Communications in Physics, Vol. 34, No. 3 (2024), pp. 203-225
DOI: https://doi.org/10.15625/0868-3166/21033

REVIEW PAPER

New Physics in the 3-3-1 models

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Received 26 June 2024

Accepted for publication 5 August 2024

Published 30 August 2024

Abstract. *Two main ingredients of current particle physics such as local gauge symmetry and mass generation via the Higgs mechanism being basic ground of the Standard Model are widely confirmed by experimental data. However, some problems such as neutrino masses, dark matter, baryon asymmetry of Universe have clearly indicated that the Standard Model cannot be the ultimate theory of nature. To surpass the mentioned puzzles, many extensions of the Standard Model (called beyond Standard Model) have been proposed. Among beyond Standard Models, the 3-3-1 models have some intriguing features and they get wide attention. The pioneer models develop in some directions. In this paper, new main versions of the 3-3-1 models and their consequences are presented.*

Keywords: standard model; beyond standard model; Higgs and neutrino physics; dark matter; lepton flavor violation.

Classification numbers: 12.10.-g; 12.10.Dm; 12.15.-y; 12.60.-i; 12.60.Cn; 14.60.Pq; 14.80.Bn.

1. Introduction

The Standard Model (SM) has been very successful in describing observed phenomena. However, it also leaves many striking features of the physics of our world unanswered [1]. First of all, electric charge and family number in the standard model are in principle arbitrary, opposite to observation. Further, neutrino oscillations coming from the sun, the atmosphere, reactors and accelerators have been well-confirmed, implying that neutrinos have nonzero masses and mixing, in contrast to the SM that possesses massless neutrinos. Alternatively, dark matter (DM) that makes up most of the mass of galaxies and galactic clusters is completely not present in the SM. Last, but not least, the SM cannot explain the baryon-number asymmetry of the universe, for which we nowadays observe only matter, and there is no popular existence of antimatter.

That said, it is well-established that the SM of elementary particles and interactions has to be extended. More specifically, the most problems arise in the electroweak sector [2]. Among beyond Standard Models (BSMs), the models based on the $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ gauge group (called by 3-3-1 models) [3–12] have some intriguing features such as the models can give explanation for generation number (n_f to be three) due to discrimination in the quark sector (one quark family transforms differently from other twos) leading to ($n_f = 3 \times n_C$) and asymptotic freedom in QCD ($n_f \leq 5$). The electric charge quantization in the 3-3-1 models is related to their fermion representation structures under 3-3-1 symmetry. Another interesting feature should be mentioned concerning automatic satisfaction with Peccei-Quinn mechanism [13]. However, the 3-3-1 models contain one disadvantage: due to discrimination of quark generations, the scalar sector is quite complicate. There are attempts to overpass the problem.

The 3-3-1 models presently attract much attention concerning their ability to address the questions of neutrino masses and DM. Besides, they present potential signals at colliders, associated with the questions of flavour-changing neutral currents (FCNCs). It is stated that the 3-3-1 model is the first non-abelian gauge principle that recognizes dark matter stability. Additionally, compelling neutrino mass generation mechanisms, such as canonical seesaw, type II seesaw, inverse seesaw, and scotogenic scheme, are all realized in the 3-3-1 models.

The aim of this work is to summarize new developments and some interesting consequences in the 3-3-1 models. The rest of this article is organized as follows. The Sec. 2 is devoted to pioneer 3-3-1 models and their minimal scalar contents. In Sec. 3 briefly reviews the model of discrimination of leptons and model with inverse seesaw. Combination of non-Abelian discrete symmetries with the 3-3-1 models is presented in Sec. 4. Sec. 5 is devoted to newest development in direction of axion or axion-like particle (ALP). In Sec. 6, I just mention the main phenomenologies of the models such as collider physics, neutrino mass, lepton flavor violating (LFV), $g-2$ and Early Universe. The conclusions are in the last Sec. 7.

2. The pioneer 3-3-1 models

In the pioneer models, the difference mostly is in content of the lepton triplet: In the minimal 3-3-1 model (M331 model) [3–6], the right-handed charged lepton is at the bottom of the lepton triplet, while in the 3-3-1 model with right-handed neutrinos (331RN model) [7–12], right-handed neutrino is at the bottom of the lepton triplet instead.

The electric charge operator is expressed in the form of diagonalized ones:

$$Q = T_3 + \beta T_8 + X, \tag{1}$$

where the factor one (1) associated with T_3 ensures embedding in of the SM. It is worth mentioning that the 3-3-1 models are characterized by the value of β.

2.1. *The minimal 3- 3-1 model*

Leptons come in fundamental representatin of $\mathrm{SU}(3)_L$ ($\beta=-\sqrt{3}$) [5]:

$$f^T_{aL}=(\nu_{aL},l_{aL},(l^c)_{aL},)\sim(1,3,0), \tag{2}$$

where $a=1,2,3$ is generation index. Note that if *leptons lie in antitriplet, then the value of β will just change the sign*, namely, in this case $\beta=+\sqrt{3}$).

