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A SPHERICALLY SYMMETRIC SOLUTION OF $R+\lambda R^2$ GRAVITY

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Abstract. *We shortly review the metric formalism for the $f(R)$ gravity. Based on the metric formalism, we study the spherically symmetric static empty space solutions with the gravity Lagrangian $L = R+\lambda R^2$. We found the general metric that described the static empty space with the spherically symmetry. Our result is more general than Schwarzschild solution, specially the predicted metric is perturbed Schwarzschild metric of the Einstein theory.*

Keywords: Einstein theory, Extended Einstein theory, Schwarzschild metric, Extended Schwarzschild metric..

I. INTRODUCTION

Early time inflation is a very attractive idea in order to solve many serious problems of the Hot Big Bang cosmological model originating in the mystery of the initial conditions of our Universe such as flatness, horizon, monopole problems [1]. The inflation can be created by only one scalar field. In particular, many simplest models give different predictions about the scalar perturbation spectrum n_s and the ratio r of squared amplitudes of tensor and scalar perturbations. Thus the predicted results can be distinguished experimentally by CMB observations [2]. On the contrary, if predictions for (n_s,r) coincide with experimental results, it is very difficult to determine which model is realized in Nature.

We would like to emphasize that the scalar-field models of inflation correspond to a modification of the energy-momentum tensor in Einstein equations. However there is another approach to explain the acceleration of the universe. This can be done by modifying gravitational theory compared to general relativity (GR). In GR, Lagrangian is a linear function of the Ricci scalar R namely, $L = R-2\Lambda$, with Λ is the cosmological constant. The presence of Λ can gives an exponential expansion of the universe. However, we can not use this solution for inflation because there is no connection between the inflationary period and the radiation period. One of the simplest modification to GR is the $f(R)$ gravity in which the Lagrangian density is a function of the Ricci scalar R. For more details, the reader can see in [3–6].

In fact, there are two formalisms to derive the equation of motion from the action in $f(R)$ gravity. The first is the standard metric formalism in which the the equation of motion is derived by the variation of the action with respect to the metric tensor $g_{\mu\nu}$ and the affine connection $\Gamma^{\alpha}_{\mu\nu}$ depends on $g_{\mu\nu}$. The second is the Palatini formalism in which $g_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ are two independent variables if we take the variation of the action. Two formalisms give different equations of motion for a non-linear Lagrangian density in R but for GR action these equations of motion are identified [7].

The simple Lagrangian density contained non-linear R terms is model with $f(R) = R+\lambda R^2$. This model of inflation was first proposed by Starobinsky in 1980 [8]. This model is well consistent with the temperature anisotropies observed in CMB and thus it can be a viable alternative to the scalar- field models of inflation. In this work, we concentrate to study spherically symmetric solutions based on the gravity model $f = R+\lambda R^2$.

The work is organized as follows. In Section **II**, we shortly review the method to obtain the equations of motion $f(R)$ gravity based on the metric formalism. In Section **III**, we apply the equation of motion $f(R)$ to find the spherically symmetric solution of $R+\lambda R^2$ gravity.

II. THE METRIC FORMALISM FOR $f(R)$ GRAVITY

The total action for $f(R)$ theory takes the form

$$S_{met} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_M(g_{\mu\nu},\psi) \tag{1}$$

In order to derive the equations of motion, the variation of the action with respect to the metric tensor $g_{\mu\nu}$ and the affine connection $\Gamma^{\alpha}_{\mu\nu}$ depending on $g_{\mu\nu}$ are taken. However, the boundary condition for fixing metric such as $\delta g_{\mu\nu}|_{\partial V} = 0$ is not enough to derive the equation of motion because the surface terms in the variation of the action (1) do not consist of total derivative of some quantities. Therefore, we have to add more terms in the action in order to cancel the surface term. The way comes from the fact that the action includes higher-order derivatives of the metric and, therefore, it should be possible to fix more degrees of freedom on the boundary than those of the metric itself. This work was done by Gibbons, Hawking and York, specially the adding term is given as

$$S_{GYH} = 2\oint_{\partial V} d^3y\varepsilon\sqrt{|h|}K, \text{ where } K = \nabla_\mu n^\mu \text{ and } g^{\mu\nu} = h^{\mu\nu} + \varepsilon n^\mu n^\nu. \tag{2}$$

After taking variation with respect to the metric element and some manipulations and modulo surface terms, we obtain the equation of motion as follows

$$f'_R R_{\mu\nu} + g_{\mu\nu}\Box f'_R - \nabla_\mu\nabla_\nu f'_R - \frac{1}{2}f g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3}$$

with $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}$. The prime denotes the partial derivatives with respect to R, ∇ is the covariant derivative and $\Box = \nabla_\mu\nabla^\mu$.

