $T_1(x,y) = (x_1y_1, x_2y_2, x_3 + y_3 - x_3y_3)$,

$$T_2(x,y) = \Big(\frac{x_1y_1}{\lambda_1 + (1-\lambda_1)(x_1+y_1-x_1y_1)}, \frac{x_2y_2}{\lambda_2 + (1-\lambda_2)(x_2+y_2-x_2y_2)}, (x_3^a + y_3^a - x_3^a y_3^a)^{\frac{1}{a}}\Big),$$

$\lambda_1, \lambda_2, a \in [1, +\infty)$,

4.2. Nipoltent-nipoltent-nipoltent t-norms subclass, denoted by Δ_{nnn}

Definition 4.2. A picture fuzzy t-norm T is called *nipoltent-nipoltent-nipoltent* iff

$$T\left(x,y\right)=\left(t_{1}\left(x_{1},y_{1}\right),t_{2}\left(x_{2},y_{2}\right),s_{3}\left(x_{3},y_{3}\right)\right),\forall x,y\in D^{*},$$

where t_1, t_2 are nipoltent fuzzy t-norms on [0,1] and s_3 is a nipoltent fuzzy t-conorm on [0,1].

Examples 4.2.

$$T_3(x,y)=(0\vee(x_1+y_1-1),0\vee(x_2+y_2-1),1\wedge(x_3+y_3)),$$

$$\begin{aligned}T_4(x,y)=&(((x_1+y_1-1)(1+\lambda_1)-\lambda_1x_1y_1)\vee0,((x_2+y_2-1)(1+\lambda_2)-\lambda_2x_2y_2)\vee0,\\&1\wedge(x_3^a+y_3^a)^{\frac{1}{a}}),\ \lambda_1,\lambda_2\in[0,+\infty),a\geq1,\end{aligned}$$

$$T_5(x,y)=((0\vee(x_1^a+y_1^a-1))^{\frac{1}{a}},(0\vee(x_2^b+y_2^b-1))^{\frac{1}{b}},1\wedge(x_3^c+y_3^c)^{\frac{1}{c}}),\ a,b,c\geq1,$$

$$\begin{aligned}T_6(x,y)=&((\tfrac{1}{a}(x_1+y_1-1+(a-1)x_1y_1)\vee0),(\frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2)\vee0),\\&1\wedge(x_3^c+y_3^c)^{\frac{1}{c}}),\ a,b\in(0,1];c\geq1,\end{aligned}$$

$$\begin{aligned}T_7(x,y)=&((\frac{1}{a}(x_1+y_1-1+(a-1)x_1y_1)\vee0),((x_2+y_2-1)(1+\lambda)-\lambda x_2y_2)\vee0,\\&1\wedge(x_3^b+y_3^b)^{\frac{1}{b}}),\ a\in(0,1],\lambda\geq0,b\geq1,\end{aligned}$$

$$\begin{aligned}T_8(x,y)=&(((x_1+y_1-1)(1+\lambda)-\lambda x_1y_1)\vee0,(\frac{1}{a}(x_2+y_2-1+(a-1)x_2y_2)\vee0),\\&1\wedge(x_3^b+y_3^b)^{\frac{1}{b}}),\ a\in(0,1],b\geq1,\lambda\geq0,\end{aligned}$$

$$\begin{aligned}T_9(x,y)=&((\frac{1}{a}(x_1+y_1-1+(a-1)x_1y_1)\vee0),0\vee(x_2^b+y_2^b-1)^{\frac{1}{b}},\\&1\wedge(x_3^c+y_3^c)^{\frac{1}{c}}),\ a\in(0,1],b,c\geq1,\end{aligned}$$

$$\begin{aligned}T_{10}(x,y)=&(0\vee(x_1^a+y_1^a-1)^{\frac{1}{a}},(\frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2)\vee0),\\&1\wedge(x_3^c+y_3^c)^{\frac{1}{c}}),\ b\in(0,1],a,c\geq1,\end{aligned}$$

$$\begin{aligned}T_{11}(x,y)=&(((x_1+y_1-1)(1+\lambda)-\lambda x_1y_1)\vee0,0\vee(x_2^a+y_2^a-1)^{\frac{1}{a}},\\&1\wedge(x_3^b+y_3^b)^{\frac{1}{b}}),\ \lambda\geq0,a,b\geq1,\end{aligned}$$

$$\begin{aligned}T_{12}(x,y)=&(0\vee(x_1^a+y_1^a-1)^{\frac{1}{a}},((x_2+y_2-1)(1+\lambda)-\lambda x_2y_2)\vee0,\\&1\wedge(x_3^b+y_3^b)^{\frac{1}{b}}),\ \lambda\geq0,a,b\geq1.\end{aligned}$$

4.3. Nipoltent-nipoltent-strict t-norms subclass, denoted by Δ_{nns}

Definition 4.3. A picture fuzzy t-norm T is called *nipoltent-nipoltent-strict* iff

$$T(x,y) = (t_1(x_1,y_1), t_2(x_2,y_2), s_3(x_3,y_3)), \forall x,y \in D^*,$$

where t_1, t_2 are nipoltent fuzzy t-norms on [0,1] and s_3 is a strict fuzzy t-conorm on [0,1].

Examples 4.3.

$$T_{13}(x,y) = (0 \vee (x_1+y_1-1), 0 \vee (x_2+y_2-1), x_3+y_3-x_3y_3),$$

$$T_{14}(x,y) = (\frac{1}{2}(x_1+y_1-1+x_1y_1) \vee 0, \frac{1}{2}(x_2+y_2-1+x_2y_2) \vee 0, (x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}),$$

$a \geq 1$.

$$T_{15}(x,y) = (((x_1+y_1-1)(1+\lambda_1)-\lambda_1x_1y_1) \vee 0, ((x_2+y_2-1)(1+\lambda_2)-\lambda_2x_2y_2) \vee 0,$$
$$(x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}),\ \lambda_1,\lambda_2 \in [0,+\infty), a \geq 1,$$

$$T_{16}(x,y) = (0 \vee (x_1^a+y_1^a-1)^{\frac{1}{a}}, (x_2^b+y_2^b-1)^{\frac{1}{b}} \vee 0, (x_3^c+y_3^c-x_3^cy_3^c)^{\frac{1}{c}}),\ a,b,c \geq 1,$$

$$T_{17}(x,y) = (\tfrac{1}{a}(x_1+y_1-1+(a-1)x_1y_1) \vee 0, 0 \vee (x_2^b+y_2^b-1)^{\frac{1}{b}}, (x_3^c+y_3^c-x_3^cy_3^c)^{\frac{1}{c}}),$$
$$a \in (0,1]; b,c \geq 1,$$