The third quark generation is in triplet and two others are in anttriplet:

$$\begin{aligned}
Q^T_{3L} &= (u_{3L},d_{3L},T_L)\sim(3,3,2/3),\\
Q^T_{iL} &= (d_{iL},-u_{iL},D_{iL})\sim(3,\tilde{3},-1/3),\\
u_{iR} &\sim (3,1,2/3), d_{iR}\sim(3,1,-1/3), D_{iR}\sim(3,1,-4/3),\ i=1,2,\\
u_{3R} &\sim (3,1,2/3), d_{3R}\sim(3,1,-1/3), T_R\sim(3,1,5/3).
\end{aligned} \tag{3}$$

The spontaneous symmetry breakdown (SSB) of the model requires three scalar triplets and one sextet

$$\begin{aligned}
\chi^T &= \left(\chi_1^-,\chi_2^{--},\chi_3^0\right)\sim(1,3,-1), \qquad (4)\\
\rho^T &= \left(\rho_1^+,\rho_2^0,\rho_3^{++}\right)\sim(1,3,1),\\
\eta^T &= \left(\eta_1^0,\eta_2^-,\eta_3^+\right)\sim(1,3,0),\\
S &= \begin{pmatrix} \sigma_1^0 & h_2^-/\sqrt{2} & h_1^+/\sqrt{2}\\ h_2^-/\sqrt{2} & H_1^{--} & \Sigma_2^0/\sqrt{2}\\ h_1^+/\sqrt{2} & \Sigma_2^0/\sqrt{2} & H_2^{++}\end{pmatrix}\sim(1,6,0).
\end{aligned}$$

with VEVs:

$$\begin{aligned}
\langle\chi\rangle^T &= \left(0,0,\omega/\sqrt{2}\right),\ \langle\rho\rangle^T=\left(0\ ,v/\sqrt{2}\ ,0\right),\ \langle\eta\rangle^T=\left(u/\sqrt{2},0,0\right),\\
\sqrt{2}\langle S\rangle &= \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & v'\\ 0 & v' & 0\end{pmatrix}.
\end{aligned} \tag{5}$$

Without right-handed charged lepton, the lepton mass matrix is antisymmetric 3×3 matrix containing one massless eigenvalue. To solve this puzzle, the sextet with symmetric interaction to leptonic triplets, has been added [5].

The SSB of the model is in two steps as follows:

$$\mathrm{SU}(3)_L\otimes\mathrm{U}(1)_X \xrightarrow{\langle\chi_3^0\rangle} \mathrm{SU}(2)_L\otimes\mathrm{U}(1)_Y \xrightarrow{u,v,v'} \mathrm{U}(1)_Q, \tag{6}$$

The ratio between couplings constant of $U(1)_X$ and $SU(3)_L$ is given by [5]

$$t=\frac{g_X}{g}=\frac{\sin^2_W}{1-4\sin^2_W}. \tag{7}$$

Denominator in Eq. (7) tends to zero (Landau pole) [6, 14] at 5 TeV. In recent work, by adding scalar leptoquarks, this pole has been searched at 100 TeV [15].

2.2. *The 3-3-1 model with right-handed neutrinos*

In the 331RN model, leptons are in triplet [8–12] ($\beta = -\frac{1}{\sqrt{3}}$):

$$f^T_{aL} = (\nu_{aL}, l_{aL}, (\nu^c_L)_a) \sim (1,3,-1/3), l^a_R \sim (1,1,-1), \tag{8}$$

where $a = 1,2,3$ is a generation index. Two first generation of quarks comes in the anti-fundamental representation of $\mathrm{SU}(3)_L$, and the third one is in triplet:

$$Q^T_{iL} = (d_{iL}, -u_{iL}, D_{iL}) \sim (3,\tilde{3},0), \tag{9}$$

$$u_{iR} \sim (3,1,2/3), d_{iR} \sim (3,1,-1/3), D_{iR} \sim (3,1,-1/3),\ i = 1,2,$$
$$Q^T_{3L} = (u_{3L}, d_{3L}, T_L)^T \sim (3,3,1/3),$$
$$u_{3R} \sim (3,1,2/3), d_{3R} \sim (3,1,-1/3), T_R \sim (3,1,2/3).$$

To spontaneous symmetry breaking, we need three Higgs triplets

$$\begin{aligned} \chi^T &= \left(\chi^0, \chi^-, \chi'^0\right) \sim (1,3,-1), \\ \rho^T &= \left(\rho^+, \rho^0, \rho'^+\right) \sim (1,3,2), \\ \eta^T &= \left(\eta^0, \eta^-, \eta'^0\right) \sim (1,3,-1). \end{aligned} \tag{10}$$

The SSB follows the scheme as below

$$\mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X \stackrel{\langle\chi'^0\rangle}{\longrightarrow} \mathrm{SU}(2)_L \otimes \mathrm{U}(1)_Y \stackrel{\langle\rho\rangle,\langle\eta\rangle}{\longrightarrow} \mathrm{U}(1)_Q, \tag{11}$$

where

$$\sqrt{2}\langle\chi\rangle = (0,0,\omega)^T, \quad \sqrt{2}\langle\rho\rangle = (0\ , v\ , 0)^T, \quad \sqrt{2}\langle\eta\rangle = (u,0,0)^T. \tag{12}$$

Note that the model in which $\beta = 0$ has been constructed in [16] and its phenomenology in quark sector was studied in [17]. If so, the model contains extra lepton E_a with electric charge equal to $-1/2$.

The models with arbitrary β (331β) were proposed in Ref. [18].

As above mentioned, the 3-3-1 models contain many intriguing features, but they face one limitation in large scalar content which prevent their predictability. A reason for this is that due to discrimination in quark/lepton representations, more scalar multiplets are required. There are attempts in this direction and results are two models: economical (E331) and simple (S331) models.

2.3. *The economical 3-3-1 model*

Note that in the model with $\beta = -\frac{1}{\sqrt{3}}$, the exist two triplets η and χ with identical quantum. Hence, we can omit one η [19–21], then

$$\chi^T = \left(\chi^0, \chi^-, \chi'^0\right) \sim (1,3,-1), \tag{13}$$
$$\rho^T = \left(\rho^+, \rho^0, \rho'^+\right) \sim (1,3,2). \tag{14}$$

To provide masses of fermions and gauge bosons, one needs to provide χ with the following VEV:

$$\sqrt{2}\langle\chi\rangle^T = (u,\ 0, \omega). \tag{15}$$

It is worth mentioning that the triplet χ contains the VEV ω which responds for the first step of SSB.

