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BETHE-SCHWINGER EFFECTIVE RANGE THEORY AND LEHMANN AND WEINBERG CHIRAL PERTURBATION THEORIES

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Dedicated to the Memory of Prof. K. Nishijima

Abstract. *This paper is a brief review of low energy soft hadronic physics, starting from the invention of the low energy effective range theory in the late 40's due to Bethe and Schwinger for nucleon-nucleon scattering, and its generalization to the static Chew-Low model for pion nucleon scattering, to the present development of the Lehmann and Weinberg Chiral Perturbation Theories. It is pointed out that a consistent low energy calculation can be achieved with the incorporation of the unitarity relation in the Chiral Perturbation Theory.*

I. INTRODUCTION

It is appropriate to write a brief review of the low energy non relativistic effective range theory as it was first started out in 1948 [1, 2, 3] and the recent development of the Chiral Perturbation Theory by Weinberg and others (ChPT)[4], the Unitarized Chiral Perturbation Theory by Lehmann(UChPT)[5] and others [6, 7], i.e the Chiral Perturbation Theory (ChPT) for pions with the incorporation of unitarity relation. I want to argue that the UChPT is a logical follow-up of the non relativistic effective range theory for nucleon-nucleon scattering developed in the late 40's by Bethe and Schwinger and 10 years later, the Chew-Low theory for the πN scattering, Δ resonance. [9].

In fact the development of UChPT by Lehmann [5] was done 7 years earlier than the perturbation approach of Weinberg [4]. Unfortunately, because Lehmann calculation was done in the chiral limit and that his unitarisation of the partial wave amplitudes for the pion pion scattering was done in the old-fashioned effective range theory, his line of approach to the chiral theories was not appreciated by workers in ChPT.

The important question is whether one can use perturbation theory for strong interaction physics? On one hand we have the effective range theory which implies that strong interaction cannot be treated perturbatively, on the other hand it is now a fashion to treat chiral theories by the perturbation theory (ChPT) as advocated by Weinberg [4] and his followers. The question is which line of approach is correct or are they both correct?

ChPT may be valid at a very low energy where the constraint of the elastic unitarity could be unimportant (see, however, the discussion in the section on the form factor calculation), this situation is no longer satisfactory as the energy region of interest is closer to the resonance region. How close or far from the resonant region is difficult to

define precisely. It is therefore useful to have a theory which is also valid at low energy and also in the resonance region. As we shall see, the failure of the ChPT approach in calculating of the phases of the pion, pion Kaon scalar and vector form factors, in its early development stage at *one loop level*, reflects the lack of the unitarity in the theory [6]. Recent ChPT calculations of these processes at two loop level, in my opinion, have removed to some extend these difficulties. It is regrettable, however, that in recent ChPT calculations the *phases* of the matrix elements are not calculated in order to compare with experiments or ChPT theoretical prescription. For example for $\pi\pi$ elastic scattering, the phase shifts are identified with the real part of the amplitudes which is in itself the unitarization prescription in ChPT [6, 49]; it can also, however, also be identified with the ratio of the real to imaginary parts of the partial wave matrix element. The difference of these two calculations reflects the accuracy of the ChPT approach.

This review is rather partial, emphasizing mostly only on the dispersive approach where unitarity is respected which I have made a number of contributions and which I am familiar with. I must admit that I am not familiar with most recent enormous ChPT works and hence I have to concentrate on my previous publications. Most of the discussions are given by older calculations which I have not time to update with new experimental results. I think it is useful to summarize some these old calculations before they got lost because the current fashion of using perturbation calculation for strong interaction, although this may be incorrect. Whenever possible, I shall compare the results of the non-perturbation approach with those of ChPT. I have to apologize to many authors whose related works are not discussed in this review.

Being a physicist belonging to the older generation and being brought up during the early day of the development of strong interaction particle physics, I was deeply influenced by such general principles as the unitarity and dispersion relation because they were extensively used at that time.

During my second year in the Graduate School at Cornell, I had the privilege to follow the nuclear physics course taught by Professor Bethe who lectured on his low energy effective range theory [3] and also to attend the Particle Physics course where he lectured on the static Chew-Low effective range theory of the Δ resonance [9] and also on the application of dispersion relations. Most physicists nowadays would dismiss these topics are no longer of interest, but I think quite contrary.

My involvement with chiral symmetry did not come until the late 70's after the discovery of the τ lepton. My collaboration with Pham and Roiesnel resultsed in the publication of the current algebra calculations for the hadronic decays of the τ lepton with a follow-up calculation of $e^+e^- \to 4\pi$ where the discrepancy of a factor 20 in the cross section between the soft pion theorem was explained [15, 16, 17]. This result was independently rediscovered 24 years later by Ecker and Unterdorfer [18]. After these works I became interested in the Ke_4 problem and pointed out the role of analyticity and unitarity, in particular the role of the threshold square root singularity for the S-wave pion pion scattering [19]. (It is regrettable that this non-relativistic quantum mechanics name of the square root singularity is nowadays replaced by the name of chiral logarithm which has the threshold square root singularity!). Following this work, using the idea of the square root threshold singularity and with the collaboration of C. Roiesnel, we gave a

resolution to the $\eta \to 3\pi$ rate which was previously calculated by Weinberg giving a too low rate by a factor of 5 [20, 21].

Most of my works on Chiral Symmetry, unlike in the standard ChPT approach [4], were based on Current Algebra and on the supposition that the chiral power series Effective Lagrangian is an effective theory which contains all features of Chiral Symmetry and Current Algebra; the calculated matrix element is valid whenever a power series expansion in momenta of the Nambu-Goldstone boson is legitimate i.e in the region where there are no singularities, e.g. no unitarity cut, and is usually in the unphysical region of energy. The calculated S-Matrix element from the chiral effective lagrangian has to be analytically continued to the physical region (on the cut) with the constraint of unitarity and analyticity. Because we are interested in a low energy theory in the elastic region, the elastic unitarity relation has to be imposed using its full form and not the perturbation unitarity relation as usually done in ChPT [20].

Fortunately in this approach, the elastic unitarity relation enables us to generate the low energy resonance ρ, K* ... which are the main features of the low energy of the soft pion and kaon physics. The elastic unitarity relation can be implemented in the dispersion relation approach by the inverse amplitude method (IAM), the N/D method or simply the resummation of the perturbation series by the Pade aproximant method (valid also for the multichannel problems). This last approach may be a good compromise for those who love perturbation theory and have neglected the unitarization procedure in their perturbation results. Most of my works on K_{l4}, pion, $K\pi$ form factors, hadronic and rare K decays, hadronic τ decays and the $\eta \to 3\pi$ which was done either by myself or with my collaborators followed this line of approach; the calculation can be done on a few page of papers, if not on the back of an envelope as compared with the enormous length of the ChPT calculations [20].

This viewpoint of the effective lagrangian is similar to the problem dealt previously in the literature on the question of deducing physical consequences in the time like region of the form factor (on the cut)from a knowledge of a few terms of a Taylor's series expansion in the momentum transfer of the space like form factors (below the cut)[22]. There is of course no satisfactory answer to this problem. For the pion form factor at low energy, below the inelastic region, our answer is that the elastic unitarity via dispersion relation must be imposed and not the the technique of conformal mapping of mapping the form factor cut plane into a unit circle to do the analytic continuation.

In related processes, the constraint of the elastic relations forces us to make use of the solution of the Muskelishvilli-Omnes [23] integral equation, the inverse amplitude method, the N/D method and also the Pade approximant method if the perturbation method is used. I shall show that the pure perturbation theory is not applicable in the presence of the ρ resonance for the vector pion form factor. Although the elastic unitarity constraint can generate the ρ resonance, it is however not sufficiently accurate to explain the experimental data, and hence we are forced to introduce additional parameters to simulate the inelastic effects at low energy. This is so because in the dispersive approach, the imaginary part of the form factor or scattering amplitudes, gets contribution from of all higher mass intermediate states, a satisfactory low energy theory must minimize their contribution as they are difficult to calculate.