$$T_{18}(x,y) = (\tfrac{1}{a}(x_1+y_1-1+(a-1)x_1y_1) \vee 0, \tfrac{1}{b}(x_2+y_2-1+(b-1)x_2y_2) \vee 0,$$
$$(x_3^c+y_3^c-x_3^cy_3^c)^{\frac{1}{c}}),\ a,b \in (0,1]; c \geq 1,$$

$$T_{19}(x,y) = (\frac{1}{a}(x_1+y_1-1+(a-1)x_1y_1) \vee 0, ((x_2+y_2-1)(1+b)-bx_2y_2) \vee 0,$$
$$(x_3^c+y_3^c-x_3^cy_3^c)^{\frac{1}{c}}),\ a \in (0,1]; b \geq 0; c \geq 1,$$

$$T_{20}(x,y) = (0 \vee (x_1^a+y_1^a-1)^{\frac{1}{a}}, \frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2) \vee 0,$$
$$(x_3^c+y_3^c-x_3^cy_3^c)^{\frac{1}{c}}),\ b \in (0,1]; a,c \geq 1,$$

$$T_{21}(x,y) = (((x_1+y_1-1)(1+a)-ax_1y_1) \vee 0, \frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2) \vee 0,$$
$$(x_3^c+y_3^c-x_3^cy_3^c)^{\frac{1}{c}}),\ a \geq 0; b \in (0,1]; c \geq 1,$$

$$T_{22}(x,y) = (((x_1+y_1-1)(1+\lambda)-\lambda x_1y_1) \vee 0, 0 \vee (x_2^a+y_2^a-1)^{\frac{1}{a}},$$
$$(x_3^b+y_3^b-x_3^by_3^b)^{\frac{1}{b}}),\ \lambda \geq 0, a,b \geq 1,$$

$$T_{23}(x,y) = (0 \vee (x_1^a+y_1^a-1)^{\frac{1}{a}}, ((x_2+y_2-1)(1+\lambda)-\lambda x_2y_2) \vee 0,$$
$$(x_3^b+y_3^b-x_3^by_3^b)^{\frac{1}{b}}),\ \lambda \geq 0, a,b \geq 1.$$

4.4. Strict-nipoltent-strict t-norms subclass, denoted by Δ_{sns}

Definition 4.4. A picture fuzzy t-norm T is called *strict -nipoltent-strict* iff

$$T(x,y) = (t_1(x_1,y_1), t_2(x_2,y_2), s_3(x_3,y_3)), \forall x,y \in D^*,$$

where t_1 is a strict fuzzy t-norm on [0,1], t_2 is a nipoltent fuzzy t-norm on [0,1] and s_3 is a strict fuzzy t-conorm on [0,1].

Example 4.4.

$$T_{24}(x,y) = (x_1y_1, 0 \vee (x_2+y_2-1), x_3+y_3-x_3y_3)$$

,

$$T_{25}(x,y) = (\frac{x_1y_1}{\lambda_1+(1-\lambda_1)(x_1+y_1-x_1y_1)}, ((x_2+y_2-1)(1+\lambda_2)-\lambda_2x_2y_2) \vee 0,$$
$$(x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}),\ \lambda_1 \geq 1, \lambda_2 \geq 0, a \geq 1,$$

$$T_{26}(x,y) = (\frac{x_1y_1}{\lambda_1+(1-\lambda_1)(x_1+y_1-x_1y_1)}, 0 \vee (x_2^a+y_2^a-1)^{\frac{1}{a}},$$
$$(x_3^b+y_3^b-x_3^by_3^b)^{\frac{1}{b}}),\ \lambda_1 \geq 1, a,b \geq 1,$$

$$T_{27}(x,y) = (\frac{x_1y_1}{\lambda_1+(1-\lambda_1)(x_1+y_1-x_1y_1)}, \frac{1}{a}(x_2+y_2-1+(a-1)x_2y_2) \vee 0,$$
$$(x_3^b+y_3^b-x_3^by_3^b)^{\frac{1}{b}}),\ \lambda_1, b \geq 1, a \in (0,1].$$

4.5. Nipoltent-strict-strict t-norms subclass, denoted by Δ_{nss}

Definition 4.5. A picture fuzzy t-norm T is called *nipoltent-strict-strict* iff

$$T(x,y) = (t_1(x_1,y_1), t_2(x_2,y_2), s_3(x_3,y_3)), \forall x,y \in D^*,$$

where t_1 is a nipoltent fuzzy t-norm on [0,1], t_2 is a strict fuzzy t-norm on [0,1] and s_3 is a strict fuzzy t-conorm on [0,1].

Example 4.5.

$$T_{28}(x,y) = (0 \vee (x_1+y_1-1), x_2y_2, x_3+y_3-x_3y_3),$$

$$T_{29}(x,y) = (\frac{1}{a}(x_1+y_1-1+(a-1)x_1y_1) \vee 0, \frac{x_2y_2}{\lambda+(1-\lambda)(x_2+y_2-x_2y_2)},$$
$$(x_3^b+y_3^b-x_3^by_3^b)^{\frac{1}{b}}),\ a \in (0,1]; b,\lambda \geq 1,$$

$$T_{30}(x,y) = (0 \vee (x_1^a+y_1^a-1)^{\frac{1}{a}}, \frac{x_2y_2}{\lambda+(1-\lambda)(x_2+y_2-x_2y_2)},$$
$$(x_3^b+y_3^b-x_3^by_3^b)^{\frac{1}{b}}),\ a,b,\lambda \geq 1,$$

$$T_{31}(x,y) = (((x_1+y_1-1)(1+\lambda_1)-\lambda_1x_1y_1, \frac{x_2y_2}{\lambda_2+(1-\lambda_2)(x_2+y_2-x_2y_2)},$$
$$(x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}),\ a, \lambda_2 \geq 1, \lambda_1 \in (0,1].$$

Proposition 4.6. *There doesn't exist representable picture fuzzy t-norm T*

$$T(x,y) = (t_1(x_1,y_1), t_2(x_2,y_2), s_3(x_3,y_3)), \forall x,y \in D^*,$$

where t_1 or t_2 is a strict fuzzy t-norm on $[0,1]$, and s_3 is a nipoltent fuzzy t-conorm on $[0,1]$.

Proof. Assume $T(x,y) = (t_1(x_1,y_1), t_2(x_2,y_2), s_3(x_3,y_3)), \forall x,y \in D^*$, with t_1 is a strict t-norm and there exist $x_3, y_3 \in (0,1)$ such that $S_3(x_3,y_3) = 1$. Let $x_1, x_2 \neq 0 | x_1 + x_2 + x_3 \leq 1; y_1, y_2 \neq 0 | y_1 + y_2 + y_3 \leq 1$, and since t_1 is strict t-norm then $t_1(x_1,y_1) > 0$.