One of the VEVs u carrying lepton number two is a source of lepton-number violations and a reason for the mixing between W and singly-charged bilepton gauge boson Y as well as between X^0 and the photon, the Z and the Z'. Thus, this leads to $u \ll v$, and there are three quite different scales for the VEVs of the model: one is very small $u \simeq O(1)$ GeV - a lepton-number violating parameter, the second v is close to the SM one : $v \simeq v_{weak} = 246$ GeV and the last is in the range of new physics scale about $O(1)$ TeV.

Due to $W - Y$ mixing, there exist lepton number violating (LNV) couplings in both W and Y interactions are given as follows [20]

$$\sqrt{2}H^{\rm CC} = g\left(j_W^{\mu-}W_\mu^+ + j_Y^{\mu-}Y_\mu^+ + j_X^{\mu 0*}X_\mu^0 + H.c.\right) \tag{16}$$

where

$$\begin{aligned}
j_W^{\mu-} &= c_\theta(\overline{\nu}_{iL}\gamma^\mu e_{iL} + \overline{u}_{iL}\gamma^\mu d_{iL}) + s_\theta(\overline{\nu}^c_{iL}\gamma^\mu e_{iL} + \overline{U}_L\gamma^\mu d_{1L} + \overline{u}_{\alpha L}\gamma^\mu D_{\alpha L}), & (17)\\
j_Y^{\mu-} &= c_\theta(\overline{\nu}^c_{iL}\gamma^\mu e_{iL} + \overline{U}_L\gamma^\mu d_{1L} + \overline{u}_{\alpha L}\gamma^\mu D_{\alpha L}) - s_\theta(\overline{\nu}_{iL}\gamma^\mu e_{iL} + \overline{u}_{iL}\gamma^\mu d_{iL}), & (18)\\
j_X^{\mu 0*} &= (1-t_{2\theta}^2)(\overline{\nu}_{iL}\gamma^\mu \nu^c_{iL} + \overline{u}_{1L}\gamma^\mu U_L - \overline{D}_{\alpha L}\gamma^\mu d_{\alpha L}) - t_{2\theta}^2(\overline{\nu}^c_{iL}\gamma^\mu \nu_{iL} + \overline{U}_L\gamma^\mu u_{1L} - \overline{d}_{\alpha L}\gamma^\mu D_{\alpha L})\\
&\quad + \frac{t_{2\theta}}{\sqrt{1+4t_{2\theta}^2}}(\overline{\nu}_i\gamma^\mu \nu_i + \overline{u}_{1L}\gamma^\mu u_{1L} - \overline{U}_L\gamma^\mu U_L - \overline{d}_{\alpha L}\gamma^\mu d_{\alpha L} + \overline{D}_{\alpha L}\gamma^\mu D_{\alpha L}). & (19)
\end{aligned}$$

Note that the LNV couplings lie in the second term of Eq. (17) and in the first term of Eq. (18).

Some interesting phenomenologies can be illustrated by figure 1 bellows

Fig. 1. Feynman diagrams for wrong muon decay ($\mu^- \to e^- \nu_e \tilde{\nu}_\mu$). The new contribution is presented in the second diagram (only in the economical 3-3-1 model)

.

It is emphasized that the first diagram exists in both pioneer and E331 models, while the second diagram exists only in the E331 model.

Due to the minimal scalar sector, some quarks have vanishing masses at the tree-level, and they get masses at loop level [22–24].

The model is very rich in neutrino physics, and it has only one problem: no candidate for dark matter. For this purpose, the inner triplets were added [25].

2.4. The simple 3-3-1 model

This model belongs to class with $\beta = -\sqrt{3}$ and the third generation of quarks transforms differently from the first two generations [26]:

$$\begin{aligned}
\psi^T_{aL} &\equiv (\nu_{aL}, l_{aL}, (l_{aR})^c) \sim (1,3,0),\ Q^T_{\alpha L} \equiv (d_{\alpha L}, -u_{\alpha L}, D_{\alpha L}) \sim (3,\tilde{3},-1/3),\\
Q^T_{3L} &\equiv (u_{3L}, d_{3L}, T_L) \sim (3,3,2/3), & (20)\\
u_{aR} &\sim (3,1,2/3),\quad d_{aR} \sim (3,1,-1/3),\ D_{\alpha R} \sim (3,1,-4/3),\quad T_R \sim (3,1,5/3),
\end{aligned}$$

where $a=1,2,3$ and $\alpha=1,2$ are family indices. The new quarks possess exotic electric charges as $Q(D_\alpha)=-4/3$ and $Q(T)=5/3$.

The model has only two scalar triplets [26,27]:

$$\eta^T=\left(\eta_1^0,\eta_2^-,\eta_3^+\right)\sim(1,3,0),\quad \chi^T=\left(\chi_1^-,\chi_2^{--},\chi_3^0\right)\sim(1,3,-1),\tag{21}$$

with VEVs

$$\sqrt{2}\langle\eta\rangle^T=(u,0,0),\quad \sqrt{2}\langle\chi\rangle^T=(0,0,\omega).\tag{22}$$

To have DM candidate, sextet with $X=1$ was added [26]:

$$S=\begin{pmatrix} h_{11}^+ & \frac{\Sigma_{12}^0}{\sqrt{2}} & \frac{h_{13}^{++}}{\sqrt{2}} \\ \frac{\sigma_{12}^0}{\sqrt{2}} & h_{22}^- & \frac{h_{23}^+}{\sqrt{2}} \\ \frac{h_{13}^{++}}{\sqrt{2}} & \frac{h_{23}^+}{\sqrt{2}} & h_{33}^{+++} \end{pmatrix}\sim(1,6,1).\tag{23}$$

Under a Z_2 symmetry, this sextet is odd, whereas all the other fields are even.

Because of the minimal scalar sector, some fermions are massless at the tree-level. However, they can get corrections coming from the effective interactions.

3. The 3-3-1 model specific for leptons and neutrinos

According LHCb data in 2014, the original 3-3-1 models face non-universality in lepton sector and neutrino physics. The attempts for solving these troubles were made and some of them are presented below.