III. A SPHERICALLY SYMMETRIC SOLUTION OF $R+\lambda R^2$ GRAVITY

Let us take into account the case of spherical-symmetry in $f(R)=R+\lambda R^2$-gravity. First, applying the Eq.(3) for the case $f(R)=R+\lambda R^2$ we obtain

$$(1+2\lambda R)R_{\mu\nu}+2\lambda g_{\mu\nu}\Box R-2\lambda\nabla_\mu\nabla_\nu R-\frac{1}{2}(R+\lambda R^2)g_{\mu\nu}=\kappa T_{\mu\nu} \tag{4}$$

Multiplying both sides of the Eq. (4) with $g^{\mu\nu}$, we get the result

$$R=6\lambda\Box R-\kappa T,\ where\ T=g^{\mu\nu}T_{\mu\nu} \tag{5}$$

Inserting the scalar Ricci R from the Eq.(5) into the Eq.(4), we get a result as follows

$$(1-2\lambda\kappa T)R_{\mu\nu}-\frac{1}{6}g_{\mu\nu}R=g_{\mu\nu}(\frac{1}{3}\kappa T-\frac{\lambda}{2}\kappa^2T^2)-2\lambda\nabla_\mu\nabla_\nu(\kappa T) \tag{6}$$

Because of small λ, we omit terms proportional to λ^2 and multiply both sides of the Eq. (6) with $g^{\mu\nu}$. The Eq. (7) can be written as follows

$$R=-\kappa T-6\lambda\Box(\kappa T) \tag{7}$$

From Eqs. (6), (7), we obtain the result as follows

$$(1-2\lambda\kappa T)R_{\mu\nu}+\frac{1}{6}g_{\mu\nu}(\kappa T+6\lambda\Box(\kappa T))=\kappa T_{\mu\nu}-g_{\mu\nu}(\frac{1}{3}\kappa T-\frac{\lambda}{2}k^2T^2)-2\lambda\nabla_\mu\nabla_\nu(\kappa T). \tag{8}$$

Performing multiplication both sides of the equation (8) with $(1+2\lambda\kappa T)$ and also omit terms proportional to λ^2, we get

$$\begin{aligned} R_{\mu\nu} &= k(1+2\lambda\kappa T)T\mu\nu-\frac{1}{2}g_{\mu\nu}(\kappa T+\lambda k^2T^2)-2\lambda\nabla_\mu\nabla_\nu(\kappa T)-\lambda g_{\mu\nu}\Box(\kappa T),\\ R^\mu_\nu &= k(1+2\lambda\kappa T)T^\mu_\nu-\frac{1}{2}\delta^\mu_\nu(\kappa T+\lambda k^2T^2)-2\lambda\nabla^\mu\nabla_\nu(\kappa T)-\lambda\delta^\mu_\nu\Box(\kappa T). \end{aligned} \tag{9}$$

We studied spherically symmetric static space. It means that all the components T^ν_μ are valid except $T^o_o=\rho(r)c^2$. In this case, we g et $T\equiv Tr(T^{\mu\nu})=\rho c^2$. Hence, the Eq.(9) can be written:

$$R^0_0 = \frac{1}{2}\kappa T+\frac{3}{2}\lambda k^2T^2-2\lambda\nabla^0\nabla_0(\kappa T)-\lambda\Box(\kappa T). \tag{10}$$

Because $\frac{\partial(\kappa T)}{\partial x^0}=0$ and $\lambda\Gamma^\sigma_{\mu\nu}$ is very small, we can write the Eq. (10) as follows

$$R^0_0=\frac{1}{2}\kappa c^2\rho(r)+\frac{3}{2}\lambda k^2c^4\rho^2(r)-\lambda\kappa c^2\Delta\rho(r), \tag{11}$$

with $\Delta\rho(r)=\frac{\partial^2\rho(r)}{(\partial x^i)^2}$, $(i=1,2,3)$.