Let $x = (x_1,x_2,x_3), y = (y_1,y_2,y_3)$, we have a contradiction $t_1(x_1,y_1) + t_2(x_2,y_2) + s_3(x_3,y_3) > 1$.

Similarly, if t_2 is strict t-norm and s_3 is nipoltent t-conorm then we have a contradiction. ■

Proposition 4.7. *If T belongs to one of four classes Δ_{sss}, Δ_{nns}, Δ_{sns}, Δ_{nss} then T is strict.*

Proof. Assume for all $x,y \in D^*, s_3$ is a strict fuzzy t-conorm on [0,1], T is a representable picture fuzzy t-norm $T(x,y) = (t_1(x_1,y_1), t_2(x_2,y_2), s_3(x_3,y_3))$ and T is nipoltent.

Then $\exists x,y \in D^* \setminus \{0_{D^*}\}$, $T(x,y) = 0_{D^*}$, and it implies $t_1(x_1,y_1) = 0$, $t_2(x_2,y_2) = 0$, $s_3(x_3,y_3) = 1$. Since s_3 is a strict fuzzy t-conorm on [0, 1], then $x_3 = 1$ or $y_3 = 1$, which is a contradiction. ■

Proposition 4.8. *If T belongs to the class Δ_{nnn} then T is a nipoltent picture fuzzy t-norm.*

Proof. Assume $T \in \Delta_{nnn}, \forall x,y \in D^* : T(x,y) = (t_1(x_1,y_1), t_2(x_2,y_2), s_3(x_3,y_3))$.

Since t_1, t_2 are nipoltent fuzzy t-norms on [0, 1], we have

$\exists x_1, y_1, x_2, y_2 | t_1(x_1,y_1) = 0, t_2(x_2,y_2) = 0$. Since t_1, t_2 are not decreasing, so $\forall x_1' \leq x_1, y_1' \leq y_1$; $x_2' \leq x_2, y_2' \leq y_2 | t_1(x_1',y_1') = 0$, $t_2(x_2',y_2') = 0$. Since s is a nipoltent fuzzy t-conorm on [0,1] so $\exists x_3, y_3 \neq 1 | s_3(x_3,y_3) = 1$. Let $x = (x_1', x_2', x_3), y = (y_1', y_2', y_3) \in D^*$. Then $T(x,y) = (t_1(x_1',y_1'), t_2(x_2',y_2'), s_3(x_3,y_3)) = 0_{D^*}$. T is a nipoltent picture fuzzy t-norm. ■

5. SOME SUBCLASSES OF REPRESENTABLE PICTURE FUZZY T-CONORMS

Similarly to the Section 4, we can give some subclasses of representable picture fuzzy t-conorms.

5.1. Strict-strict-strict t-conorms subclass, denoted by ∇_{sss}

Definition 5.1. A picture fuzzy t-conorm S is called *strict-strict-strict* iff

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x,y \in D^*.$$

where t_1, t_2 are strict fuzzy t-norms on [0, 1] and s_3 is a strict fuzzy t-conorm on [0, 1].

Examples 5.1.

$$S_1(x,y) = (x_1 + y_1 - x_1y_1, x_2y_2, x_3y_3),$$

$$S_2(x,y) = \Big((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, \frac{x_2 y_2}{\lambda_1 + (1-\lambda_1)(x_2 + y_2 - x_2 y_2)}, \frac{x_3 y_3}{\lambda_2 + (1-\lambda_2)(x_3 + y_3 - x_3 y_3)}\Big),$$

with $\lambda_1, \lambda_2, a \in [1, +\infty)$.

5.2. Nipoltent-nipoltent-nipoltent t-conorms subclass, denoted by ∇_{nnn}

Definition 5.2. A picture fuzzy t-conorm S is called *nipoltent-nipoltent-nipoltent* iff

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x,y \in D^*,$$

where t_1, t_2 are nipoltent fuzzy t-norms on [0, 1] and s_3 is a nipoltent fuzzy t-conorm on $[0,1]$.

Examples 5.2.

$$S_3(x,y) = (1 \wedge (x_1 + y_1),\ 0 \vee (x_2 + y_2 - 1),\ 0 \vee (x_3 + y_3 - 1)),$$

$$\begin{aligned} S_4(x,y) = (1 \wedge (x_1^a + y_1^a)^{\frac{1}{a}},\ &((x_2 + y_2 - 1)(1+\lambda_1) - \lambda_1 x_2 y_2) \vee 0, \\ &((x_3 + y_3 - 1)(1+\lambda_2) - \lambda_2 x_3 y_3) \vee 0),\ \lambda_1, \lambda_2 \in [0, +\infty), a \geq 1, \end{aligned}$$

$$\begin{aligned} S_5(x,y) = {} & (1 \wedge (x_1^a + y_1^a)^{\frac{1}{a}}, (0 \vee (x_2^b + y_2^b - 1))^{\frac{1}{b}}, 0 \vee (x_3^c + y_3^c - 1)^{\frac{1}{c}}), \\ & a, b, c \geq 1, \end{aligned}$$

$$\begin{aligned} S_6(x,y) = {} & (1 \wedge (x_1^a + y_1^a)^{\frac{1}{a}}, (\frac{1}{b}(x_2 + y_2 - 1 + (b-1)x_2 y_2) \vee 0), \\ & (\frac{1}{c}(x_3 + y_3 - 1 + (c-1)x_3 y_3) \vee 0)),\ a \geq 1; b, c \in (0,1], \end{aligned}$$

$$\begin{aligned} S_7(x,y) = {} & (1 \wedge (x_1^a + y_1^a)^{\frac{1}{a}}, (\frac{1}{b}(x_2 + y_2 - 1 + (b-1)x_2 y_2) \vee 0), \\ & ((x_3 + y_3 - 1)(1+\lambda) - \lambda x_3 y_3) \vee 0),\ a \geq 1,\ b \in (0,1],\ \lambda \geq 0, \end{aligned}$$

$$\begin{aligned} S_8(x,y) = {} & (1 \wedge (x_1^a + y_1^a)^{\frac{1}{a}}, ((x_2 + y_2 - 1)(1+\lambda) - \lambda x_2 y_2) \vee 0, \\ & (\frac{1}{b}(x_3 + y_3 - 1 + (b-1)x_3 y_3) \vee 0)),\ a \geq 1,\ b \in (0,1],\ \lambda \geq 0, \end{aligned}$$

$$\begin{aligned} S_9(x,y) = {} & (1 \wedge (x_1^a + y_1^a)^{\frac{1}{a}}, (\frac{1}{b}(x_2 + y_2 - 1 + (b-1)x_2 y_2) \vee 0), \\ & 0 \vee (x_3^c + y_3^c - 1)^{\frac{1}{c}}),\ b \in (0,1], a, c \geq 1, \end{aligned}$$

