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# AN EXACTLY SOLUBLE EQUATION FOR THE STATIONARY PROBABILITY DISTRIBUTION IN A NONLINEAR SYSTEM UNDER THE INFLUENCE OF TWO-TELEGRAPH NOISE: APPLICATION TO THE NOISE REDUCTION IN A RAMAN RING LASER

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**Abstract.** *In this paper, we will consider a nonlinear system with random telegraph noises in a Raman ring laser by modeling the laser pump light as a pre-Gaussian process and find an exactly soluble equation for the stationary probability distribution of fluctuations in this nonlinear system under the influence of two-telegraph noise. As a consequence, we will obtain the so-called noise reduction in this system: the Stokes output of this laser tends to stabilize under the influence of the broad-band two-telegraph pre-Gaussian pump and compare this result with that obtained in our previous paper (Cao Long Van, Doan Quoc Khoa, Opt. Quant. Electron.* ***43*** *(2012) 137) for the case of one-telegraph noise.*

*Keywords: Raman ring laser, two-telegraph noises, noise reduction, nonlinear system.*

*Classification numbers: 42.50.Lc; 42.65.Sf; 02.50.Ey; 02.50.Ga; 42.55.Ye.*

## I. INTRODUCTION

Laser lights are never perfectly monochromatic so they generally have fluctuations in phase and amplitude. To simplify the complicated microscopic nature of all the relevant relaxation mechanisms, we model the laser lights by classical time-dependent random processes. The dynamical equations that contain field parameters become stochastic differential equations. Except for some special cases, the obtaining of an exact solution for such stochastic equations is a very difficult task. One of the most useful stochastic models has been introduced in Refs. [1–4], which is based on the so-called pre-Gaussian process. It is composed of a finite number of independent telegraph signals mentioned above. The strength of the pre-Gaussian formalism is obtained from the exact solubility of wide classes of the stochastic integro-diferential equations. The one-telegraph pre-Gaussian noise has been used to study the symmetric and asymmetric resonance fluorescence spectrum under the influence of collisions [5–7]. The formalism used in these works proposes that one can model the environment noise affecting a given system by the pre-Gaussian processes. In other papers, one can investigate so-called decoherence of a single system, such as a qubit treated as a two-level atom [8, 9]. It is well-known that one of the central problems of quantum information is decoherence treatment.

The case of one-telegraph pre-Gaussian pump in a Raman ring laser has been investigated in our previous work [10]. We have shown that the Stokes output of this laser tends to stabilize under the influence of the one-telegraph pre-Gaussian pump. Now we extend our formalism to the case of two-telegraph pre-Gaussian pump and show that the noise reduction process is much stronger. Moreover, our results approach very well the results derived for the case of colored chaotic Gaussian pump [11].

## II. THE EQUATION FOR THE STATIONARY PROBABILITY DISTRIBUTION IN A NONLINEAR SYSTEM UNDER THE INFLUENCE OF TWO-TELEGRAPH NOISE

We start with the general nonlinear differential stochastic equation:

$$\frac{dx}{dt} = f(x) + z(t)\, g(x). \tag{1}$$

In this equation $f(x)$ and $g(x)$ are two arbitrary functions of the investigated dynamical variable $x$, $z(t)$ is the two-telegraph pre-Gaussian noise. This noise has the following properties

$$z(t) = x_1(t) + x_2(t), \tag{2}$$

$$\langle x_i(t) \rangle = 0, \tag{3}$$

$$\left\langle x_i(t)\, x_j(s) \right\rangle = \Delta^2 \exp\left(-\sigma\,|t-s|\right). \tag{4}$$

According to [12], we introduce the quantity $\rho$ of the investigated dynamical variable $x$as

$$\rho(\xi, t) = \delta(\xi - x(t)). \tag{5}$$

Then the stochastic average of $x$can be written in the form:

$$\langle x \rangle = \int \xi\, P(\xi, t)\, d\xi, \tag{6}$$

where $P(\xi,t)$ is the probability distribution of $x$that is expressed by a path integration with a functional measure given in Ref. [12]. From (5) and (1) we obtain the Liouville equation as

$$\frac{d\rho}{dt} = -\frac{\partial}{\partial x} f(x)\,\rho - z(t)\,\frac{\partial}{\partial x} g(x)\,\rho. \tag{7}$$