For this reason we can also criticize the calculation of the Chew-Low model in the sense that the approximation of the elastic unitarity is made here without introducing a subtraction constant in order to suppress the contribution from higher mass intermediate states or the inelastic effect. In the world of Chiral Symmetry, similarly to the quantum electrodynamics phenomena (e.g. low energy theorems for Compton scattering), there are also low energy theorems which can set the scale for our calculation. The complication in QCD is that the pseudoscalars π, K, η, unlike the photon have finite mass unlike the photon.

The approach of analyticity and elastic unitarity for low energy physics can be criticized for being unsystematic. This may be true, but let us point out also that although most of the matrix elements of quantum electrodynamics can undeniably be treated by the perturbation theory except the bound state problems which must be treated by the approach of the Bethe-Salpeter equation of the ladder summation. Here in UChPT, the constraints of analyticity and elastic unitarity on the matrix elements, are treated by the IAM, the N/D, the Pade approximant methods or the solution of the Muskelishvilli-Omnes integral equation [23].

The plan of this talk is organized as follows:

The first sections is devoted to the explanation of the effective range theory, which, in the Bethe's approach, is related to the strength and finite range of the potential in the Schrodinger equation. It is shown here that in fact the effective range expansion is due to a more general principle of analyticity and elastic unitarity for the partial wave amplitudes. This means that the usual perturbation expansion calculation for the partial wave amplitudes cannot be consistent with unitarity unless some summation methods have to be used or that the strength of the interaction is sufficiently weak. The presence of the low energy resonances in the pion and Kaon systems (e.g. ρ, K^*...) invalidates this possibility.

The following four sections deal with various unitarisation schemes.

A brief review of the pion pion scattering is given in section 6.

A somewhat detailed study of the vector pion form factor is given in section 7 in order to explain the difficulties of the ChPT approach. We show that there are some problems associated with calculating the pion form factor phase using perturbation theory as previously discussed in reference [6]. We give here the answer to the question of how to incorporate the ChPT calculation to the vector meson dominance (we cannot unambiguously do unless the effective lagrangian is used as advocated in our approach). Two possibilities could be tried: a) The most popular one is to use the ChPT result at some low energy as a low energy theorem to set the scale (as subtraction constants) for the dispersion relation, b) Another possibility is to add Vector Meson dominance or dispersion relation amplitudes to ChPT *amplitudes* at a given order not just at a few points.

In either possibility, we run against an amusing "theorem" stating "there is no such a thing as a small analytic function"[48], that is a small unmeasurable analytic function at low energy can become very large at a higher energy. For example, a small difference at low energy between a resonance amplitude given by the elastic unitarity calculation and that of ChPT calculation can become enormous in the ρ region.

A criterium is tentatively given to test under what circumstances the standard perturbation theory can be used.

In section 8, the Ke_4 problem is discussed.

In section 9, the $\eta \to 3\pi$ is briefly summarized.

In section 10, the $K \to \pi$, $K \to 2\pi$, $K \to 3\pi$ amplitudes are discussed.

In section 11 the $\gamma\gamma \to 2\pi$, $K_S \to 2\gamma$ and $K_L \to \pi^0\gamma\gamma$ are discussed.

In section 12, the $\tau \to K\pi\nu$ and $\tau \to 3\pi\nu$ Decays are mentioned.

Finally in section 13, I briefly discuss the problem related to the calculation of the *absolute* enhancement factor due to the final state pion pion interaction in the $K \to 2\pi$ decay which is of current interest.

II. NON RELATIVISTIC EFFECTIVE RANGE THEORY AND INVERSE AMPLITUDE METHOD

The history of the development of the Effective Range Theory is a long one. As early as 1939, Breit and collaborators [10] suspected that low energy experiments on nucleon nucleon scattering can determine only two parameters in nucleon-nucleon potential, the effective potential depth and range. Subsequently, Landau and Smorodinsky [11, 12] suggested that an effective range expansion for the the S-wave phase shifts δ :

$$k \cot \delta(k^2) = -\frac{1}{a} + \frac{1}{2} r_0 k^2 + ... \tag{1}$$

where k the relative momenta, a the scattering length and r_0 the effective range. The minus in front of a is by convention. We omit the superscripts for the singlet and triplet states for convenience. For the triplet state, a is positive because of the deuteron bound state, and a is negative for the singlet scattering.

Schwinger [1, 2] was the first person to give a general proof of the effective range expansion Eq. (1). His proof is quite complicated and was based on a variational principle. A year later Bethe [3] and others [13, 14] gave a much simpler proof.

The low energy nucleon-nucleon experimental data on the S-waves singlet and triplet states agree very well with the effective range expansion, Eq. (1).

The proof of Eq.(1) using the Shroedinger Equation depends only on the assumption of the finite range of the potential and not on the strength of the potential. It holds for potentials which are strong enough to produce a resonant virtual bound state as in the singlet scattering, or the real triplet (deuteron) bound state. It also holds for a weak scattering potential with a finite range.

Bethe's proof is based on the following physical picture. Let us divide the spatial scattering region into two separate regions, inside and outside the potential. Outside the potential range, the scattering wave function is that of the asymptotic form with the shifted phase shifts, inside the potential, the wave function is distorted under the influence of the action of the potential. The scattering length a is the zero energy wave function which intercepts on the distance axis, and the effective range is proportional to the integral of the difference of the square of the true and asymptotic wave function. Because one works with the Schroedinger equation, it is expected that the effective range expansion is consistent with unitarity.

In order to generalize Eq. (1), to a relativistic situation which is now the central point of the development of the low energy pion physics, in particular the ChPT, it is useful to derive it from the more general principles of analyticity and unitarity of the partial without referring explicitly to the type of potentials, except that they are of finite range.

Let us consider, for example, the S-wave scattering amplitude $f(\nu)$ and omit the subscripts or superscripts spin and isospin. Setting $\nu = k^2$, the elastic unitarity relation is:

$$Imf(\nu) = \rho(\nu) \mid f(\nu) \mid^2 \tag{2}$$

where $\rho(\nu) = \sqrt{\nu}$ is the non-relativistic phase space factor and ν is the square of the relative momentum k. Eq. (2) implies that:

$$f(\nu) = \frac{e^{i\delta(\nu)} \sin\delta(\nu)}{\rho(\nu)} \tag{3}$$

which is the same as:

$$f(\nu) = \frac{1}{\rho(\nu)(-i + \cot\delta(\nu))} \tag{4}$$

and hence any function representing δ, in particular, for $tan\delta$ or $cot\delta$ used in Eq. (3) or Eq. (4) would give rise to a partial wave amplitude satisfying the unitarity relation.

There is however a restriction on the choice of the appropriate function, namely the analytic property of the constructed partial wave amplitudes which can be proved from general principles [8]. The partial waves are, in fact, analytic functions in the complex ν-plane with a right cut on the real axis from 0 to ∞ and a left cut from $-\nu_c$ to $-\infty$ where ν_c is positive. On the right hand cut the unitarity relation Eq. (2) must be satisfied and hence this cut is usually referred as the unitarity cut. The discontinuity across the left cut depends on the characteristic of the potential used in the Schroedinger equation to describe the scattering process. The analytic and unitarity properties of the partial waves are well explained in an article by Blankenbecler et. al. [24].

Let us define the inverse function $g(\nu) = f^{-1}(\nu)$. This function is also analytic with the same right and left cuts, apart from isolated poles coming from the zeroes of $f(\nu)$. On the positive real axis $g(\nu)$ is given by:

$$g(\nu) = \rho(\nu)(-i + \cot\delta(\nu)) \tag{5}$$

and it can analytically be continued throughout the complex ν-plane, because its singularity are just on the real axis and isolated poles.