$$\begin{aligned} S_{10}(x,y) = {} & (1 \wedge (x_1^a + y_1^a)^{\frac{1}{a}}, 0 \vee (x_2^b + y_2^b - 1)^{\frac{1}{b}}, \\ & (\frac{1}{c}(x_3 + y_3 - 1 + (c-1)x_3 y_3) \vee 0)),\ c \in (0,1], a, b \geq 1, \end{aligned}$$

$$\begin{aligned} S_{11}(x,y) = {} & (1 \wedge (x_1^a + y_1^a)^{\frac{1}{a}}, ((x_2 + y_2 - 1)(1+\lambda) - \lambda x_2 y_2) \vee 0, \\ & 0 \vee (x_3^b + y_3^b - 1)^{\frac{1}{b}}),\ \lambda \geq 0, a, b \geq 1, \end{aligned}$$

$$\begin{aligned} S_{12}(x,y) = {} & (1 \wedge (x_1^a + y_1^a)^{\frac{1}{a}}, 0 \vee (x_2^b + y_2^b - 1)^{\frac{1}{b}}, \\ & ((x_3 + y_3 - 1)(1+\lambda) - \lambda x_3 y_3) \vee 0),\ \lambda \geq 0, a, b \geq 1. \end{aligned}$$

5.3. Strict-nipoltent-nipoltent t-conorms subclass, denoted by ∇_{snn}

Definition 5.3. A picture fuzzy t-conorm S is called *strict-nipoltent-nipoltent* iff

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x,y \in D^*,$$

where t_1, t_2 are nipoltent fuzzy t-norms on [0,1] and s_3 is a strict fuzzy t-conorm on [0,1].

Examples 5.3.

$$S_{13}(x,y) = (x_1 + y_1 - x_1y_1,\ 0 \vee (x_2+y_2-1),\ 0 \vee (x_3+y_3-1)),$$

$$\begin{aligned} S_{14}(x,y) = (&(x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, \frac{1}{2}(x_2+y_2-1+x_2y_2) \vee 0,\\ &\frac{1}{2}(x_3+y_3-1+x_3y_3) \vee 0),\ a \geq 1, \end{aligned}$$

$$\begin{aligned} S_{15}(x,y) = (&(x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, ((x_2+y_2-1)(1+\lambda_1) - \lambda_1 x_2 y_2) \vee 0,\\ &((x_3+y_3-1)(1+\lambda_2) - \lambda_2 x_3 y_3) \vee 0),\ \lambda_1, \lambda_2 \in [0,+\infty),\ a \geq 1, \end{aligned}$$

$$S_{16}(x,y) = ((x_1^c + y_1^c - x_1^c y_1^c)^{\frac{1}{c}},\ 0 \vee (x_2^a + y_2^a - 1)^{\frac{1}{a}},\ 0 \vee (x_3^b + y_3^b - 1)^{\frac{1}{b}}),\ a,b,c \geq 1,$$

$$\begin{aligned} S_{17}(x,y) = (&(x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, \frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2) \vee 0,\\ &0 \vee (x_3^c + y_3^c - 1)^{\frac{1}{c}}),\ b \in (0,1]; a,c \geq 1, \end{aligned}$$

$$\begin{aligned} S_{18}(x,y) = (&(x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, \frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2) \vee 0,\\ &\frac{1}{c}(x_3+y_3-1+(c-1)x_3y_3) \vee 0),\ b,c \in (0,1]; a \geq 1, \end{aligned}$$

$$\begin{aligned} S_{19}(x,y) = (&(x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, \frac{1}{b}(x_2+y_2-1+(b-1)x_2y_2) \vee 0,\\ &((x_3+y_3-1)(1+c) - cx_3y_3) \vee 0),\ a \geq 1,\ b \in (0,1]; c \geq 0, \end{aligned}$$

$$\begin{aligned} S_{20}(x,y) = (&(x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, (x_2^b + y_2^b - 1)^{\frac{1}{b}} \vee 0,\\ &\frac{1}{c}(x_3+y_3-1+(c-1)x_3y_3) \vee 0),\ c \in (0,1]; a,b \geq 1, \end{aligned}$$

$$\begin{aligned} S_{21}(x,y) = (&(x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, ((x_2+y_2-1)(1+b) - bx_2y_2) \vee 0,\\ &\frac{1}{c}(x_3+y_3-1+(c-1)x_3y_3) \vee 0),\ a \geq 1; b \geq 0; c \in (0,1], \end{aligned}$$

$$\begin{aligned} S_{22}(x,y) = (&(x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, ((x_2+y_2-1)(1+\lambda) - \lambda x_2y_2) \vee 0,\\ &0 \vee (x_3^b + y_3^b - 1)^{\frac{1}{b}}),\ \lambda \geq 0, a,b \geq 1, \end{aligned}$$

$$\begin{aligned} S_{23}(x,y) = (&(x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, 0 \vee (x_2^b + y_2^b - 1)^{\frac{1}{b}},\\ &((x_3+y_3-1)(1+\lambda) - \lambda x_3y_3) \vee 0),\ \lambda \geq 0, a,b \geq 1. \end{aligned}$$

5.4. Strict-nipoltent-strict t-conorms subclass, denoted by ∇_{sns}

Definition 5.4. A picture fuzzy t-conorm S is called *strict-nipoltent-strict* iff

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x,y \in D^*,$$

where t_1 is a strict fuzzy t-norm on [0, 1], t_2 is a nipoltent fuzzy t-norm on [0, 1] and s_3 is a strict fuzzy t-conorm on [0, 1].

Examples 5.4.

$$S_{24}(x,y) = (x_1 + y_1 - x_1y_1,\ 0 \vee (x_2 + y_2 - 1),\ x_3y_3),$$

$$S_{25}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}}, ((x_2 + y_2 - 1)(1 + \lambda_1) - \lambda_1 x_2 y_2) \vee 0, \frac{x_3y_3}{\lambda_2 + (1-\lambda_2)(x_3 + y_3 - x_3y_3)}),\ \lambda_1 \geq 0; \lambda_2, a \geq 1,$$

$$S_{26}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}},\ 0 \vee (x_2^b + y_2^b - 1)^{\frac{1}{b}}, \frac{x_3y_3}{\lambda_1 + (1-\lambda_1)(x_3 + y_3 - x_3y_3)}),\ \lambda_1 \geq 1; a, b \geq 1,$$

$$S_{27}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}},\ \tfrac{1}{b}(x_2 + y_2 - 1 + (b-1)x_2y_2) \vee 0, \frac{x_3y_3}{\lambda_1 + (1-\lambda_1)(x_3 + y_3 - x_3y_3)}),\ a, \lambda_1 \geq 1; b \in (0,1].$$

5.5. Strict-strict-nipoltent t-conorms subclass, denoted by ∇_{ssn}

Definition 5.5. A picture fuzzy t-conorm S is called *strict-strict-nipoltent* iff

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x,y \in D^*,$$

where t_1 is a nipoltent fuzzy t-norm on [0, 1], t_2 is a strict fuzzy t-norm on [0, 1] and s_3 is a strict fuzzy t-conorm on [0, 1].