Now, let us calculate the R^μ_ν components based on the metric formalism. If we ignore the quadratic terms of metric $g^{\mu\nu}$, the R^μ_ν components is written as

$$R^\mu_\nu=g^{\mu\rho}(\partial_\rho\Gamma^\rho_{\delta\nu}-\partial_\nu\Gamma^\rho_{\delta\rho}). \tag{12}$$

Because of the spherically symmetric, the R^0_0 component have a form as follows

$$R^0_0=g^{0\delta}\partial_i\Gamma^i_{\delta 0}=g^{00}\partial_i\Gamma^i_{00},\qquad (i=1,2,3) \tag{13}$$

For the spherically symmetric static space, we got

$$\Gamma^i_{00} = -\frac{1}{2}g^{ik}\partial_k g_{00}. \tag{14}$$

The weakness of the gravitational field is once again expressed as our ability to decompose the metric into the flat Minkowski metric plus a small perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| << 1, \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \qquad . \tag{15}$$

if we ignore the terms $\delta g^{ik}\frac{\partial(\delta g_{00})}{\partial x^k}$, we got

$$\Gamma^i_{00} = \frac{1}{2}\delta^{ik}\partial_k g_{00}. \tag{16}$$

Replacing them into Eq. (16), we have

$$R^0_0 = \frac{1}{2}\triangle g_{00}. \tag{17}$$

From the Eq.(11) with Eq. (16), we obtain

$$\triangle g_{00}(r) = \kappa c^2\rho(r) + 3\lambda k^2 c^4 \rho^2(r) - 2\lambda\kappa c^2 \triangle\rho(r). \tag{18}$$

We would like to emphasize that $g_{00} = 1 + h(r)$ and $h(r) = 0$ if $r \to \infty$. In this limit, the Eq. (18) can be written as follows

$$\triangle h(r) = -4\pi f(r) \tag{19}$$

with $f(r) = -\frac{1}{4\pi}[\kappa c^2\rho(r) + 3\lambda k^2 c^4\rho^2(r) - 2\lambda\kappa c^2\triangle\rho(r)]$.

Because of the boundary condition $h(r) = 0$ at $r \to \infty$, the solution of the Eq.(19) can be written as follows

$$h(r) = \int \frac{f(r')}{\sqrt{(\vec{r}-\vec{r'})^2}}dr', \tag{20}$$

$$\begin{aligned} \Rightarrow h(r) \quad &= \quad -\frac{\kappa c^2}{4\pi}\int \frac{\rho(r')dr'}{\sqrt{r^2+r'^2-2\vec{r}\vec{r'}}} - \frac{3\lambda k^2 c^4}{4\pi}\int \frac{\rho^2(r')dr'}{\sqrt{r^2+r'^2-2\vec{r}\vec{r'}}} \\ &\quad + \frac{2\lambda k^2 c^2}{4\pi}\int \frac{\triangle\rho(r^2)d\vec{r'}}{\sqrt{r^2+r'^2-2\vec{r}\vec{r'}}} = I_1 + I_2 + I_3. \end{aligned} \tag{21}$$

In order to calculate the integrals, we use spherical coordinates, namely

$$d\vec{r'} = r'^2\sin\theta dr' d\theta d\varphi, \quad \Delta\rho(r') = \frac{1}{r'^2}\frac{\partial}{\partial r'}(r'\frac{\rho(r')}{\partial r'}), \quad \vec{r}\vec{r'} = rr'\cos\theta. \tag{22}$$

We would like to denote that $\rho(r') = 0$ if $r' > r_o$, r_o is the radius of the object. Hence the integral I_1, I_2, I_3 can be calculated.The final result is given as follows

$$\begin{aligned} h(r) \quad &= \quad -\frac{\kappa c^2 m}{4\pi}\frac{1}{r} - \frac{9\lambda k^2 c^4 m^2}{16\pi^2 r_0^3}\frac{1}{r} + \frac{2\lambda k^2 c^2 r_0^2}{r}\frac{\partial\rho(r')}{\partial r'}\,|_{r'=r_0} \\ g_{00}(r) \quad &= \quad 1 + h(r) = 1 - \frac{\kappa c^2 m}{4\pi}\frac{1}{r} - \frac{9\lambda k^2 c^4 m^2}{16\pi^2 r_0^3}\frac{1}{r} + \frac{2\lambda k^2 c^2 r_0^2}{r}\frac{\partial\rho(r')}{\partial r'}\,|_{r'=r_0}\,. \end{aligned} \tag{23}$$