Let us now write an once subtracted dispersion relation for $g(\nu)$ using the subtraction point at $\nu = 0$. (More than one subtraction at $\nu = 0$ was not possible because the dispersion integral would diverge; more than one subtraction could, however, be made if the subtraction point ν_0 was taken in the gap $-\nu_c < \nu_0 < 0$). For simplicity, we assume that $f(\nu)$ does not have any zero in the complex ν plane:

$$g(\nu) = g(0) - \frac{\nu}{\pi}\int_0^\infty dz \frac{\sqrt{z}}{z(z-\nu-i\epsilon)} + \frac{\nu}{\pi}\int_{-\infty}^{-\nu_c} dz \frac{Img(z)}{z(z-\nu)} \tag{6}$$

where we use for $\nu > 0$, $Img(\nu) = -\sqrt{\nu}$ and for $\nu < -\nu_c$, $Img(\nu) = - \mid g(\nu) \mid^2 Imf(\nu)$. Once we know $Imf(\nu)$ on the left cut, e.g. by perturbation theory, we arrive at a non linear integral equation for $g(\nu)$, just the same as in the Chew Low theory.

The first integral on the R.H.S of this equation can readily be evaluated by considering $i\sqrt{\nu}$ as an analytic function in the cut plane with a cut on the real axis from 0 to ∞. Separating the contribution from the first integral into the principal part and the δ-function contributions, we finally arrive at:

$$g(\nu) = g(0) + L(\nu) - i\sqrt{\nu} \tag{7}$$

where $L(\nu)$ denotes the second integral on the R.H.S. of Eq.(6) which is the left cut contribution i.e. the potential contribution. Instead of solving the integral equation, for the present purpose, let us treat it phenomenologically. For $\nu > 0$, we can expand $L(\nu)$ in a power series of ν, $L(\nu) = \nu \sum_{n=0}^{\infty} \alpha_n \nu^n$ with a radius of convergence ν_c. In the special case when a first few terms are sufficient, from Eq.(7) and Eq.(5), one has:

$$\sqrt{\nu} \cot \delta(\nu) = g(0) + \alpha_0 \nu + \alpha_1 \nu^2 + ... \tag{8}$$

and is just the effective range expansion of Eq.(1). The scattering length is inversely proportional to the subtraction constant $-a^{-1} = g(0)$, the effective range is the first term in the power series; the next term α_1 is the potential shape dependence term.

The expansion of $L(\nu)$ as a power series in ν has usually a small radius of convergence. It is better to approximate this series by a Padé approximant method i.e. the ratio of two polynomials [25, 26]. The zeroes of the polynomial in the denominator become poles in the Padé approximant. If they were on the the real negative ν axis, as they should be, we would have the usual pole approximation for the left hand cut. In the the full relativistic theory for $\pi\pi$ scattering, to be discussed later, we shall treat the left hand cut contribution more realistically.

The above treatment is valid for the S-wave nucleon-nucleon in the singlet state. For the triplet S-wave nucleon-nucleon scattering, because of the deuteron bound state, our treatment must be modified to include the deuteron bound state pole. We then obtain the same effective range expansion, but the two parameters scattering length and effective range, are directly related to the binding energy of the deuteron and the residue of the deuteron pole.

What has been discussed previously is not new. In fact it was Noyes and Wong [27] who recognized first that the effective range expansion is due to analyticity and unitarity with the contribution from the left hand cut is apprroximated by a pole approximation. The pole approximation can be regarded as the Pade approximant for the low energy subtraction constants as discussed above.

It may happen that the partial wave amplitudes obtained from the inverse amplitude method (I.A.M) could develop poles on the negative cuts and in the complex ν plane. They must be removed from the constructed amplitude. This removal would result in a violation of the unitarity relation. If the unwanted poles are far from the threshold region with small residues, their effects on the violation of the unitarity would be small in the low energy region and hence could be neglected.

The IAM method, being the simplest one, is not the only method to unitarize the partial wave amplitudes. Because the partial waves have both right and left cuts, they can be written as the product of two cuts whereas the in the I.A.M., they are written as the sum of two cuts. This is the N/D method. In this method, just the same as in the I.A.M., the right cut is treated exactly, and the left cut is treated approximately.

Another method which can be quite useful is the Padé approximant method (P.A.M.) [25, 26]. In this method, the elastic unitarity relation is satisfied, and the left cut is treated perturbatively. This method is particularly useful for perturbation calculations, since the reconstructed series satisfy unitarity and analyticity. We shall come back to the P.A.M. method later when we discuss $\pi\pi$ scattering.

The non relativistic K-matrix approach has to be modified in order to take into account of the analytic properties of the partial wave amplitudes, i.e. the real and imaginary part of the partial waves are related to each other by dispersion relation or Hilbert transform [8].

III. RELATIVISTIC EFFECTIVE RANGE THEORY

For a relativistic theory such as $\pi\pi$ scattering, the phenomenological approach to the effective range theory can similarly be carried out. The phase space factor is now $\rho(\nu) = \sqrt{\frac{\nu}{\nu+\mu^2}}$ where μ is the pion mass. The partial wave amplitude $f(\nu)$ is defined as in Eq. (3), with the new expression for the phase space factor. $f(\nu)$ has the same analytic structure as the non-relativistic one:

$$f(\nu) = \frac{e^{i\delta(\nu)} \sin\delta(\nu)}{\rho(\nu)} \tag{9}$$

For simplicity let us consider from now on, as an example, the S-wave amplitude; higher partial waves because of the kinematical zeros at threshold can be straightforwardly taken care of. Following the same reasoning as in the non-relativistic situation with the same definition for the inverse amplitude $g(\nu) = f^{-1}(\nu)$, the contribution of the principal part of the first integral on the R.H.S. no longer vanishes. Instead of Eq.(8) we arrive at the relativistic effective range:

$$\rho(\nu)\cot(\delta) = g(0) + \frac{2}{\pi}\sqrt{\frac{\nu}{\nu+\mu^2}}\ln\frac{\sqrt{\nu}+\sqrt{\nu+\mu^2}}{2\mu} + \alpha_0\nu + \alpha_1\nu^2 + ... \tag{10}$$

with $g(0) = \mu/a$, where a is the scattering length and we have used the sign convention for the relativistic scattering length, i.e. $a > 0$ for an attractive interaction. In addition to α_0 which is proportional to the effective range, we have the additional logarithm term.

As in the non-relativistic case Eq. (7), $L(\nu)$ can be expanded in a power series and it is convenient to use of the Padé approximant method [25, 26] for this power series.

A generalization of the effective range expansion for the theory of the P-wave, isospin 3/2 πN scattering, the Δ resonance is obtained, using the same line of reasoning. Using the dispersion relation for the P-wave for πN scattering in isospin 3/2 and taking into account of the nucleon poles in the direct and crossed channels and requiring the elastic unitarity condition, one would get a non linear integral equation of the Chew-Low theory

[14]. Using the I.A.M., the unitarity relation can then be treated exactly, and then the left cut is to be treated by perturbation as a first iteration.

The development of the effective range Chew-Low theory [14] is therefore simply a generalization of the non-relativistic theory. This is a first triumph for using analyticty and elastic unitarity to solve a non perturbation problem in a relativistic pion physics.

We shall see below, there is also a simple method which can resum the perturbative approach in a manner that unitarity is satisfied. This is the Padé approximant method (PAM) [25, 26, 6].