The distribution function $P(t) = \langle \rho(t) \rangle$ is a solution of the integro-differential equation:

$$\frac{d}{dt} P(t) = -\frac{\partial}{\partial x} f(x)\, P(t) - \int\limits_0^t K(t-s)\, P(s)\, ds \tag{8}$$

with the Laplace transformed kernel ($t \geq 0$):

$$\hat{K}(z) = \int\limits_0^\infty e^{-zt} K(t)\, dt = i\frac{\partial}{\partial x} g(x)\, \frac{\Delta^2}{z+\sigma+\frac{\partial}{\partial x} f(x) + i\frac{\partial}{\partial x} g(x)\,\frac{\Delta^2}{z+2\sigma+\frac{\partial}{\partial x} f(x)}\, i\frac{\partial}{\partial x} g(x)}\, i\frac{\partial}{\partial x} g(x). \tag{9}$$

The steady state distribution function is then a solution of the equation:

$$\left(-\frac{\partial}{\partial x} f(x) - \hat{K}(0)\right) P_{ns} = 0. \tag{10}$$

The equation (10) leads to the following ordinary differential equation of second order:

$$\left(2g - \frac{f^2}{\Delta^2 g}\right)\frac{d^2 y}{dx^2} + \left(\frac{2\sigma g}{f} - \frac{3\sigma f}{\Delta^2 g} - \frac{1}{\Delta^2}\frac{d}{dx}\left(\frac{f^2}{g}\right) + 2\frac{dg}{dx}\right)\frac{dy}{dx} + \left(-\frac{2\sigma^2}{\Delta^2 g} - \frac{\sigma}{\Delta^2}\frac{d}{dx}\left(\frac{f}{g}\right)\right) y = 0, \tag{11}$$

with

$$y(x) = \int^x \frac{f P_{ns}}{g}. \tag{12}$$

It is worth nothing that the integro-differential equation in the steady regime leads to an ordinary differential equation of second order for different powerful methods of solving have been developed. $f(x)$ and $g(x)$ are just rational functions for many interesting problems of quantum optics, so equation (11) reduces to the Fuchs type equation.

If we consider the case which only has one telegraph then our result is exactly the same as what has been obtained by Kitahara *et al.* [13] and has been used in our previous paper [10]. The case of two telegraphs is a subject of our analysis in the next sections.

## III. NOISE REDUCTION IN A RAMAN RING LASER FOR THE CASE OF TWO-TELEGRAPH NOISE

Now we study the single mode Raman laser in a ring cavity. This laser is described by the following system of differential equations:

$$\begin{aligned} \frac{dE_p(t)}{dt} &= -E_s(t) - \lambda\left[E_p(t) - Y(t)\right], \\ \frac{dE_s(t)}{dt} &= E_p(t)\, Q^*(t) - \mu E_s(t), \\ \frac{dQ(t)}{dt} &= E_p(t)\, E_s^*(t) - \nu Q(t), \end{aligned} \tag{13}$$

where $Q(t)$ is the atomic polarization, $E_p(t)$ is the pump field, $E_s(t)$ is the Stokes field, $Y(t)$ is the injected pump inside the cavity, $1/\lambda$ and $1/\mu$ are the lifetimes of cavity eigenmodes corresponding to the Pump and Stokes fields respectively, $1/\nu$ is the polarization dephasing time. When $Y(t)$ represents a colored light we have:

$$\langle Y(t)\rangle = 0, \quad \left\langle Y(t)Y^*\left(t'\right)\right\rangle = 4\Delta^2 \exp\left(-\sigma\left|t-t'\right|\right), \tag{14}$$

where $1/\sigma$ is the coherence time of the injected pump. Under typical experimental conditions the following inequalities hold:

$$\lambda >> \nu, \mu, \sigma. \tag{15}$$

Therefore, the pump field may be eliminated adiabatically and in (13) we restrict ourselves to the case when

$$\nu >> \mu, \sigma. \tag{16}$$

Under the condition (16) $Q(t)$ can be eliminated adiabatically too. These eliminations with the notation $x = E_s^*(t)E_s(t)$ lead to the stochastic equation as

$$\frac{dx}{dt} = -2\left[\mu - \frac{\nu\lambda^2\left|Y(t)\right|^2}{(x+\nu\lambda)^2}\right]x. \tag{17}$$