3.1. The flipped 3-3-1 model

In the original 3-3-1 models, the anomaly free requires one quark generation transforms under $SU(3)_L$ differently from two the other ones. In the flipped version, the discrimination happens in lepton sector [28], namely, the lightest leptons (e) transforms as sextet (symmetric representation), while two heavy (μ) and (τ) transform as triplets. In contrast, all left-handed quarks are in antitriplets, so that the model is free gauge anomaly. To provide fermion masses, the scalar sector contains three triplets and one sextet. The particle content of the model is summarized in Table 1 [28,29].

The Lagrangian for leptons is given as

$$-\mathscr{L}_{\text{lepton}}^Y=\sum_{i=1}^{2}\sum_{\alpha=\mu,\tau}\sum_{\beta=1}^{6}y_{\alpha\beta}^{\ell(i)}\overline{l_{\beta R}}L_\alpha\phi_i^*+\sum_{\beta=1}^{6}y_\beta^{\ell\prime}\overline{e_{\beta R}}L_lS^*+y^{\ell\prime\prime}\overline{(L_l)^c}L_lS+\text{H.c.},\tag{24}$$

where the invariant term of the tensor product of the three sextets is expanded as $\overline{(L_l)^c}L_lS=\epsilon^{abc}\epsilon^{ijk}\overline{(L_l)^c}_{ai}(L_l)_{bj}S_{ck}$, $(L_l)^c_{ai}\equiv C\overline{(L_l)_{ai}}^T$. Note that ϕ_3 only appears in the Yukawa part of the quark guaranteeing that all quark are massive. In contrast, at tree level, the electron and neutrinos are massless, but one-loop contributions are enough to.

Note that there is no generation discrimination between three left-handed quarks multiplets in the flipped 3-3-1 model. Hence, there does not exist flavor changing neutral currents mediated by extra Z' at tree level. The natural mixing happening in the lepton sector of the flipped 3-3-1 predicts many LFV sources. It leads to the existence of the LFV decays, which should be investigated thoroughly elsewhere. For example the LFV decays of the SM-like Higgs boson was

Table 1. Particle content of the flipped 3-3-1 model.

Name	3-3-1 rep.	Components	# flavors
L_e	$\left(\mathbf{1},\mathbf{6},-\frac{1}{3}\right)$	$\begin{pmatrix} (\Sigma_R^-)^c & \frac{1}{\sqrt{2}}\Sigma_L^0 & \frac{1}{\sqrt{2}}\nu_{eL} \\ \frac{1}{\sqrt{2}}\Sigma_L^0 & \Sigma_L^- & \frac{1}{\sqrt{2}}l_L \\ \frac{1}{\sqrt{2}}\nu_{lL} & \frac{1}{\sqrt{2}}l_L & E_{lL} \end{pmatrix}$	1
$L_{\alpha=\mu,\tau}$	$\left(\mathbf{1},\widehat{\mathbf{2}},-\frac{1}{2}\right)+\left(\mathbf{1},\widehat{\mathbf{1}},-1\right)$	$(\nu_\alpha,l_\alpha,E_\alpha)_L^T$	2
$l_{\alpha R}$	$\left(\mathbf{1},\widehat{\mathbf{1}},-1\right)$	$l_{\alpha R}$	6
Q_α	$\left(\mathbf{3},\widehat{\mathbf{2}},\frac{1}{6}\right)+\left(\mathbf{3},\widehat{\mathbf{1}},\frac{2}{3}\right)$	$(d_\alpha,-u_\alpha,U_\alpha)_L^T$	3
$u_{\alpha R}$	$\left(\mathbf{3},\widehat{\mathbf{1}},\frac{2}{3}\right)$	$u_{\alpha R}$	6
$d_{\alpha R}$	$\left(\mathbf{3},\widehat{\mathbf{1}},-\frac{1}{3}\right)$	$d_{\alpha R}$	3
$\phi_{i=1,2}$	$\left(\mathbf{1},\widehat{\mathbf{2}},\frac{1}{2}\right)+\left(\mathbf{1},\widehat{\mathbf{1}},0\right)$	$\left(H_i^+,H_i^0,\sigma_i^0\right)^T$	2
ϕ_3	$\left(\mathbf{1},\widehat{\mathbf{2}},-\frac{1}{2}\right)+\left(\mathbf{1},\widehat{\mathbf{1}},-1\right)$	$\left(H_3^0,H_3^-,\sigma_3^-\right)^T$	1
S	$\left(\mathbf{1},\widehat{\mathbf{3}},1\right)+\left(\mathbf{1},\widehat{\mathbf{2}},\frac{1}{2}\right)+\left(\mathbf{1},\widehat{\mathbf{1}},0\right)$	$\begin{pmatrix} \Delta^{++} & \frac{1}{\sqrt{2}}\Delta^+ & \frac{1}{\sqrt{2}}H_S^+ \\ \frac{1}{\sqrt{2}}\Delta^+ & \Delta^0 & \frac{1}{\sqrt{2}}H_S^0 \\ \frac{1}{\sqrt{2}}H_S^+ & \frac{1}{\sqrt{2}}H_S^0 & \sigma_S^0 \end{pmatrix}$	1

discussed in Ref. [29]. Also, the new heavy leptons in the electron sextet may gives significant one-loop contributions to $(g-2)_e$ anomalies.

Note that, in the flipped 3-3-1 model, due to lepton generation discrimination, there exists lepton flavor changing neutral current mediated by extra Z' at tree level.