IV. PADÉ APPROXIMANT METHOD

Let us write the partial wave perturbation series as:

$$f^{pert}(\nu) = f^0 + f^1(\nu) + ... \tag{11}$$

where f^0 is the tree amplitude which is assumed here, for simplicity, a constant or a real polynomial (otherwise it has only the left hand cut singularity). $f^1(\nu)$ is the one loop amplitude satisfying the perturbative unitarity, for $\nu > 0$:

$$Imf^1(\nu) = \rho(\nu)(f^0)^2 \tag{12}$$

Let us construct the $[0,1]$ Padé approximant:

$$f^{[0,1]}(\nu) = \frac{f^0}{1 - \frac{f^1(\nu)}{f^0}} \tag{13}$$

This equation gives rise to a geometric series constructed out of f^0 and f^1. Expanding the denominator of Eq.(13) in a power series of f^1/f^0, its first two terms agree with the perturbation expansion, Eq.(11). The presence of the remaining terms is to preserve the elastic unitarity condition because:

$$Imf^{[0,1]}(\nu) = \rho(\nu) \mid f^{[0,1]}(\nu) \mid^2 \tag{14}$$

for $\nu > 0$. For $\nu < -\nu_c$, the discontinuity of $f^{[0,1]}$ across the left hand cut is $2i$ times the imaginary part of the Padé amplitude is given by:

$$Imf^{[0,1]}(\nu) = Imf^1(\nu)\frac{\mid f^{[0,1]}(\nu) \mid^2}{(f^0)^2} \tag{15}$$

and hence the same as the perturbation result but it is modified by a factor $\mid f^{[0,1]} \mid^2 /(f^0)^2$.

The phase shifts given by the Padé amplitude is:

$$\rho(\nu)\cot\delta(\nu) = \frac{1}{f^0} - \frac{Ref^1(\nu)}{(f^0)^2} \tag{16}$$

and is a generalisation of the relativistic effective range expansion of Eq.(10). At some energy, if there was a cancellation of the two terms on the R.H.S. of this equation, a resonant state is generated.

The Padé approximant method is similar to the bubble summation for the partial wave [28], but is more general, because it has both unitary (right) and left cuts, whereas the bubble summation has only the unitarity cut.

There are few methods in the relativistic particle physics to treat the non-perturbation problem: the infinite geometric series of the bubble summation used in the study of the Nambu-Jona-Lassinio model, or the infinite ladder summation of the Bethe-Salpeter equation used to treat the bound state problem in Quantum Electrodynamics. These treatments are not as systematic as the perturbation series, but they have successfully been used to treat the non perturbation phenomena.

V. RELATION BETWEEN PADÉ APPROXIMANT METHOD (P.A.M) AND INVERSE AMPLITUDE METHOD (I.A.M)

Within some approximation, the Padé approximant method can be derived from the more general I.A.M.. To see this let us write down the dispersion for the inverse of the partial wave $g(\nu) = f^{-1}(\nu)$ and use the elastic unitarity condition:

$$g(\nu) = g(0) - g(\nu) = g(0) - \frac{\nu}{\pi}\int_0^\infty dz \frac{\rho(\nu)}{z(z-\nu-i\epsilon)} - \frac{\nu}{\pi}\int_{-\infty}^{-\nu_c} dz \frac{\mid g(z) \mid^2 Imf(z)}{z(z-\nu)} \quad (17)$$

where it is assumed that there are no zeroes in $f(\nu)$. Unless $Imf(\nu)$ is given on the left cut, we cannot proceed. It is usually assumed that $Imf(\nu)$ is given by the perturbation series on the left cut and hence Eq.(17) becomes a non linear integral equation for $g(\nu)$. An iteration procedure can be used to solve this non linear integral equation.

If in the first iteration cycle, one sets on the left cut, $Imf(\nu) = Imf^1(\nu)$ and $\mid g(\nu) \mid^2 = (f^0)^{-2}$ then one would get the Padé result, Eq.(13). Hence there is a closed connection between the P.A.M and the I.A.M. In using the P.A.M method one thus avoids the problem of solving the non-linear integral equation as a first approximation.

The N/D method is an attempt to linearize the non-linear integral equation obtained by the I.A.M.. One write in this case $f(\nu) = N(\nu)/D(\nu)$ with $N(\nu)$ contains only the left cut and $D(\nu)$ only the right cut:

$$N(\nu) = \frac{1}{\pi}\int_{-\infty}^{-\nu_c} dz \frac{ImN(z)}{z-\nu} \quad (18)$$

and:

$$D(\nu) = 1 + \frac{\nu}{\pi}\int_0^\infty dz \frac{ImD(z)}{z(z-\nu)} \quad (19)$$

with $ImN(\nu) = D(\nu)Imf(\nu)$ and $ImD(\nu) = -\rho(\nu)N(\nu)$. Using these relations in Eq. (18) and in Eq. (19), we arrive at a coupled linear integral equation, instead at a non-linear one. Its connection with the I.A.M. or P.A.M. is not obvious. One could try, for example, to lump all functions in the perturbation series with the left cut singularity with the N function and then use the N/D method to unitarize the perturbation series.

VI. LOW ENERGY PION PION ELASTIC SCATTERING

The S and P-waves pion-pion scattering were first calculated by Weinberg [31] using current algebra technique and the assumption of a power series expansion for the scattering amplitude.

The effective range for P-wave $\pi\pi$ scattering was first proposed by Brown and Goble [51] in the approximation where the left hand cut contribution was neglected. In a systematics approach, there was no reason to neglect this contribution because it is of the same order as the as the logarithm term coming from the unitarity (right hand cut). Within this approximation Brown and Goble obtained the Kawarabayashi, Suzuki, Riazuddin and Fayyazuddin (KSRF) relation relating the ρ width with its mass [52]. Subsequently, Lehmann [5], in order to understand the gross features of $\pi\pi$ scattering up to $0.8GeV$ or so, below which, the assumption of the elastic unitarity is still valid, did the one loop ChPT calculation by perturbation theory, but in the limit of the pions as zero mass Goldstone bosons, using the non-linear σ model ($NL\sigma M$). Because working in the chiral limit, the results of his calculation depend on only two parameters. He then unitarized his results by the effective range expansion in $\cot\delta$ in its simplest form (a power series expansion in ν), he got scattering amplitudes which are consistent with analyticity and unitarity. Because his calculations were done with a zero mass pion, threshold parameters such as scattering lengths could not be calculated.

Lehmann [5]got an overall satisfactory agreement with the experimental data. The P-wave amplitude, has a reonance ρ at the right mass, and its width, satisfies the KSRF relation [52]. The I=0 S-wave phase shifts around 0.5 GeV are large and attractive, the I=2 phase shifts at the same energy are repulsive and small, in agreement with the experimental data. One should consider these results as impressive, considering that there are only two parameters in the calculation.

The lesson from Lehmann calculation [5], just the same as the classical calculation of the Chew Low theory [9] and the non relativistic effective range theory, analyticity in combination of unitarity, enables us to handle the long range (soft) strong interaction problem, even in the presence of bound states and resonances.

Lehmann [5] was puzzled by the P-wave $\pi\pi$ scattering calcualtion of Brown and Goble [51] where there was a presence of the logarithm term, whereas in his own calculation there was no such a term. The answer to this question is due to the approximation of neglecting the left hand cut contribution in Brown and Goble calculation. In the chiral limit the logarithm term from the right and left cut due to the contribution of the pion loops cancels each other out to get the Lehmann result[53]. The neglect of the left hand cut contribution for P-wave, as assumed by Brown and Goble, can be justified using the Roy equation and taking into account of the contribution of the scalar and vector mesons σ and ρ in the t and u channels. The sign of the crossing matrix is such that their contribution tends to cancel each other. This is the general justification for the KSRF relation [53].

A few years later, Jhung and Willey [29], improved the Lehmann $\pi\pi$ scattering calculation using the large σ mass limit of the linear-σ model, with chiral symmetry breaking taken into account. They calculated the partial wave amplitudes to one loop, and then the unitarization procedure was made by the Padé method. A good agreement with experimental data were obtained, in particular, the ρ resonance.