It has been shown in Ref. [10] that the stochastic differential equation (17) is quite difficult to solve and some further supposition should be proposed. Now we show that this equation is exactly solved for the case of two-telegraph noise. For this purpose we suppose that $Y(t) = \frac{1}{\sqrt{2}}(Y_1 + iY_1)$, where $Y_1 = Y_0 + I_t$, $I_t$ is two-telegraph noise, in which $I_t$ can receive two values $2\Delta$ and $-2\Delta$. Then we receive the equation in the form:

$$\frac{dx}{dt} = f(x) + I_t g(x) \tag{18}$$

with

$$f(x) = -2\mu x + \frac{2\nu\lambda^2\left(Y_0^2+4\Delta^2\right)}{(x+\nu\lambda)^2}x \quad , \tag{19}$$

$$g(x) = \frac{4\nu\lambda^2 Y_0}{(x+\nu\lambda)^2}x. \tag{20}$$

Substituting equations (19) and (20) into the expression (11), we receive the following second order linear differential equation:

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = 0, \tag{21}$$

where

$$\begin{aligned} A(x) =& 2x^2(x+\nu\lambda)\left[\mu(x+\nu\lambda)^2 - \nu\lambda^2\left(Y_0^2+4\Delta^2\right)\right] \\ &\times\left[32\nu^2\lambda^4Y_0^2\Delta^2 - \mu^2(x+\nu\lambda)^4 + 2\mu\nu\lambda^2(x+\nu\lambda)^2\left(Y_0^2+4\Delta^2\right) - \nu^2\lambda^4\left(Y_0^2+4\Delta^2\right)^2\right], \end{aligned} \tag{22}$$

$$
\begin{aligned}
B(x) =x \Big[ & 3\sigma\mu^2 (x+\nu\lambda)^7 - 32\sigma\nu^2\lambda^4 Y_0^2\Delta^2 (x+\nu\lambda)^3 - 6\sigma\mu\nu\lambda^2 (x+\nu\lambda)^5 \left(Y_0^2+4\Delta^2\right) \\
& + 3\sigma\nu^2\lambda^4 (x+\nu\lambda)^3 \left(Y_0^2+4\Delta^2\right)^2 - 2\mu^3 (x+\nu\lambda)^7 + 4\mu^2\nu\lambda^2 x (x+\nu\lambda)^4 \left(Y_0^2+4\Delta^2\right) \\
& + 6\mu^2\nu\lambda^2 (x+\nu\lambda)^5 \left(Y_0^2+4\Delta^2\right) - 4\mu^3 x (x+\nu\lambda)^6 + 4\mu\nu^2\lambda^4 x (x+\nu\lambda)^2 \left(Y_0^2+4\Delta^2\right)^2 \\
& - 6\mu\nu^2\lambda^4 (x+\nu\lambda)^3 \left(Y_0^2+4\Delta^2\right)^2 - 4\nu^3\lambda^6 x \left(Y_0^2+4\Delta^2\right)^3 + 2\nu^3\lambda^6 (x+\nu\lambda) \left(Y_0^2+4\Delta^2\right)^3 \\
& - 128\mu\nu^2\lambda^4 x Y_0^2\Delta^2 (x+\nu\lambda)^2 + 128\nu^3\lambda^6 x Y_0^2\Delta^2 \left(Y_0^2+4\Delta^2\right) \\
& + 64\mu\nu^2\lambda^4 Y_0^2\Delta^2 (x+\nu\lambda)^3 - 64\nu^3\lambda^6 Y_0^2\Delta^2 (x+\nu\lambda) \left(Y_0^2+4\Delta^2\right) \Big],
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
C(x) =&\sigma (x+\nu\lambda)^3 \left[\mu (x+\nu\lambda)^2 - \nu\lambda^2 \left(Y_0^2+4\Delta^2\right)\right] \\
&\times \left[2\mu\nu\lambda x - \sigma\nu^2\lambda^2 - 2\sigma\nu\lambda x - \gamma x^2 + 2\mu x^2\right].
\end{aligned} \tag{24}
$$

For one-telegraph noise, we obtain expression of the stationary probability density which is exactly the same as in [10]. In the case of two-telegraph noise, solving the equation (21) by series method we receive its solution in the following form:

$$
\begin{aligned}
y(x) =&\frac{C_1}{x^\alpha} \left(1 + a_1 x + b_1 x^2 + c_1 x^3 + d_1 x^4 + e_1 x^5 + \ldots\right) \\
&+ \frac{C_2}{x^\beta} \left(1 + a_2 x + b_2 x^2 + c_2 x^3 + d_2 x^4 + e_2 x^5 + \ldots\right).
\end{aligned} \tag{25}
$$

From formula (12), we receive expression of the stationary probability density of the form:

$$
P_{ns}(x) = \frac{2\nu\lambda^2 Y_0}{\nu\lambda^2 \left(Y_0^2+4\Delta^2\right) - \mu (x+\nu\lambda)^2} \frac{d}{dx} y(x). \tag{26}
$$