3.2. The 3-3-1 model with inverse seesaw neutrinos

It is well known that the seesaw mechanism happens at very high energy scale. The inverse seesaw (ISS) mechanism is one of the mechanisms with low energy scale, which can explain the neutrino oscillation as well as the lepton flavor violating decay rates of charged leptons (cLFV) hunted by experiments. The 3-3-1 model with ISS neutrinos (331ISS) for $\beta = \frac{1}{\sqrt{3}}$ has been constructed in Refs. [30–33]. The quark sector is similar and quark discussion is referred to Ref . [32]. Each lepton family consists of a $SU(3)_L$ triplet

$$\psi_{aL}^T = (\nu_a,\ l_a, N_a)_L \sim \left(1,3,-\frac{1}{3}\right),$$

and a right-handed charged lepton $l_{aR} \sim (1,1,-1)$ with $a = 1,2,3$. Each left-handed neutrino $N_{aL} = (N_{aR})^c$ is equivalent with a new right-handed neutrinos defined in previous 331RN model [8]. The 331ISS model contains three more RH neutrinos transforming as gauge singlets, $X_{aR} \sim (1,1,0)$, $a = 1,2,3$. They couple with the $SU(3)_L$ Higgs triplets to generate the neutrino mass term relating with the ISS mechanism.

The Higgs sector is the same as in Section 2.2.

The Lagrangian for generating lepton masses is [34]:

$$\mathscr{L}_{lep}^{\mathrm{Y}} = -h^e_{ab}\overline{\psi_{aL}}\rho l_{bR} + h^\nu_{ab}\epsilon^{ijk}\overline{(\psi_{aL})_i}(\psi_{bL})^c_j\rho^*_k - Y^{ab}\overline{\psi_{aL}}\chi X_{bR} - \frac{1}{2}(\mu_X)^*_{ba}\overline{(X_{aR})^c}X_{bR} + \mathrm{H.c.}. \quad (25)$$

Assuming the model respects a new lepton number symmetry $\mathscr{L}$ discussed in [35] so that the term $\overline{\psi_{aL}}\eta X_{bR}$ is not allowed in the above Yukawa Lagrangian, while the soft-breaking term $(\mu_X)^*_{ba}\overline{(X_{aR})^c}X_{bR}$ is allowed with small $(\mu_X)_{ba}$. The new lepton number $\mathscr{L}$ called by generalized lepton number [36] is defined as $L = \frac{4}{\sqrt{3}}T^8 + \mathscr{L}\mathbb{I}$, where L is the normal lepton number.

The first term in Lagrangian (25) generates charged lepton masses $m_{e_a} \equiv \frac{h^e_{ab}v_1}{\sqrt{2}}\delta_{ab}$, i.e, the mass matrix of the charged leptons is assumed to be diagonal, hence the flavor states of the charged leptons are also the physical ones. In the basis $\nu'^T_L = (\nu_L, N_L, (X_R)^c)$ and $(\nu'_L)^c = ((\nu_L)^c, (N_L)^c, X_R)^T$ of the neutral leptons, Lagrangian (25) gives a neutrino mass term corresponding to a block form of the mass matrix [33], namely

$$-\mathscr{L}^\nu_{\mathrm{mass}} = \frac{1}{2}\overline{\nu'_L}M^{\nu\dagger}(\nu'_L)^c + \mathrm{H.c., where} \quad M^{\nu\dagger} = \begin{pmatrix} 0 & m_D & 0 \\ m_D^T & 0 & M_R \\ 0 & M_R^T & \mu_X^\dagger \end{pmatrix}, \quad (26)$$

where M_R is a 3×3 matrix $(M_R)_{ab} \equiv Y_{ab}\frac{w}{\sqrt{2}}$, $(m_D)_{ab} \equiv \sqrt{2}h^\nu_{ab}v_1$ with $a,b = 1,2,3$. Neutrino subbases are denoted as $\nu_R^T = ((\nu_{1L})^c, (\nu_{2L})^c, (\nu_{3L})^c)$, $N_R^T = ((N_{1L})^c, (N_{2L})^c, (N_{3L})^c)$, and $X_L^T = ((X_{1R})^c, (X_{2R})^c, (X_{3R})^c)$. The mass matrix M_R does not appear in the 331RN model. The Dirac neutrino mass matrix m_D must be antisymmetric. The matrix μ_X defined in Eq. (25) is symmetric and it can be diagonalized by a transformation U_X:

$$U_X^T\mu_X U_X = \mathrm{diag}\left(\mu_{X,1}, \mu_{X,2}, \mu_{X,3}\right). \quad (27)$$

The matrix U_X will be absorbed by redefinition the states X_a, therefore μ_X will be set as the diagonal matrix given in the right-hand side of Eq. (27).

The mass matrix M^ν is diagonalized by a 9×9 unitary matrix U^ν,

$$U^{\nu T}M^\nu U^\nu = \hat{M}^\nu = \mathrm{diag}(m_{n_1}, m_{n_2}, ..., m_{n_9}) = \mathrm{diag}(\hat{m}_\nu, \hat{M}_N), \quad (28)$$

where m_{n_i} $(i = 1,2,...,9)$ are masses of the nine physical neutrino states n_{iL}. They consist of three active neutrinos n_{aL} $(a = 1,2,3)$ corresponding to the mass submatrix $\hat{m}_\nu = \mathrm{diag}(m_{n_1},\ m_{n_2},\ m_{n_3})$, and the six extra neutrinos n_{IL} ($I = 4,5,..,9$) with $\hat{M}_N = \mathrm{diag}(m_{n_4},\, m_{n_5}, ...,\, m_{n_9})$. The ISS mechanism leads to the following approximation solution of U^ν,

$$U^\nu = \Omega \begin{pmatrix} U_{\rm PMNS} & \mathbf{O} \\ \mathbf{O} & V \end{pmatrix}, \;\; \Omega = \exp \begin{pmatrix} \mathbf{O} & R \\ -R^\dagger & \mathbf{O} \end{pmatrix} = \begin{pmatrix} 1-\frac{1}{2}RR^\dagger & R \\ -R^\dagger & 1-\frac{1}{2}R^\dagger R \end{pmatrix} + \mathscr{O}(R^3), \tag{29}$$

where

$$R^* \simeq \left(-m_D^* M^{-1}, \quad m_D^*(M_R^\dagger)^{-1}\right), \quad M \equiv M_R^* \mu_X^{-1} M_R^\dagger, \tag{30}$$

$$m_D^* M^{-1} m_D^\dagger \simeq m_\nu \equiv U_{\rm PMNS}^* \hat{m}_\nu U_{\rm PMNS}^\dagger, \tag{31}$$

$$V^* \hat{M}_N V^\dagger \simeq M_N + \frac{1}{2} R^T R^* M_N + \frac{1}{2} M_N R^\dagger R, \; M_N \equiv \begin{pmatrix} 0 & M_R^* \\ M_R^\dagger & \mu_X \end{pmatrix}. \tag{32}$$