Lehmann and Jhung and Willey works were done later than the previous works by Lee and Lee and Basdevant [30]and others [26] based on linear σ model and the Padé approximant method [26]. There were not much works thereafter on the low energy $\pi\pi$ scattering until we publish a paper on the pion form factors, using the I.A.M. or the

P.A.M. methods to take into account of the unitarity [6]. A few years later, a unitarized version of the one loop ChPT, with chiral symmetry breaking, was published by Dobado, Herrero and Truong [7]. Excellent agreements with experimental data were obtained. The general KSFR relation was recovered here.

The revival of ChPT was due to Weinberg [4]. He outlined a systematic perturbation program with chiral symmetry breaking taken into account. This program has been carried out by Gasser and Leutwyler [32] and others for scattering processes and also for the form factor problems [33]. ChPT for $\pi\pi$ scattering was later carried out even to two loops order which requires considerable amount of effort [44, 45]. The main emphasis in these calculations is the systematics approach of the ChPT which is undisputable. However the crucial point for strongly interacting physics is the unitarity constraint which was left untouched. Probably, in analogy with the calculations in Quantum Electrodynamics, it has been assumed that perturbation unitarity is sufficient. Our discussion above shows that this is not so.

As far as the $\pi\pi$ scattering is concerned, the perturbative approach, which naturally led to an expansion in essentially a power series of $\tan\delta$, is not on the right direction to treat the non perturbation effect such as the resonant scattering as explained in section 7. The relevant expansion should be an expansion for $\cot\delta$ as a power series of energy as explained in section 7. (For this reason, Lehmann [5], in the same line of approach as the non-relativistic effective range theory and the Chew Low theories, was successful in calculating the ρ resonance). Gasser and Leutwyler [32] and others, could not get the ρ resonance.

Recent important works incorporating unitarity and analyticity by the inverse amplitude and the Padé approximant methods was done by Hannah [46] who made a careful study of various methods and compared them together. The problems of the chiral zeroes, which we ignored until now because of its complications, could be straightforwardly taken into account and were well treated by Hannah. For readers who wished to understand this subject better, the papers published by Hannah or his thesis could be quite useful [46, **?**, 50].

More recent analysis of the $\pi\pi$ scattering problem in ChPT was done together with the use of the Roy Equation [60] and very good predictions on the S-waves scattering lengths[47] were obtained. The question is whether this approach can be simplified and be carried out with the interpretation of the effective lagrangian as proposed in the introduction.

VII. VECTOR PION FORM FACTOR CALCULATION

The standard procedure of testing ChPT calculation of the pion form factor [34], which claims to support the perturbative scheme, is shown here to be unsatisfactory. This is so because the calculable terms are extremely small, less than 1.5% of the uncalculable terms at an energy of 0.5 GeV or lower whereas the experimental errors are of the order 10-15%.

Although dispersion relation (or causality) has been tested to a great accuracy in the forward pion nucleon and nucleon nucleon or anti-nucleon scatterings at low and high energy, there is no such a test for the form factors. This problem is easy to understand.

In the former case, using unitarity of the S-matrix, one rigourously obtained the optical theorem relating the imaginary part of the forward elastic amplitude to the total cross section which is a measurable quantity. This result together with dispersion relation establish a general relation between the real and imaginary parts of the forward amplitude [6, 42, 43].

There is no such a rigourous relation, valid to all energy, for the form factor. In low energy region, the unitarity of the S-matrix in the elastic region gives a relation between the phase of the form factor and the P-wave pion pion phase shift, namely they are the same [35]. Strictly speaking, this region is extended from the two pion threshold to $16m_\pi^2$ where the inelastic effect is rigourously absent. In practice, the region of the validity of the phase theorem can be extended to 1.1-1.3 GeV because the inelastic effect is negligible. Hence, using the measurements of the modulus of the form factor and the P-wave phase shifts, both the real and imaginary parts of the form factors are known. Beyond this energy, the imaginary part is not known. Fortunately for the present purpose of testing of locality (dispersion relation) and of the validity of the perturbation theory at low energy, thanks to the use of subtracted dispersion relations, the knowledge of the imaginary part of the form factor beyond 1.3 GeV is unimportant.

Because the vector pion form factor $V(s)$ is an analytic function with a cut from $4m_\pi^2$ to ∞, the n^{th} times subtracted dispersion relation for $V(s)$ reads:

$$V(s) = a_0 + a_1 s + ...a_{n-1}s^{n-1} + \frac{s^n}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ImV(z)dz}{z^n(z-s-i\epsilon)} \tag{20}$$

where $n \geq 0$ and, for our purpose, the series around the origin is considered. Because of the real analytic property of $V(s)$, it is real below $4m_\pi^2$. By taking the real part of this equation, $ReV(s)$ is related to the principal part of the dispersion integral involving the $ImV(s)$ apart from the subtraction constants a_n.

The polynomial on the R.H.S. of Eq. (20) will be referred in the following as the subtraction constants and the last term on the R.H.S. as the dispersion integral (DI). The evaluation of DI as a function of s will be done later. Notice that $a_n = V^n(0)/n!$ is the coefficient of the Taylor series expansion for $V(s)$, where $V^n(0)$ is the nth derivative of $V(s)$ evaluated at the origin. The condition for Eq. (20) to be valid was that, on the real positive s axis, the limit $s^{-n}V(s) \to 0$ as $s \to \infty$. By the Phragmen Lindeloff theorem, this limit would also be true in any direction in the complex s-plane and hence it is straightforward to prove Eq. (20). The coefficient a_{n+m} of the Taylor's series is given by:

$$a_{n+m} = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} \frac{ImV(z)dz}{z^{(n+m+1)}} \tag{21}$$

where $m \geq 0$. The meaning of this equation is clear: under the above stated assumption, not only the coefficient a_n can be calculated but all other coefficients a_{n+m} can also be calculated. The larger the value of m, the more sensitive is the value of a_{n+m} to the low energy values of $ImV(s)$. In theoretical work such as in ChPT approach, to be discussed later, the number of subtraction is such that to make the dispersion integral converges.

The elastic unitarity relation for the pion form factor is $ImV(s) = V(s)e^{-i\delta(s)}sin\delta(s)$ where $\delta(s)$ is the elastic P-wave pion pion phase shifts. Below the inelastic threshold

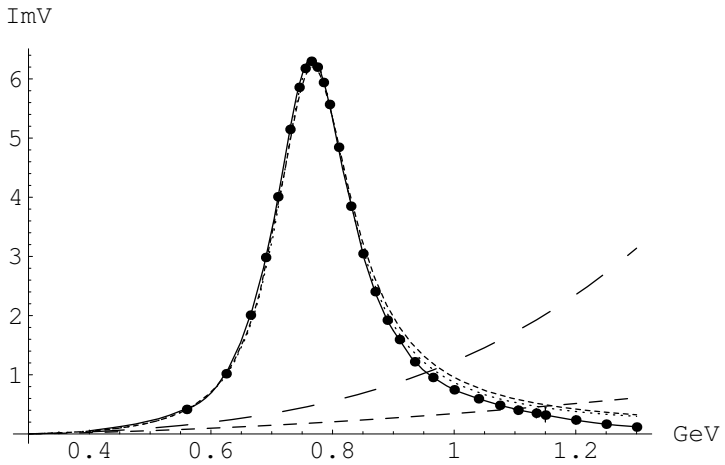

Fig. 1. Imaginary Parts of Pion Form Factor ImV as a function of energy in GeV unit. The solid curve is a fit to experimental results with experimental errors; the long-dashed curve is the two-loop ChPT calculation; the medium long-dashed curve is the one-loop ChPT calculation; the short-dashed curve is from the one-loop UChPT calculation [6] with presumably inelastic effects taken into account; the dotted curve is from the UChPT of Hannah

of $16m_\pi^2$ where m_π is the pion mass, $V(s)$ must have the phase of $\delta(s)$ [35]. It is an experimental fact that below $1.3GeV$ the inelastic effect is very small, hence, to a good approximation, the phase of $V(s)$ is δ below this energy scale.

$$ImV(z) =\mid V(z) \mid \sin\delta(z) \tag{22}$$

and

$$ReV(z) =\mid V(z) \mid \cos\delta(z) \tag{23}$$

where δ is the strong elastic P-wave $\pi\pi$ phase shifts. Because the real and imaginary parts are related by dispersion relation, it is important to know accurately $ImV(z)$ over a large energy region. Below 1.3 GeV, $ImV(z)$ can be determined accurately because the modulus of the vector form factor [36, 37] and the corresponding P-wave $\pi\pi$ phase shifts are well measured [38, 39, 40] except at very low energy.