The formula for $P_{ns}(x)$ of the model described here is very long and much more complicated in comparison with the case of one-telegraph noise, so we do not present it here. To compare our results with those from our previous paper [10], we assume the same values for the parameters (in $Y_0$ units) involved in the problem and boundaries $x_1$ and $x_2$ as

$$
x_1 = \lambda \left[\sqrt{\frac{\nu}{\mu}} \left|Y_0 - \Delta\right| - \nu\right], \tag{27}
$$

$$
x_2 = \lambda \left[\sqrt{\frac{\nu}{\mu}} \left|Y_0 + \Delta\right| - \nu\right]. \tag{28}
$$

Figures 1 - 3 show explicitly noise reduction in the nonlinear system for both the case of one- and two-telegraph pre-Gaussian pump. Figure 4 is added of figures 1-3 for the case of two-telegraph pre-Gaussian pump.

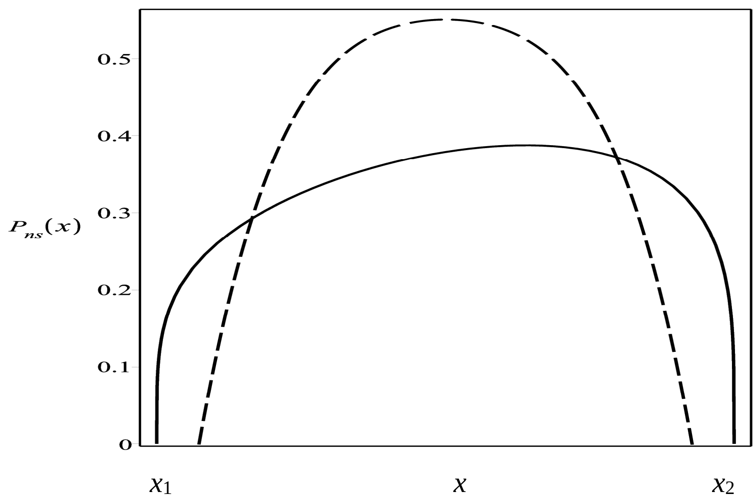


**Fig. 1.** The function of stationary probability distribution $P_{ns}(x)$ (in $Y_0^2$unit) near the boundaries $x_1$ and $x_2$ for $\sigma = 10^{-6}$, $\Delta = 8$, $\lambda = 0.1$, $\nu = 10^{-4}$, $\mu = 10^{-7}$. Solid line is for the case of one-telegraph noise, whereas dashed line is for the case of two-telegraph noise.

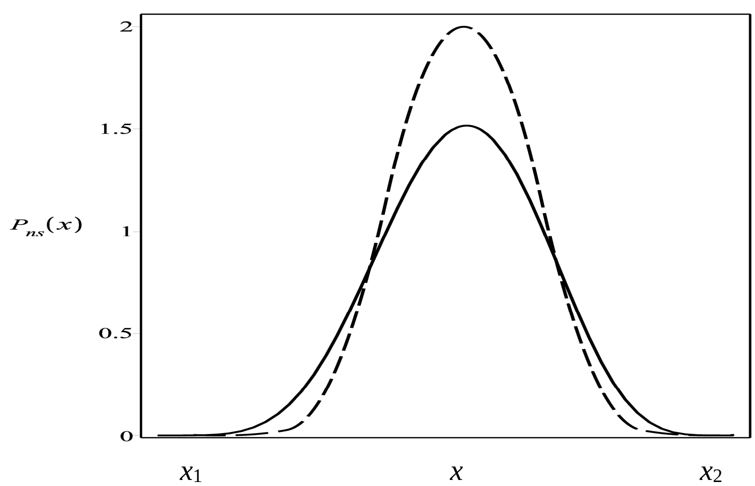


**Fig. 2.** The function of stationary probability distribution $P_{ns}(x)$ (in $Y_0^2$unit) near the boundaries $x_1$ and $x_2$ for $\sigma = 5.10^{-6}$, $\Delta = 8$, $\lambda = 0.1$, $\nu = 10^{-4}$, $\mu = 10^{-7}$. Solid line is for the case of one-telegraph noise, whereas dashed line is for the case of two-telegraph noise.