Because the density of the material is uniform, $\frac{\partial \rho(r')}{\partial r'} = 0$ and out side of gravity object, the energy of momentum tensor equals zero. Hence, the function $g_{00}(r)$ can be written as follows

$$g_{00}(r) = 1 - \frac{\kappa c^2 m}{4\pi}\frac{1}{r} - \frac{9\lambda k^2 c^4 m^2}{16\pi^2 r_0^3}\frac{1}{r}. \tag{24}$$

On the other hand the best way that we can do for a general metric in a spherically symmetric space time is writing the metric elements in form

$$g_{\mu\nu} = -diag(-c^2 e^{\beta}, e^{\alpha}, r^2, r^2 \sin^2\theta) \tag{25}$$

. The next step is to actually solve gravity equations, which will allow us to determine explicitly the functions $\alpha(t,r)$ and $\beta(t,r)$. For space outside the gravity object, $r > r_o$, the energy momentum tensor equals zero. Hence the Eqs. (8) and (7) lead to

$$R^{\mu}_{\nu} = 0, \qquad R = 0. \tag{26}$$

So, we have $R^{\mu}_{\nu} - \frac{1}{2}R = 0$. This is Einstein's equations for the spherically symmetric vacuum. The exactly solutions are given as

$$g_{00} = 1 - \frac{C}{r}, \qquad g_{11} = -\frac{1}{1-\frac{C}{r}}, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta. \tag{27}$$

Because $g_{00}(r) = 1 - \frac{\kappa c^2 m}{4\pi}\frac{1}{r} - \frac{9\lambda k^2 c^4 m^2}{16\pi^2 r_0^3}\frac{1}{r}$ then $g_{11} = \frac{-1}{1-\frac{\kappa c^2 m}{4\pi}\frac{1}{r} - \frac{9\lambda k^2 c^4 m^2}{16\pi^2 r_0^3}\frac{1}{r}}$. Our final result is the celebrated Schwarzschild metric,

$$ds^2 = (1 - \frac{\kappa c^2 m}{4\pi}\frac{1}{r} - \frac{9\lambda k^2 c^4 m^2}{16\pi^2 r_0^3}\frac{1}{r})dt^2 + \frac{-1}{1-\frac{\kappa c^2 m}{4\pi}\frac{1}{r} - \frac{9\lambda k^2 c^4 m^2}{16\pi^2 r_0^3}\frac{1}{r}}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\varphi^2. \tag{28}$$

Let us remain that the Schwarzschild metric is an exact spherically symmetric vacuum solution to Einstein equations. The solution given in (28) is general solution which is perturbed Schwarzschild metric. This solution can be recovered the Minkowski space as the same as that of the spherically symmetric vacuum solution to Einstein equations. Especially, in the limit $M \to 0$ and $r \to \infty$ the solution given in (28) recovers the Minkowski space. On the other hand, the Schwarzchild geometry of the Einstein equation contains the singularities, namely the metric coefficients become infinite at $r = 0$ and $r = \frac{\kappa c^2 m}{4\pi}$. In the case of the spherically symmetry static solution to $f(R)$ equation with $f(R) = R + \lambda R^2$, the metric coefficients become infinite at $r = 0$ and $r = \frac{\kappa c^2 m}{4\pi} + \frac{9\lambda \kappa^2 c^4 m^2}{16\pi^2 r_0^3}$.

In conclusion, based on the context of the metric formalism of the $f(R)$ -gravity, the Lagrangian $L = R + \lambda R^2$ has been developed to study dynamics of spherically symmetric metrics. We found the general metric that described the static empty space with the spherically symmetry. The singularities happened at $r = 0$ and $r = \frac{\kappa c^2 m}{4\pi} + \frac{9\lambda \kappa^2 c^4 m^2}{16\pi^2 r_0^3}$. These singularities are different from those of the Schwarzchild metric. The metric formalism reveals a useful approach to select consistent $f(R)$-models and to find out exact spherically symmetric vacuum solution.

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