It is possible to estimate the high energy contribution of the dispersion integral by fitting the asymptotic behavior of the form factor by the expression, $V(s) = -(0.25/s)ln(-s/s_\rho)$ where s_ρ is the ρ mass squared.

Using Eq. (22) and Eq. (23), $ImV(z)$ and $ReV(s)$ are determined directly from experimental data and are shown, respectively, in Fig.1 and Fig.2.

In the following, for definiteness, one assumes $s^{-1}V(s) \to 0$ as $s \to \infty$ on the cut, i.e. $V(s)$ does not grow as fast as a linear function of s. This assumption is a very mild one because theoretical models assume that the form factor vanishes at infinite energy as s^{-1}. In this case, one can write a once subtracted dispersion relation for $V(s)$, i.e. one sets $a_0 = 1$ and $n = 1$ in Eq. (20).

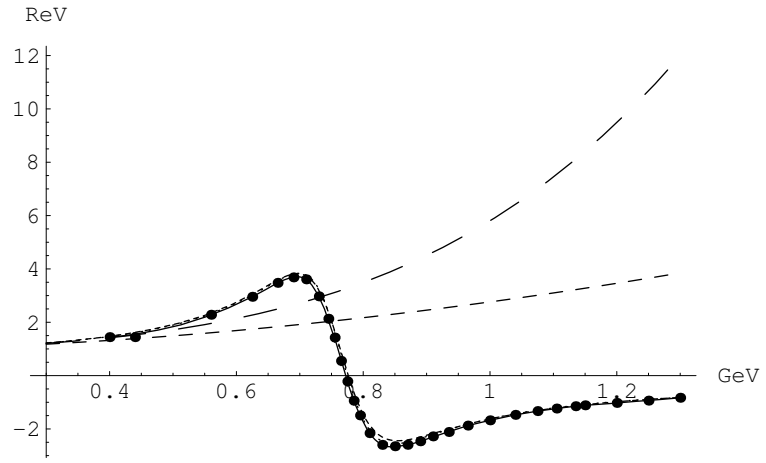

Fig. 2. Real Parts of Pion Form Factor ReV as a function of energy in GeV unit. The label of the curves are the same as in Fig. 1; the calculated Real Part of the pion form factor by the once subtracted dispersion relation using the experimental imaginary part cannot be distinguished from the solid line experimental curve

From this assumption on the asymptotic behavior of the form factor, the derivatives of the form factor at $s = 0$ are given by Eq. (21) with n=1 and m=0. In particular one has:

$$< r_V^2 >= \frac{6}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ImV(z)dz}{z^2} \tag{24}$$

where the standard definition $V(s) = 1 + \frac{1}{6} < r_V^2 > s + cs^2 + ds^3 + ...$ is used. Eq.(24) is a sum rule relating the pion rms radius to the magnitude of the time like pion form factor and the P-wave $\pi\pi$ phase shift measurements. Using these data, the derivatives of the form factor are evaluated at the origin:

$$< r_V^2 >= 0.45 \pm 0.015 fm^2; c = 3.90 \pm 0.20 GeV^{-4}; d = 9.70 \pm 0.70 GeV^{-6} \tag{25}$$

where the upper limit of the integration is taken to be $1.7GeV^2$. By fitting $ImV(s)$ by the above mentioned asymptotic expression, the contribution beyond this upper limit is completely negligible. From the 2 π threshold to $0.56GeV$ the experimental data on the the phase shifts are either poor or unavailable, an extrapolation procedure based on some model calculations to be discussed later, has to be used. Because of the threshold behavior of the P-wave phase shift, $ImV(s)$ obtained by this extrapolation procedure is small. They contribute, respectively, 5%, 15% and 30% to the a_1, a_2 and a_3 sum rules. The results of Eq. (25) change little if the $\pi\pi$ phase shifts below $0.56GeV$ was extrapolated using an effective range expansion and the modulus of the form factor using a pole or Breit-Wigner formula.

The only experimental data on the derivatives of the form factor at zero momentum transfer is the root mean square radius of the pion, $r_V^2 = 0.439 \pm .008 fm^2$ [41]. This value is very much in agreement with that determined from the sum rules. In fact the sum rule for the root mean squared radius gets overwhelmingly contribution from the ρ resonance as can be seen from Fig.1. The success of the calculation of the r.m.s. radius is a first indication that causality is respected and also that the extrapolation procedures to low energy for the P-wave $\pi\pi$ phase shifts and for the modulus of the form factor are legitimate.

Dispersion relation for the pion form factor is now shown to be well verified by the data over a wide energy region. Using $ImV(z)$ as given by Eq. (22) together with the once subtracted dispersion relation, one can calculate the real part of the form factor $ReV(s)$ in the time-like region and also $V(s)$ in the space like region. Because the space-like behavior of the form factor is not sensitive to the calculation schemes, it will not be considered here. The result of this calculation is given in Fig.2. As it can be seen, dispersion relation results are well satisfied by the data.

The i-loop ChPT result can be put into the following form, similar to Eq. (20):

$$V^{pert(i)}(s) = 1 + a_1 s + a_2 s^2 + ... + a_i s^i + D^{pert(i)}(s) \tag{26}$$

where $i+1$ subtraction constants are needed to make the last integral on the RHS of this equation converges and

$$DI^{pert(i)}(s) = \frac{s^{1+i}}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ImV^{pert(i)}(z)dz}{z^{1+i}(z - s - i\epsilon)} \tag{27}$$

with $ImV^{pert(i)}(z)$ calculated by the ith loop perturbation scheme.

Similarly to these equations, the corresponding experimental vector form factor $V^{exp(i)}(s)$ and $DI^{exp(i)}(s)$ can be constructed using the same subtraction constants as in Eq. (26) but with the imaginary part replaced by $ImV^{exp(i)}(s)$, calculated using Eq. (22).

The one-loop ChPT calculation requires 2 subtraction constants. The first one is given by the Ward Identity, the second one is proportional to the r.m.s. radius of the pion. In Fig. 1, the imaginary part of the one-loop ChPT calculation for the vector pion form factor is compared with the result of the imaginary part obtained from the experimental data. It is seen that they differ very much from each other. One expects therefore that the corresponding real parts calculated by dispersion relation should be quite different from each other.

In Fig.3 the full real part of the one loop amplitude is compared with that obtained from experiment.