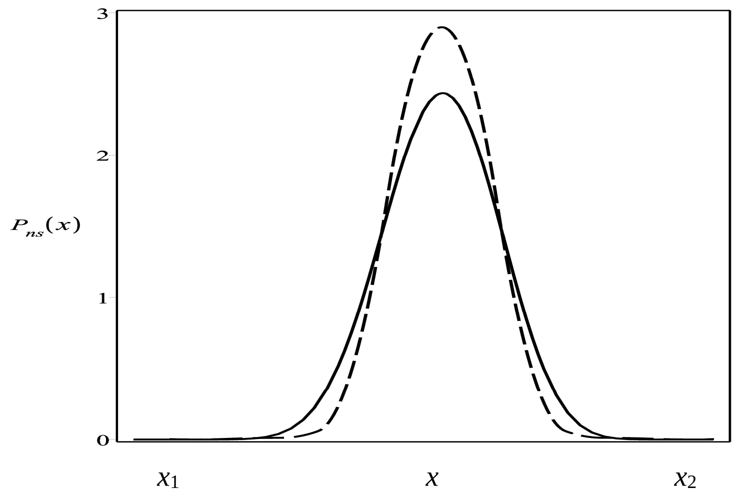


**Fig. 3.** The function of stationary probability distribution $P_{ns}(x)$ (in $Y_0^2$unit) near the boundaries $x_1$ and $x_2$ in which $\sigma = 10^{-5}$, $\Delta = 8$, $\lambda = 0.1$, $\nu = 10^{-4}$, $\mu = 10^{-7}$. Solid line is for the case of one-telegraph noise, whereas dashed line is for the case of two-telegraph noise.

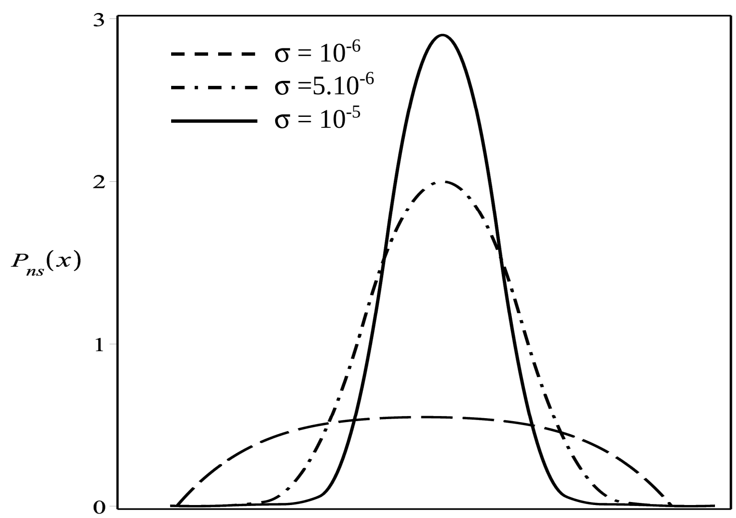


**Fig. 4.** The function of stationary probability distribution $P_s(x)$ (in $Y_0^2$ unit) near the boundaries $x_1$ and $x_2$ for the case of two-telegraph noise with the other values of the $\sigma$in which $\Delta = 8$, $\lambda = 0.1$, $\nu = 10^{-4}$, $\mu = 10^{-7}$.

Figures 1-3 show the steady probability distribution for $\sigma = 10^{-6}$, $\sigma = 5.10^{-6}$ and $\sigma = 10^{-5}$respectively, when the rest of the parameters are constant. In reality when $\sigma$increases, the distribution is narrower. Thus, although alike to noise reduction in the nonlinear system for the case of one-telegraph noise but here the noise reduction is faster than one. By the broad-band pump, the Stokes output tends to be stabilized. In our opinion, for the construction of a Raman laser with a broad-band pump that has been discussed in Refs. [14,15], the creation of the telegraph noise is much easier compared with Gaussian one, so our results are of more direct experimental relevance than in the case of Gaussian noise.

## IV. CONCLUSION

In this paper, the model of a nonlinear system with random telegraph noises is considered and the exactly soluble equations for the stationary probability distribution of fluctuations in this nonlinear system under the influence of two-telegraph noise are derived. In its application, we also showed seemly interesting phenomenon that is the so-called noise reduction in a Raman ring laser. This phenomenon only appears in the nonlinear system, namely, if the value of the noise parameter in the nonlinear system increases then the distribution of stationary probability shrinks. It has been shown that for the case of two-telegraph noise, the noise reduction in a Raman ring laser is faster compared with those in the case of one-telegraph noise. The tending of the Stokes output of a Raman ring laser to the stabilization under influence of the broad-band telegraph pump has been shown. This phenomenon could be realized experimentally in a much easier manner than for the case of Gaussian pump because the construction of the injected telegraph pump signal is much easier compared with the Gaussian signal.

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