The relations between the flavor and mass eigenstates are

$$\nu'_L = U^\nu n_L, \quad \text{and } (\nu'_L)^c = U^{\nu *}(n_L)^c, \tag{33}$$

where $n_L^T \equiv (n_{1L}, n_{2L}, ..., n_{9L})$ and $(n_L)^c \equiv ((n_{1L})^c, (n_{2L})^c, ..., (n_{9L})^c)^T$. The standard form of the lepton mixing matrix $U_{\rm PMNS}$ is a function of three angles θ_{ij}, one Dirac phase δ and two Majorana phases α_1, and α_2, [1], namely

$$\begin{aligned} U_{\rm PMNS} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{diag}\left(1, e^{i\alpha_1}, e^{i\alpha_2}\right) \\ &= U_{\rm PMNS}^0 \,\mathrm{diag}\left(1, e^{i\alpha_1}, e^{i\alpha_2}\right), \end{aligned} \tag{34}$$

where $s_{ij} \equiv \sin\theta_{ij}$, $c_{ij} \equiv \cos\theta_{ij}$, $i,j = 1,2,3$ ($i<j$), $0 \le \theta_{ij} < 90$ [Deg.] and $0 < \delta \le 720$ [Deg.]. The Majorana phases are chosen in the range $-180 \le \alpha_i \le 180$ [Deg.]

The 331ISS model inherits interesting consequences in the neutrinos sector that does not exist in other 3-3-1 models with $\beta \neq \frac{1}{\sqrt{3}}$. In particularly, the Dirac mass matrix m_D is always antisymmetric, hence predicts one massless active neutrino at tree level. In addition, m_D cannot be parameterized using the most popular form introduced in Ref. [37]. Instead, m_D has a very special form [32], leading to strict relations of cLFV rates if all heavy neutrinos have the same masses, such as $\mathrm{Br}(\mu \to e\gamma)/\mathrm{Br}(\tau \to \mu\gamma)$ = constant and depends precisely on the neutrino oscillation data. On the other hand, non degenerate heavy neutrino masses can relax this cLFV relations, but the recent experimental constraint of $\mathrm{Br}(\tau \to \mu\gamma)$ results in the largest value of the deviation of $(g-2)_\mu$ anomaly from the SM is around 10^{-9}, not enough to explain the experimental $(g-2)_\mu$ data [33]. A solution for this problems is adding a new singly charged Higgs boson so that new Yukawa couplings will lead to new one-loop contributions that cancel the large cLFV amplitudes, while increase the $(g-2)_\mu$ value. The allowed parameter regions explaining successfully the $(g-2)_\mu$ experimental data were shown in Ref. [33]. A study discussing on the regions explaining both experimental data of $(g-2)_{e,\mu}$ anomalies as well as LFV decays of charged leptons and the SM-like Higgs boson was introduced [38]. Finally, the 331ISS model has rich physical consequences in the lepton sector because of the very special form of m_D, which is deserved for further studies other LFV decays such as $Z \to l_b l_a$, $l_b \to l_a l_c l_d$, etc.

4. The 3-3-1 model with non-Abelian discrete symmetries

It is well known that neutrinos are massive and their mixing with very special forms. A prospective research direction has been studied extensively, to explain lepton mixing pattern, the small quark mixing angles, the tiny masses of neutrinos and other important phenomenologies, that is the combination of discrete symmetries and the SM or it's extensions. In this treatment, discrete symmetries, such as A_4, S_3, S_4, D_4, T_7 and so on, have been added to the BSMs such as the 3-3-1 models, $B-L$ models, etc [39–44]. On the other hand, the most natural solution of the quark and lepton mass hierarchies probably are by the Froggatt-Nielsen mechanism or by non-Abelian discrete symmetries [45–47]. Within the Froggatt-Nielsen mechanism, the obtained values of the three neutrino mixing angles are quite small compared to the experimental data, and while the quark and charged-lepton mass hierarchies can be explained within this mechanism, the specific predictions suffer from relatively large errors.

Apart from neutrino phenomenology, 3-3-1 model also naturally accommodate potential candidate for DM which have been applied with discrete symmetry [48–50]. Moreover, in recent years, the anomalous magnetic moment (AMM) of charged leptons being an interesting problem that goes BSMs. There is an inconsistency in ae between theoretical and experimental values. The experimental magnitude of the differences owns both negative and positive signs, with $\Delta a_e = (-8.8 \pm 3.6)10^{-13}$ [51,52] and $\Delta a_e = (4.8 \pm 3.0)10^{-13}$ [52,53]. Therefore, we need further investigations of electron's AMM (ae) to confirm the sign of Δa_e. Further, the of the muon's AMM (a_μ) plays a central role in precision tests of the SM. The difference between experiment and theory on a_μ is determined as $\Delta a_\mu = (251 \pm 59)10^{-11}$ [52]. The difference between theoretical and experimental values of ae and $a\mu$ will open the great prospects for physics beyond the SM which can be addressed within the framework of the 3-3-1 model [54] and the 3-3-1 model with discrete symmetry [55].

5. The 3-3-1 model with cosmological inflation

The axion which is very attractive to experimental searches, is a popular topic in the modern physics. This is arised from the spontaneous breaking of the the global $U(1)_{Q_A}$ symmetry that was implemented by Peccei - Quinn (PQ) to solve the Strong *CP* problem [56, 57].