At very low energy one cannot distinguish the perturbative result from the experimental one due to the dominance of the subtraction constants. At an energy around $0.56 GeV$ there is a definite difference between the perturbative result and the experimental data. This difference becomes much clearer in Fig. 3 where only the real part of the perturbative DI, $ReDI^{pert(1)}(s)$, is compared with the corresponding experimental quantity, $ReDI^{exp(1)}(s)$. It is seen that even at 0.5 GeV the discrepancy is clear. Supporters of

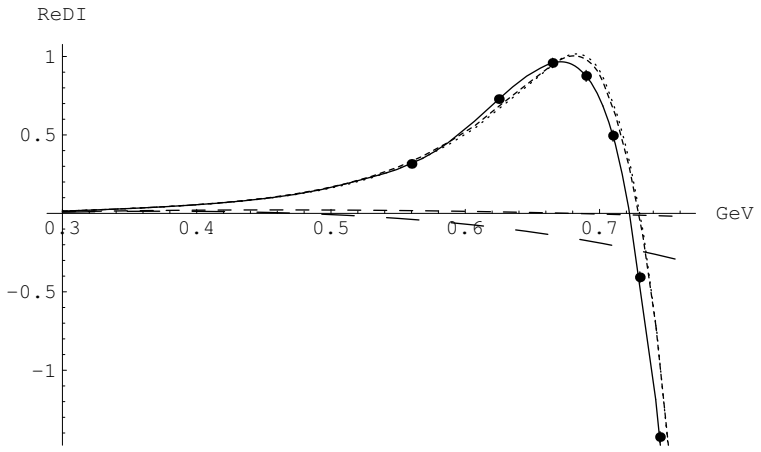

Fig. 3. Real Parts of the Dispersion Integral ReDI as a function of energy. The label of the curves are just the same as in Fig. 1

ChPT would argue that ChPT would not be expected to work at this energy. One would have to go to a lower energy where the data became very inaccurate.

This argument is false as can be seen by comparing the ratio $R_1 = ReDI^{pert(1)}/ReDI^{exp(1)}$ i.e.the ratio of the one loop ChPT result to its value calculated using the experimental result. We have: $R_1 = 0.16$ for $0 \leq s \leq 4m_\pi^2$, i.e. the one-loop ChPT calculated term is too small by a factor of 7 as compared with that calculated using experimental data for the imaginary part Above the two pion threshold and up to 600 MeV this value becomes even less.

This ratio becomes better with the inclusion of the two loop effects; instead of being less by a factor of 7 as for the one loop calculation it becomes a factor of 2.5 below the two pion threshold much larger and becomes larger above the threshold.

These results illustrate the "amusing" theorem on the small analytic function discussed in the introduction.

Similarly to the one-loop calculation, the two-loop results are plotted in Fig. (1) - Fig. (3) [34]. Although the two-loop result is better than the one-loop calculation, because more parameters are introduced, calculating higher loop effects will not explain the data.

It is seen that perturbation theory is inadequate for the vector pion form factor even at very low momentum transfer. This fact is due to the very large value of the pion r.m.s. radius or a very low value of the ρ mass s_ρ (see below). In order that the perturbation theory to be valid the calculated term by ChPT should be much larger than the non perturbative effect. At one loop, by requiring the perturbative calculation dominates over the non-perturbative effects at low energy, one has $s_\rho >> \sqrt{960}\pi f_\pi m_\pi = 1.3 GeV^2$ which is far from being satisfied by the physical value of the ρ mass.

The unitarized models are now examined. It has been shown a long time ago that to take into account of the unitarity relation, it is better to use the inverse amplitude $1/V(s)$ or the Pade approximant method [6, 8].

The first model is obtained by introducing a zero in the calculated form factor in the ref. [6, 55] to get an agreement with the experimental r.m.s. radius . The pion form factor is now multiplied by $1 + \alpha s/s_\rho$ where s_ρ is the ρ mass squared [6]: A more insight to the existence of the zero is possibly that the unitarity relation was truncated with the two particle state, or the elastic approximation. The solution of the Muskelishvilli Omnes integral equation with the inelastic contribution was previously studied [54, 55] . Its solution can be written as the product form of the standard form of the elastic unitarity, i.e. the Omnes function, and the inelastic contribution [54]. Below the inelastic threshold, the inelastic contribution can be written as a power series whose leading term is the factor $1 + \alpha s/s_\rho$. This phenomenological description of the pion form factor with the inelastic contribution was first given in the reference [55].

The experimental data can be fitted with a ρ mass equal to $0.773 GeV$ and $\alpha = 0.14$. These results are in excellent agreement with the data [37, 41].

The second model, which is more complete, at the expense of introducing more parameters, is based on the two-loop ChPT calculation with unitarity taken into account. It has the singularity associated with the two loop graphs. Using the same inverse amplitude method as was done with the one-loop amplitude, but generalizing this method to two-loop calculation, Hannah has recently obtained a remarkable fit to the pion form factor in the time-like and space-like regions. His result is equivalent to the (0,2) Padé approximant method as applied to the two-loop ChPT calculation [46]. Both models contain ghosts which can be shown to be unimportant [46]. At this moment there seems to be no preference for one of either two models, but it would be surprising that the inelastic effect can completely be neglected in the dispersive approach.

It is interesting for a given lagrangian and one or two low energy experimental parameters how do we know whether perturbation theory was valid. There is of course no general answer to this question. But by looking at the expression for the one loop perturbation result and its unitarized version of the pion form factor [6], one can realize that the r.m.s. radius of the pion, related to the inverse of the ρ mass, is far too large to make the perturbation theory valid. This question was discussed in some details in the reference [57].

Another interesting question is how to incorporate the ChPT calculation with the vector meson ρ dominance for the pion form factor which is needed to analyze the experimental data for the hadronic τ decays for example. We simply cannot add the ChPT result to the expression of the vector meson contribution even with care not to violate the "low energy theorem" of ChPT. This is so because this procedure would amount to a double counting. Furthermore, in the time like region, although a rough fit to the form factor can be made due to the dominance of the vector meson dominance term, in the space-like region both the ChPT term and the vector meson dominance term are roughly equal in magnitude at moderate momentum transfer and their sum would give a wrong prediction of the form factor at moderate momentum transfer.

It is usually done in ChPT is to do some matching at a few points of the vector meson dominance term with ChPT the one or two-loop calculation. Here we encounter the "amusing" theorem and the inaccuracy of ChPT calculation.

To the best of our knowledge, only the unitarized approach can preserve the low energy theorem and at the same time gives correctly the vector meson dominance as required by data,i.e. can avoid the double counting problem.

As can be seen from Figs. 1, 2 and 3 the imaginary and real parts of these two models are very much in agreement with the data. A small deviation of $ImV(s)$ above $0.9GeV$ is due to a small deviation of the phases of $V(s)$ in these two models from the data of the P-wave $\pi\pi$ phase shifts.

In conclusion, higher loop perturbative calculations do not solve the unitarity problem. The perturbative scheme has to be supplemented by the well-known unitarisation schemes such as the inverse amplitude, N/D and Padé approximant methods as discussed in the preceeding sections.

VIII. K_{e4} DECAY

Although this work was done a long time ago [59], using the current algebra technique, reduction formula, dispersion relation and the equal time current algebra commutation relations, it is still the only work where the phases and magnitudes and slope parameters of the S and P-waves form factors can be calculated and agree with experiments. Because of the lack of the ChPT two loop calculation for these amplitudes, as pointed out in [6], the question of calculating the phases of the relevant form factors were ignored in ChPT calculations.

It is useful to summarize briefly the calculation technique, which was due originally to Weinberg where the unitarity correction was neglected [61]. After extracting the equal time commutation relation (ETCR) terms, there remains terms which are proportional to the pion four-momenta which tend to zero in the soft pion limit; the ETCR terms do not tend to zero and therefore becomes the low energy theorems and are simply the Born terms. The terms proportional to the pion momenta, can be shown to obey a dispersion relation. An integral equation of the Muskelishvilli-Omnes type can be written with the Born terms(ETCR) to take into account of the elastic unitarity condition for the pion pion rescattering. The scale of the relevant amplitudes are set by the ETCR are in the unphysical region of the Ke_4 decay (the limit of the pion 4-momenta vanishes.