Examples 5.5.

$$S_{28}(x,y) = (x_1 + y_1 - x_1y_1,\ x_2y_2,\ 0 \vee (x_3 + y_3 - 1)),$$

$$S_{29}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}},\ \frac{x_2y_2}{\lambda + (1-\lambda)(x_2 + y_2 - x_2y_2)}, \frac{1}{b}(x_3 + y_3 - 1 + (b-1)x_3y_3) \vee 0),\ a, \lambda \geq 1; b \in (0,1],$$

$$S_{30}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}},\ \frac{x_2y_2}{\lambda + (1-\lambda)(x_2 + y_2 - x_2y_2)}, 0 \vee (x_3^b + y_3^b - 1)^{\frac{1}{b}}),\ a, b, \lambda \geq 1,$$

$$S_{31}(x,y) = ((x_1^a + y_1^a - x_1^a y_1^a)^{\frac{1}{a}},\ \frac{x_2y_2}{\lambda + (1-\lambda)(x_2 + y_2 - x_2y_2)}, ((x_2 + y_2 - 1)(1 + b) - bx_2y_2) \vee 0),\ a, \lambda \geq 1; b \geq 0.$$

Proposition 5.6. *There doesn't exist representable picture fuzzy t-conorm S*

$$S(x,y) = (s_3(x_1,y_1), t_2(x_2,y_2), t_1(x_3,y_3)), \forall x,y \in D^*,$$

here t_1 or t_2 is strict fuzzy t-norm on $[0,1]$ *and s_3 is a nipoltent fuzzy t-conorm on* $[0,1]$.

Proposition 5.7. *If S belongs to one of four classes ∇_{sss}, ∇_{snn}, ∇_{sns}, ∇_{ssn} then S is strict.*

Proposition 5.8. *If S belongs to the class ∇_{nnn} then S is nipoltent.*

6. SOME NEW DE MORGAN PICTURE OPERATOR TRIPLES IN PICTURE FUZZY LOGIC

De Morgan picture operator triples is a basic algebra of the Picture Fuzzy Logic. The notion of t-norm plays the role of intersection , or in logical terms, "and". The duality of that notion is that of union , or "or" . In the case of sets, union and intersection are related via complements. The well-known De Morgan formulas do that. They are

$$(A \cup B)^C = (A^C \cap B^C),\ (A \cap B)^C = (A^C \cup B^C).$$

Let $T(x,y)$ be a picture fuzzy t-norm and let $S(x,y)$ be a picture fuzzy t-conorm and $n(x)$ be a picture negation operator, The De Morgan formulas now become the new equations

$$n(S(x,y)) = T(n(x), n(y)),\ \forall x, y \in D^*, \tag{$a,*$}$$

$$n(T(x,y)) = S(n(x), n(y)),\ \forall x, y \in D^*. \tag{$b,*$}$$

Definition 6.1. The triple of operators (T, S, n) is called *a De Morgan picture operator triple* if they satisfy both the equation $(a,*)$ and the equation $(b,*)$. Then we say that T and S are *dual corresponding to* the negation operator $n(x)$.

With an involutive picture negation operator, De Morgan triples of picture fuzzy operators satisfy the following equations

$$S(x,y) = n(T(n(x), n(y))),\ \forall x, y \in D^*, \tag{$a,**$}$$

and

$$T(x,y) = n(S(n(x), n(y))),\ \forall x, y \in D^*. \tag{$b,**$}$$

Some De Morgan picture operator triples were given in [9].

Now we give some new De Morgan picture operator triples (T, S, n_0) corresponding the picture negation operator $n_0(x)$.

Proposition 6.2. *The triple $(T_{\min}, S_{\max}, n_0)$ corresponding the picture negation operator $n_0(x)$ is a De Morgan picture operator triple.*

Proof. We have

$$T_{\min}(x,y) = (\min(x_1,y_1), \min(x_2,y_2), \max(x_3,y_3)).$$

And

$$S_{\max}(x,y) = (\max(x_1,y_1), \min(x_2,y_2), \min(x_3,y_3)).$$

$$\begin{aligned} n_0(S_{\max}(x,y)) &= n_0(\max(x_1,y_1), \min(x_2,y_2), \min(x_3,y_3)) \\ &= (\min(x_3,y_3), 0, \max(x_1,y_1)), \end{aligned}$$

and

$$\begin{aligned} & n_0(x) = (x_3, 0, x_1),\ n_0(y) = (y_3, 0, y_1) \Rightarrow \\ & T_{\min}(n_0(x), n_0(y)) = (\min(x_3,y_3), \min(0,0), \max(x_1,y_1)) \\ & \qquad = (\min(x_3,y_3), 0, \max(x_1,y_1)) = n_0(S_{\max}(x,y)). \end{aligned}$$

It means that we have the equation $(a, *)$. Analogously

$$n_0(T_{\min}(x, y)) = (\max(x_3, y_3), 0, \min(x_1, y_1)),$$

$$\begin{aligned} S_{\max}(n_0(x), n_0(y)) &= (\max(x_3, y_3), \min(0, 0), \mathrm{mn}(x_1, y_1)) \\ &= (\max(x_3, y_3), 0, \min(x_1, y_1)) = n_0(T_{\min}(x, y)). \end{aligned}$$

We have the equation $(b, *)$. ∎

Proposition 6.3. *Consider the picture t-norm* $T_{02}(x, y) = (\min(x_1, y_1), x_2y_2, \max(x_3, y_3))$ *and the picture t-conorm* $S_{02}(x, y) = (\max(x_1, y_1), x_2y_2, \min(x_3, y_3))$.

The triple (T_{02}, S_{02}, n_0) is a De Morgan picture operator triple.

Proof. The proof is analogous to the proof of the Proposition 6.2.

Proposition 6.4. *Let* $t_2(x, y)$, $t_3(x, y)$ *be fuzzy t-norms on* $[0, 1]$. *Consider the picture fuzzy t-norm*

$$T_{\min,t_2}(x, y) = (\min(x_1, y_1), t_2(x_2y_2), \max(x_3, y_3))$$

and the picture fuzzy t-conrm

$$S_{\max,t_3}(x, y) = (\max(x_1, y_1), t_2(x_2y_2), \min(x_3, y_3)).$$

The triple of operators $(T_{\min,t_2}, S_{\max,t_3}, n_0)$ corresponding the picture negation operator $n_0(x)$ is a De Morgan picture operator triple.