5.1. The 3-3-1 model with axion like particle

In the frameworks of the 3-3-1 models, the PQ formalism has been considered, for the version with $\beta = -\frac{1}{\sqrt{3}}$, almost two decades ago [58–60]. The main ingredient is a singlet scalar ϕ carrying lepton number two. However, in the above mentioned works, the singlet ϕ is expanded as

$$\phi = \frac{1}{\sqrt{2}}(v_\phi + R_\phi + iI_\phi)\,. \tag{35}$$

It is worth mentioning that within above expansion of scalar ϕ, its pseudoscalar I_ϕ should be called ALP but not axion. In this case, the ALP mixs with other CP odd scalars. The axion is the pure imaginary part of ϕ only in the limit $v_\phi \gg v_\chi$, which is the VEV responsible for breaking from $SU(3)_L$ to SM subgroup. In Ref. [60], the PQ symmetry was considered for two main versions: M331 and 331RN models [61, 62].

New development in this direction was done seven years ago in Ref. [63], where the discrete symmetry $Z_{11} \times Z_2$ is imposed, and Majorana right-handed neutrinos are introduced. To provide mass of Majorana neutrinos, a complex scalar transforming as a SU(3)$_L$ singlet ϕ is added. As a consequence, the model also contains a heavy CP even scalar with mass in the range of 10^{10} GeV identified as an inflaton. However, Ref. [63] still contains some flawed points such as no identification of the SM-like Higgs boson and incorrect mixing of CP odd scalars. The mentioned problems have been recently solved in Ref. [64]. As a result, the model contains the expected inflaton with mass around 10^{11} GeV, a heavy scalar with mass at TeV scale labeled by H_χ, one scalar with mass at the EW scale labeled by (h_5), and of course the SM-like Higgs boson (h).

The particles and their transforms [63,64] under SU(3)$_C \times$ SU(3)$_L \times$ U(1)$_N \times Z_{11} \times Z_2$ group are presented in Table 2.

Table 2. $SU(3)_C \otimes SU(3)_L \otimes U(1)_N \otimes Z_{11} \otimes Z_2$ charge assignments of the particles in the model. Here $w_k = e^{ik2\pi/11}$, $a = 1,2,3$ and $\alpha = 1,2$.

	Q_{3L}	Q_{nL}	u_{aR}	d_{aR}	T_{3R}	D_{nR}	ψ_{aL}	l_{aR}	N_{aR}	η	χ	ρ	ϕ
$SU(3)_C$	**3**	**3**	**3**	**3**	**3**	**3**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
$SU(3)_L$	$\bar{\mathbf{3}}$	**3**	**1**	**1**	**1**	**1**	**3**	**1**	**1**	**3**	**3**	**3**	**1**
$U(1)_N$	$\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	-1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	0
Z_{11}	ω_0	ω_4^{-1}	ω_5	ω_2	ω_3	ω_4	ω_1	ω_3	ω_5^{-1}	ω_5^{-1}	ω_3^{-1}	ω_2^{-1}	ω_1^{-1}
Z_2	1	1	-1	-1	1	1	1	-1	-1	-1	1	-1	1

Masses of fermions and gauge bosons request VEVs of three triplets and one singlet

$$\begin{aligned} \sqrt{2}\langle\eta\rangle^T &= (v_\eta,0,0)\,,\ \sqrt{2}\langle\chi\rangle^T = \left(0,0,v_\chi\right)\,, \\ \sqrt{2}\langle\rho\rangle^T &= \left(0,v_\rho,0\right)\,,\quad \sqrt{2}\langle\phi\rangle = v_\phi\,. \end{aligned} \tag{36}$$

The symmetry breaking of the model under consideration is in three steps by following scheme:

$$\begin{array}{c} SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes Z_{11} \otimes Z_2 \otimes U(1)_{Q_A} \\ \downarrow \langle\phi\rangle \\ SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes Z_2 \\ \downarrow \langle\chi\rangle \\ SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \\ \downarrow v_\eta, v_\rho \\ SU(3)_C \otimes U(1)_Q. \end{array}$$

The version with axion [65] is presented in the next section.

5.2. *The 3-3-1 model with axion*

The new format of writing PQ transforms is given in Refs. [66–69] and in Refs. [70, 71]. According to Ref. [59], and using notations in Refs. [70, 71] for an arbitrary fermion and scalar boson, the PQ transformations are as follows For future presentation, here we write explicitly the scalar ϕ

$$\phi = \frac{1}{2}(v_\phi + R_\phi)\, e^{i\frac{a}{2f_a}c_\phi}\,. \tag{37}$$

The new format of writing PQ transforms is given in Refs. [66–69] and in Refs. [70, 71]. According to Ref. [59], and using notations in Refs. [70, 71] for an arbitrary fermion and scalar boson, the PQ transformations are as follows

$$f \;\rightarrow\; f' = e^{i\left(\frac{c_f}{2f_a}\right)\gamma_5 a} f\,, \quad \bar{f} \rightarrow \bar{f}' = \bar{f} e^{i\left(\frac{c_f}{2f_a}\right)\gamma_5 a}\,, \quad \varphi \rightarrow \varphi' = e^{i\left(\frac{c_\varphi}{2f_a}\right)a}\varphi\,. \tag{38}$$