Using the experimental knowledge of the $\pi\pi$ S and P-wave phase shifts, in the solution of the Muskelishvilli-Omnes integral equation, the magnitude and phase of the form factors can be calculated and agree with experiments.

Because the scale of the problem is set by the low energy current algebra theorem which is below the two pion threshold, and the measurement is done at above the two pion threshold, the analytic continuation of the current algebra result has to pass through the two pion branch point giving rise to the effect of the square root threshold singularity [59] which is unfortunately called, nowadays, as the logarithm singularity.

Such an effect enhances the current algebra S-wave result by a factor of 1.4 (in amplitude) and by a factor of 1.2 for the P-wave form factor near the two pion threshold in the physical region. This same enhancement factor is also found in the low energy $\pi\pi$ scattering and a larger enhancement factor in $\eta \to 3\pi$ problem due to the 3 pion final states.

IX. $\eta \to 3\pi$ DECAY

Weinberg [62], using tree Lagrangian and Dashen theorem : [63] to calculate the rate for $\eta \to \pi^+\pi^-\pi^0$ rate and found its width to be 65 eV as compared to the present experimental rate of 295 eV. At one loop level the ChPT calculation by Gasser and Leutwyler to be 160 eV . Recent calculations this value is increased to 220 $\pm$20 eV.[67, 68, 69]. There are however, difficulties with odd pion slope calculation.

Our approach to this problem was done a long time ago [21] using the integral equation of the type Khuri Treiman and Sawyer Wali [66]taking into account only of the S-wave pion pion interaction. Due to the poor approximation made for the treatment of the multiple pion pion scattering a width of 430 eV was obtained.

The result of the $\eta \to 3\pi$ calculation is the same as that of $K \to 3\pi$ Decay [57, 58] to be discussed later, where both the I=0 S-wave and the I=1 P-wave pion pion interactions are taken into account(the P-wave pion pion interaction of the pair $\pi^+\pi^0$ and $\pi^-\pi^0$ are allowed but one has to symmetrize the amplitudes; their contribution only affects the odd pion slope and was neglected in the Khuri-Treiman integral equation). This calculation yields a width for the charged pion mode of η to be 240 eV and a correct linear slope of the odd pion was obtained which was not possible with the ChPT calculation. With the η and $\eta^{'}$ mixing, the value of the width is increased to 350 eV.

X. $K \to \pi$, $K \to 2\pi$, $K \to 3\pi$ AMPLITUDES

A precise knowledge of the $K\pi$ amplitude and its relation with $K \to 2\pi$ and $K \to 3\pi$ are of a fundamental importance in the study of the origin of the $\Delta I = 1/2$ rule, the CP violation effect in the standard model and the rare decays of K_L and K_S.

The simplest way to implement chiral symmetry and SU(3) on $K \to 2\pi$ and $K \to 3\pi$ is to use the chiral lagrangian [56]. The amplitude for $M(K_S \to \pi^+\pi^-)$ is given by:

$$M(K_S(k) \to \pi^+(p)\pi^-(q)) = \frac{1}{2} iC f_\pi (2k^2 - p^2 - q^2) \tag{28}$$

The amplitude for $M(K_L \to \pi^+\pi^-\pi^0)$ is

$$M(K_L \to \pi^+(q_1)\pi^-(q_2)\pi^0(q_3)) = \frac{C}{\sqrt{2}}(s - \mu^2) \tag{29}$$

and the $K - \pi$ amplitude is:

$$M(K_L \to \pi^0) = -\frac{C}{\sqrt{2}} f_\pi^2 q(\pi) q(K) \tag{30}$$

where $s = (q_1 + q_2)^2, t = (q_1 + q_3)^2, u = (q_2 + q_3)^2$ with $s + t + u = 3s_0 = 3\mu^2 + m^2$ where m and μ are respectively K and π masses.

Using the experimental determination of the $\Delta = 1/2$ amplitude $a_1/2 = (0.469 \pm 0.006).10^-3 MeV$ it is found that $C = 1.26.10^-11 MeV$. The $K \to 3\pi$ as given by Eq. (29)is:

$$M(K_L \to \pi^+\pi^-\pi^0) = 7.43.10^-7(1 + 0.233(s - s_0)/\mu^2) \tag{31}$$

This result is to be compared with the experimental value:

$$M(K_L \to \pi^+\pi^-\pi^0) = 9.10.10^-7(1 + 0.264(s - s_0)/\mu^2) \tag{32}$$

It is seen that the tree lagrangian yields a prediction for the $K_L \to 3\pi$ amplitude a value too low by 20% and the odd pion slope is 12% too low. Such discrepancies are due to the neglect of unitarity. For example the Eq.(28)is purely real while unitarity requires it to have a phase of approximately of 40^0 onthe K mass. The resolution of these problems was given by taking unitarity into account for the rescattering in the S-wave as well as P-wave pion pion interactions [57]. It should be remarked that due to the 3-body in the final state, we can have both S and P-waves pion pion interaction.

$$M(K_L \to \pi^+\pi^-\pi^0) = 8.86.10^-7(1 + 0.250(s - s_0)/\mu^2) \tag{33}$$

and is very much in agreement with the experimental result Eq.(32)

XI. STUDY OF $\gamma\gamma \to 2\pi$, $K_S \to 2\gamma$ AND $K_L \to \pi^0\gamma\gamma$

The result of the calculation of the $K \to \pi$ calculation discussed in the previous section allowed us to calculate the rare decay modes of $K_S \to 2\gamma$ and $K_L \to \pi^0\gamma\gamma$. These were done in the reference [58] and there are agreements between theory and experiments and will not be discussed here. I would like to point out two old papers on $\gamma\gamma \to 2\pi$ which are still relevant for further study on this subject [70, 71]. A more recent paper is also relevant[72].

XII. $\tau \to K\pi\nu$ and $\tau \to 3\pi\nu$ DECAYS

The process $\tau \to K\pi\nu$ illustrate the usefulness of the combination of the current algebra low energy theorems with unitarity and dispersion relation [73]. The S and P-wave $K - \pi$ form factor were calculated using elastic unitarity in combination with dispersion relation. The KSRF relation were found to be valid, and the forward-backward asymmetry due to the interference of the S and P wave form factors were predicted and to be verified by future experimental results.

A similar calculation was done for the $\tau \to 3\pi\nu$ decay [74].

XIII. ENHANCEMENT FACTOR IN $K \to 2\pi$

Discussions in the previous sections are based on Current Algebra or Effective Lagrangian low energy theorems. Given these theorems at low energy, usually in the unphysical region, with elastic unitarity and dispersion relations we analytically continue these theorems to the elastic region. This is a small extrapolation. In a different line of physics, we want to ask a much more difficult question what is the difference in amplitudes, e.g. in the $K \to 2\pi$ with the final state interaction of the two pion interaction when it is switched on and off. This last question has no answer and is dependent on the input assumption.

This question was asked a long time ago by Isgur et al [75]. The answer they gave was that the calculation of the matrix element with the final state interaction switched off has to be multiplied with the S-wave pion pion interaction wave function at the origin. This question was examined [76], and the answer depends on what one assumes when writing down the Muskelshivilli-Omnes integral equation. Namely, one assumes that at infinite energy the pi pi interaction does not affect the result of calculation, in other words, the pi pi interaction are completely switched off. and that the elastic unitarity is

valid for writing down the dispersion relation for the imaginary part of the enhancement factor. Under these assumptions, the enhancement factor is simply the inverse of the Jost function. The Jost function has the property that it becomes unity at infinite energy which is an extremely long extrapolation from the elastic pi pi region to an infinite energy. The answer given needs not be reliable. For a detailed discussion on this problem, the reader is referred to the reference [76].

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