Proof.

$$S_{max,t_3}(x, y) = (\max(x_1, y_1), t_3(x_2, y_2), \min(x_3, y_3)).$$

$$\begin{aligned} n_0(S_{max,t_3}(x, y)) &= n_0(\max(x_1, y_1), t_3(x_2, y_2), \min(x_3, y_3)) \\ &= (\min(x_3, y_3), 0, \max(x_1, y_1)), \end{aligned}$$

and

$$\begin{aligned} &n_0(x) = (x_3, 0, x_1),\ n_0(y) = (y_3, 0, y_1) \Rightarrow \\ &T_{\min,t_2}(n_0(x), n_0(y)) = (\min(x_3, y_3), t_2(0, 0), \max(x_1, y_1)) \\ &\qquad = (\min(x_3, y_3), 0, \max(x_1, y_1)) = n_0(S_{\max,t_3}(x, y)). \end{aligned}$$

It means that we have the equation $(a, *)$. Analogously

$$n_0(T_{\min,t_2}(x, y)) = (\max(x_3, y_3), 0, \min(x_1, y_1)),$$

$$\begin{aligned} S_{\max,t_3}(n_0(x), n_0(y)) &= (\max(x_3, y_3), t_3(0, 0), \mathrm{mn}(x_1, y_1)) \\ &= (\max(x_3, y_3), 0, \min(x_1, y_1)) = n_0(T_{\min,t}(x, y)). \end{aligned}$$

We have the equation $(b, *)$. ∎

We easily receive the following proposition.

Proposition 6.5. *Consider the picture t-norm* T_{04} *and the picture t-conorm* S_{04}

$$T_{04}(x, y) = (x_1y_1, x_2y_2, x_3 + y_3 - x_3y_3),$$

and

$$S_{04}(x, y) = (x_1 + y_1 - x_1y_1, x_2y_2, x_3y_3).$$

The triple (T_{04}, S_{04}, n_0) *is a De Morgan picture operator triple.*

Proof.

$$\begin{aligned} n_0(S_{04}(x,y)) &= n_0(x_1+y_1-x_1y_1, x_2y_2, x_3y_3)\\ &= (x_3y_3, 0, x_1+y_1-x_1y_1), \end{aligned}$$

and

$$\begin{aligned} &n_0(x) = (x_3,0,x_1),\ n_0(y) = (y_3,0,y_1) \Rightarrow\\ &T_{04}(n_0(x), n_0(y)) = (x_3y_3, 0.0, x_1+y_1-x_1y_1) = (x_3y_3, 0, x_1+y_1-x_1y_1) = n_0(S_{04}(x,y)). \end{aligned}$$

It means that we have the equation $(a,*)$. Analogously

$$n_0(T_{04}(x,y)) = (x_3+y_3-x_3y_3, 0, x_1y_1),$$

$$\begin{aligned} S_{04}(n_0(x), n_0(y)) &= (x_3+y_3-x_3y_3, 0.0, x_1y_1)\\ &= (x_3+y_3-x_3y_3, 0, x_1y_1) = n_0(T_{04}(x,y)). \end{aligned}$$

We have the equation $(b,*)$. ■

Now we consider the case where picture t-norm T belongs to the nilpotent, nilpotent, nilpotent subclass Δ_{nnn} and S belongs to the subclass ∇_{nnn}.

Proposition 6.6. *Consider the picture t-norm T_3 and the picture t-conorm S_3*

$$T_3(x,y) = (0 \vee (x_1+y_1-1), 0 \vee (x_2+y_2-1), 1 \wedge (x_3+y_3)),$$

$$S_3(x,y) = (1 \wedge (x_1+y_1), 0 \vee (x_2+y_2-1), 0 \vee (x_3+y_3-1)).$$

The triple (T_3, S_3, n_0) is a De Morgan picture operator triple.

Proof.

$$\begin{aligned} n_0(S_3(x,y)) &= n_0((1 \wedge (x_1+y_1), 0 \vee (x_2+y_2-1), 0 \vee (x_3+y_3-1)))\\ &= (0 \vee (x_3+y_3-1), 0, (1 \wedge (x_1+y_1)), \end{aligned}$$

$$\begin{aligned} &n_0(x) = (x_3,0,x_1),\ n_0(y) = (y_3,0,y_1) \Rightarrow\\ &T_3(n_0(x), n_0(y)) = (((x_3+y_3-1) \vee 0), (0+0-1) \vee 0), 1 \wedge (x_1+y_1))\\ &\qquad = ((0 \vee (x_3+y_3-1), 0, 1 \wedge (x_1+y_1)) = n_0(S_3(x,y)). \end{aligned}$$

It means that we have the equation $(a,*)$. Analogously

$$n_0(T_3(x,y)) = (1 \wedge (x_3+y_3)), 0, (0 \vee (x_1+y_1-1)),$$

$$\begin{aligned} S_3(n_0(x), n_0(y)) &= (1 \wedge (x_3+y_3), (0 \vee (0+0-1), (0 \vee (x_1+y_1-1))\\ &= (1 \wedge (x_3+y_3), 0, (0 \vee (x_1+y_1-1)) = n_0(T_3(x,y)). \end{aligned}$$

We have the equation $(b,*)$. ■

Proposition 6.7. *Consider the picture t-norm T_4 of subclass Δ_{nnn} and S_4 belongs to the subclass ∇_{nnn}.*

$T_4(x,y) = (((x_1+y_1-1)(1+\lambda_1)-\lambda_1x_1y_1) \vee 0, ((x_2+y_2-1)(1+\lambda_2)-\lambda_2x_2y_2) \vee 0, 1 \wedge (x_3^a+y_3^a)^{\frac{1}{a}}), S_4(x,y) = (1 \wedge (x_1^a+y_1^a)^{\frac{1}{a}}, ((x_2+y_2-1)(1+\lambda_2)-\lambda_2x_2y_2) \vee 0, ((x_3+y_3-1)(1+\lambda_1)-\lambda_1x_3y_3) \vee 0)$, *where* $\lambda_1, \lambda_2 \in [0,+\infty), a \geq 1$. *The triple (T_4, S_4, n_0) is a De Morgan picture operator triple.*

Proof.

$$
\begin{aligned}
&n_0(S_4(x,y))\\
&= n_0((1\wedge(x_1^a+y_1^a)^{\frac{1}{a}}, ((x_2+y_2-1)(1+\lambda_2)-\lambda_2x_2y_2)\vee 0, ((x_3+y_3-1)(1+\lambda_1)-\lambda_1x_3y_3)\vee 0))\\
&= ((x_3+y_3-1)(1+\lambda_1)-\lambda_1x_3y_3)\vee 0), 0, (1\wedge(x_1^a+y_1^a)^{\frac{1}{a}}),
\end{aligned}
$$

$$
\begin{aligned}
&T_4(n_0(x), n_0(y))\\
&= (((x_3+y_3-1)(1+\lambda_1)-\lambda_1x_3y_3)\vee 0, ((0+0-1)(1+\lambda_2)-\lambda_2 0.0)\vee 0, 1\wedge(x_1^a+y_1^a)^{\frac{1}{a}})\\
&= (((x_3+y_3-1)(1+\lambda_1)-\lambda_1x_3y_3)\vee 0, 0, 1\wedge(x_1^a+y_1^a)^{\frac{1}{a}}) = n_0(S_4(x,y)).
\end{aligned}
$$

It means that we have the equation $(a,*)$. Analogously

$$n_0(T_4(x,y)) = ((1\wedge(x_3^a+y_3^a)^{\frac{1}{a}}), 0, (x_1+y_1-1)(1+\lambda_1)-\lambda_1x_1y_1)\vee 0)),$$

$$
\begin{aligned}
&S_4(n_0(x), n_0(y))\\
&= ((1\wedge(x_3^a+y_3^a)^{\frac{1}{a}}, ((0+0-1)(1+\lambda_2)-\lambda_2 0.0))\vee 0, ((x_1+y_1-1)(1+\lambda_1)-\lambda_1x_1y_{13})\vee 0))\\
&= ((1\wedge(x_3^a+y_3^a)^{\frac{1}{a}}, 0, ((x_1+y_1-1)(1+\lambda_1)-\lambda_1x_1y_{13})\vee 0)) = n_0(T_4(x,y)).
\end{aligned}
$$

We have the equation $(b,*)$. ∎

Proposition 6.8. *We consider the case picture t-norm T_{11} belongs to Δ_{nns} - the nilpotent, nilpotent, strict subclass and S_{11} belongs to the subclass ∇_{snn}.*

$$T_{11}(x,y) = (0\vee(x_1+y_1-1), 0\vee(x_2+y_2-1), x_3+y_3-x_3y_3),$$

$$S_{11}(x,y) = (x_1+y_1-x_1y_1, 0\vee(x_2+y_2-1), 0\vee(x_3+y_3-1)).$$

The triple (T_{11}, S_{11}, n_0) is a De Morgan picture operator triple.

Proof.

$$
\begin{aligned}
n_0(S_{11}(x,y)) &= n((x_1+y_1-x_1y_1, 0\vee(x_2+y_2-1), 0\vee(x_3+y_3-1)))\\
&= ((0\vee(x_3+y_3-1)), 0, (x_1+y_1-x_1y_1)),
\end{aligned}
$$

$$
\begin{aligned}
T_{11}(n_0(x), n_0(y)) &= (0\vee(x_3+y_3-1), 0\vee(0+0-1), x_1+y_1-x_1y_1)\\
&= (0\vee(x_3+y_3-1), 0, x_1+y_1-x_1y_1)\\
&= n_0(S_{11}(x,y)).
\end{aligned}
$$

It means that we have the equation $(a,*)$. Analogously

$$
\begin{aligned}
n_0(T_{11}(x,y)) &= n_0((0\vee(x_1+y_1-1), 0\vee(x_2+y_2-1), x_3+y_3-x_3y_3))\\
&= ((x_3+y_3-x_3y_3), 0, (0\vee(x_1+y_1-1)),
\end{aligned}
$$

$$
\begin{aligned}
S_{11}(n_0(x), n_0(y)) &= ((x_3+y_3-x_3y_3), 0\vee(0+0-1), 0\vee(x_1+y_1-1))\\
&= ((x_3+y_3-x_3y_3), 0, (0\vee(x_1+y_1-1)) = n_0(T_{11}(x,y)),
\end{aligned}
$$

we have the equation $(b,*)$. ∎

Now we consider the case where picture t-norm T_{12} belongs the nilpotent, nilpotent, strict subclass Δ_{nns} and the picture t-conorm S_{12} belongs to the subclass ∇_{snn}.

Proposition 6.9. *Consider*

$$T_{12}(x,y) = (\frac{1}{2}(x_1+y_1-1+x_1y_1)\vee 0, \frac{1}{2}(x_2+y_2-1+x_2y_2)\vee 0, (x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}),$$

$$S_{12}(x,y) = ((x_1^a+y_1^a-x_1^ay_1^a)^{\frac{1}{a}}, \frac{1}{2}(x_2+y_2-1+x_2y_2)\vee 0, \frac{1}{2}(x_3+y_3-1+x_3y_3)\vee 0),$$

where $a \geq 1$.

The triple (T_{12}, S_{12}, n_0) is a De Morgan picture operator triple.

Proof.

$$\begin{aligned} &n_0(S_{12}(x,y)) \\ &= n_0(((x_1^a+y_1^a-x_1^ay_1^a)^{\frac{1}{a}}, \frac{1}{2}(x_2+y_2-1+x_2y_2)\vee 0, \frac{1}{2}(x_3+y_3-1+x_3y_3)\vee 0)) \\ &= ((\frac{1}{2}(x_3+y_3-1+x_3y_3)\vee 0), 0, (x_1^a+y_1^a-x_1^ay_1^a)^{\frac{1}{a}}), \end{aligned}$$

$$\begin{aligned} &T_{12}(n_0(x), n_0(y)) \\ &= ((\frac{1}{2}(x_3+y_{31}-1+x_3y_3)\vee 0), \frac{1}{2}(0+0-1+0.0)\vee 0, (x_1^a+y_1^a-x_1^ay_1^a)^{\frac{1}{a}})) \\ &= ((\frac{1}{2}(x_3+y_{31}-1+x_3y_3)\vee 0), 0, (x_1^a+y_1^a-x_1^ay_1^a)^{\frac{1}{a}}) = n_0(S_{12}(x,y)). \end{aligned}$$

It means that we have the equation $(a,*)$. Analogously

$$\begin{aligned} &n_0(T_{12}(x,y)) \\ &= n_0((\frac{1}{2}(x_1+y_1-1+x_1y_1)\vee 0), \frac{1}{2}(x_2+y_2-1+x_2y_2)\vee 0, (x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}})) \\ &= ((x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}), 0, (\frac{1}{2}(x_1+y_1-1+x_1y_1)\vee 0)) = S_{12}(n_0(x), n_0(y)), \end{aligned}$$

$$\begin{aligned} &S_{12}(n_0(x), n_0(y)) \\ &= ((x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}, (\frac{1}{2}(0+0-1+0.0)\vee 0), (\frac{1}{2}(x_1+y_1-1+x_1y_3)\vee 0)) \\ &= ((x_3^a+y_3^a-x_3^ay_3^a)^{\frac{1}{a}}, 0, (\frac{1}{2}(x_1+y_1-1+x_1y_3)\vee 0)) = n_0(T_{12}(x,y)). \end{aligned}$$

It means that we have the equation $(b,*)$. ∎

Some other De Morgan picture operator triples can be seen in [9, 8].

7. CONCLUSION

Conjunction operations (fuzzy t-norms) and disjunction operations (fuzzy t-conorms) are basic operators of the fuzzy logics [22, 13]. Picture fuzzy t-norms and picture fuzzy t-conorms firstly were defined and studied in 2015 [6, 9]. In this paper we give some algebraic properties of the picture fuzzy t-norms and the picture fuzzy t-conrms on picture fuzzy sets, including some classes of representable picture fuzzy t-norms and and some classes of representable picture fuzzy t-conorms. Then we study the De Morgan picture operator triples of the Picture Fuzzy Logics. Some new classes of De Morgan picture operator triples were presented. In the following papers new other issues of the Picture Fuzzy Logic should be considered.

Acknowledgment

This research is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 102.01-2017.02.

REFERENCES

[1] K. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, pp. 87–96, 1986.

[2] ——, *On Intuitionistic Fuzzy Sets Theory.* Berlin: Springer, 2012.

[3] B. C. Cuong, "Picture fuzzy sets - first results," in *Part 1, Seminar: Neuro-Fuzzy Systems with Applications, Preprint 03/2013, May 2013, Institute of Mathematics, Hanoi, Vietnam*, 2013.

[4] ——, "Picture fuzzy sets - first results," in *Part 2, Seminar: Neuro-Fuzzy Systems with Applications, Preprint 04/2013, June 2013, Institute of Mathematics, Hanoi Vietnam*, 2013.

[5] ——, "Picture fuzzy sets," *Journal of Computer Science and Cybernetics*, vol. 30, no. 4, pp. 409–420, 2014.

[6] B. C. Cuong and P. V. Hai, "Some fuzzy logic operators for picture fuzzy sets," in *the Proceedings of the 2015 IEEE International Conference on Knowledge Systems Engineering, KSE 2015, IEEE Computer Society Publications and CPS, Washington*, 2015, pp. 132–137.

[7] B. C. Cuong and V. Kreinovich, "Picture fuzzy sets - a new concept for computational intelligence problems," in *The Proceedings of the World Congress on Information and Communication Technologies (WICT 2013), Hanoi, Vietnam, IEEE CS*, 2013, pp. 1–6.

[8] B. C. Cuong and R. T. Ngan, "Some new de morgan triples of operators in the picture fuzzy sets theory," in *Seminar: Neuro-Fuzzy Systems with Applications, Preprint 08/2017, June 2017, Institute of Mathematics, Hanoi, Vietnam*, 2017.

[9] B. C. Cuong, R. T. Ngan, and B. D. Hai, "An involutive picture fuzzy negation on picture fuzzy sets and some de morgan triples," in *the Proceedings of the 2015 IEEE International Conference on Knowledge and Systems Engineering, KSE 2015, IEEE Computer Society Publications and CPS, Washington*, 2015, pp. 126–131.

[10] G. Deschrijver, C. Cornelis, and E. E. Kerre, "On the representation of intuitionistic fuzzy t-norms and t-conorms," *IEEE Transactions on Fuzzy Systems*, vol. 12, no. 1, pp. 45–61, 2004.

[11] F.Smaranddache, *A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic.* Rehoboth: American Research Press, 1999.

[12] J. Goguen, "L-fuzzy sets," *J. Math. Anal. Appl.*, vol. 18, pp. 145–174, 1967.

[13] E. P. Klement and R. Mesiar, *Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms.* Amsterdam: Elsevie, 2005.

[14] P. H. Phong and B. C. Cuong, "Some intuitionistic linguistic aggregation operators," *Journal of Computer Science and Cybernetics*, vol. 30, no. 3, pp. 216–226, 2014.

[15] ——, "Multi-criteria group decision making with picture linguistic numbers," *VNU Journal of Science: Computer Science and Communication Engineering*, vol. 32, no. 3, pp. 38–51, 2016.

[16] ——, "Symbolic computational models for intuitionistic linguistic information," *Journal of Computer Science an Cybernetics*, vol. 32, no. 1, pp. 30–44, 2016.

[17] L. H. Son, "DPFCM: A novel distributed picture fuzzy clustering method on picture fuzzy sets," *Expert systems with applications*, vol. 42, pp. 51–66, 2015.

[18] ——, "Generalized picture distance measure and applications to picture fuzzy clustering," *Applied Soft Computing*, vol. 46, p. 284295, 2016.

[19] ——, "Measuring analogousness in picture fuzzy sets: From picture distance measures to picture association, measures," *Fuzzy Optimization and Decision Making*, pp. 1–20, 2017.

[20] L. H. Son and P. H. Thong, "Some novel hybrid forecast methods based on picture fuzzy clustering for weather nowcasting from satellite image sequences," *Applied Intelligence*, vol. 46, no. 1, pp. 1–15, 2017.

[21] L. H. Son, P. V. Viet, and P. V. Hai, "Picture inference system: A new fuzzy inference system on picture fuzzy set," *Applied Intelligence*, vol. 46, no. 3, pp. 652–669, 2017.

[22] T.H.Nguyen and E.Walker, *A first course in fuzzy logic, Second Edition.* CRC, 2000, Boca Raton: Chapman & Hall, 2000.

[23] P. H. Thong and L. H. Son, "Picture fuzzy clustering: A new computational intelligence method," *Soft Computing*, vol. 20, no. 9, pp. 3544–3562, 2016.

[24] C. Wang, X. Zhou, H. Tu, and S. Tao, "Some geometric aggregation operators based on picture fuzzy sets and their application in multiple attribute decision making," *Italian Journal of Pure and Applied Mathematics*, no. 37, pp. 477–492, 2017.

[25] G. Wei, "Picture fuzzy cross-entropy for multiple attribute decision making problems," *Journal of Business Economics and Management*, vol. 17, no. 4, pp. 491–502, 2016.

[26] ——, "Picture 2-tuple linguistic bonferroni mean operators and their application to multiple attribute decision making," *International Journal of Fuzzy Systems*, vol. 19, no. 4, pp. 997–1010, 2017.

[27] L. A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, pp. 338–353, 1965.

[28] ——, "The concept of a linguistic variable and its application to approximate reasoning," *Information Sciences*, vol. 8, pp. 199–249, 1975.

Received on September 19, 2017
Revised on December 01, 2017