For chiral fermions in the model under consideration, this is

$$\begin{aligned} f_L \;&\rightarrow\; f'_L = e^{-i\left(\frac{c_f}{2f_a}\right)a} f_L\,, \quad \bar{f}_L \rightarrow \bar{f}'_L = \bar{f}_L e^{i\left(\frac{c_f}{2f_a}\right)a}\,,\\ f_R \;&\rightarrow\; f'_R = e^{i\left(\frac{c_f}{2f_a}\right)a} f_R\,, \quad \bar{f}_R \rightarrow \bar{f}'_R = \bar{f}_R e^{-i\left(\frac{c_f}{2f_a}\right)a}\,, \end{aligned} \tag{39}$$

where c_f is PQ charge of fermion and $f_a \sim 10^{11}$ GeV is axion decay constant relating to the scale of symmetry breaking of $U(1)_{PQ}$ global group. The values c_F for transformations of fermions under the Z_{11} symmetry are given as [63]

$$c_u = c_T = -c_d = -c_D = c_l = -c_{lR} = -c_\nu = c_{\nu_R} = -c_N \equiv R\,. \tag{40}$$

For scalars, from Yukawa couplings, it follows that charged scalars have vanishing PQ charge since they connect *up and down* particles with opposite values, while electrically neutral scalar has PQ charge duplicate charge (with opposite sign) of fermion to which it provides mass, because it connects to *both up or down* particles:

$$\begin{aligned} \eta_1^0 \;&\rightarrow\; e^{i\frac{a}{f_a}}\eta_1^0\,, \quad \chi_1^0 \rightarrow e^{i\frac{a}{f_a}}\chi_1^0\,, \quad \rho_1^+ \rightarrow \rho_1^+\,,\\ \phi \;&\rightarrow\; e^{i\frac{a}{f_a}}\phi\,, \quad \rho_2^0 \rightarrow e^{-i\frac{a}{f_a}}\rho_2^0\,, \quad \chi_2^- \rightarrow \chi_2^-\,. \end{aligned} \tag{41}$$

The term $\lambda_\phi \epsilon^{ijk}\eta_i\rho_j\chi_k\phi^*$ shows that ρ has the same value and opposite to that of η, χ and ϕ [65]. This is explicitly clarified in Table 3. Hereafter, the PQ charge is renamed by Q_A charge.

Table 3. $U(1)_{Q_A}$ charge assignments of the particle content of the model. Here $Q_A(F) = c_F = c_{F_R} = -c_{F_L}$.

	u	d	T	D_α	l	ν	ν_R	N_R	η_1^0	η_3^0	χ_1^0	χ_3^0	ρ^0	ϕ	η_2^-	χ_2^-	ρ_1^+	ρ_3^+
$U(1)_{Q_A}$	1	-1	1	-1	-1	$+1$	1	-1	2	2	2	2	-2	2	0	0	0	0

Note that the Higgs sector in this model is similar to that in Ref. [64] with just difference in the *CP* odd sector, where the component I_ϕ is decoupled.

In the limit $v_\phi \gg v_\chi \gg v_\rho \gg v_\eta$, the scalar content of the model can be presented as follows [64]

$$\begin{aligned} \rho \;&\simeq\; \begin{pmatrix} G_{W^+} \\ \frac{1}{\sqrt{2}}(v + h + iG_Z) \\ H_2^+ \end{pmatrix}, \eta \simeq \begin{pmatrix} \frac{1}{\sqrt{2}}(u + h_5 + iA_5) \\ H_1^- \\ G_{X^0} \end{pmatrix}, \chi \simeq \begin{pmatrix} \chi_1^0 \\ G_{Y^-} \\ \frac{1}{\sqrt{2}}(v_\chi + H_\chi + iG_{Z'}) \end{pmatrix},\\ \phi \;&=\; \frac{1}{\sqrt{2}}\left(v_\phi + \Phi\right) e^{-i\frac{a}{f_a}}\,. \end{aligned} \tag{42}$$

Combination of Table 2 and Eq. (42) leads to the following interesting consequences: Firstly, the SM-like Higgs boson h has Yukawa couplings with only SM fermions. Secondly, the heavy scalar H_χ and pseudoscalar A_5 can have Yukawa couplings with not only exotic quarks but also SM quarks and leptons.

For singlets of right-handed fermions and scalar ϕ, the generators T_3 and T_8 produce the zero, but the general PQ charge $\mathscr{X}_{pq}$ for right handed fermions takes opposite value of left-handed partner, while for ϕ, as usually, it is followed from Yukawa couplings, i.e., $\mathscr{X}_{pq}(f_R) = -Q_A(f_L)$ and $\mathscr{X}_{pq}(\phi) = 2\,,\Rightarrow Q_A(\phi) = 2$. To be noted that there exists the same situation when one deals with electric charges of right-handed fermions.

Note that the parameter R in Eq. (40) in principle, is any non-zero integer. However, as seen later, within notations in Eq. (38), the absolute value of R should be equal to the unit, i.e., $|R| = 1$. From kinetic part of scalar ϕ, it follows

$$\left(\frac{c_\phi}{2f_a}\right)^2 = \left(\frac{1}{f_a}\right)^2, \tag{43}$$

$$f_a = v_\phi\,. \tag{44}$$

From (43), one gets $c_\phi = \pm 2$. From (40) and (41), one obtains $c_\phi = 2c_u$. Therefore $|R| = 1$, and we choose the sign is plus, i.e., $R = 1$.

i) Formula for PQ charge

The PQ charges given in Table 3 allow us to write some nice formula similar to generalized lepton number in [35,36]. From Table 3, and Eq. (41), we can formulate PQ charge assignment for triplets in the model under consideration as follows

$$Q_A = 2T_3 - \frac{2}{\sqrt{3}}T_8 + \mathscr{C}_f\,. \tag{45}$$

Let us call $\mathscr{C}_f$ by hyper-chirality. It is worth mentioning that the formula (45) is firstly constructed for BSMs. Compering with Eq. (9) in Ref. [62], one can see that our formula is much better.

Therefore, the PQ transformation in Eqs. (39) and (41) can be written in form of PQ charge operator labeled by Q_A as follows

$$U(1)_{Q_A}: \quad f \to f' = e^{i\left(\frac{a}{2f_a}\right)\gamma_5 Q_A} f\,, \quad \bar{f} \to \bar{f}' = \bar{f} e^{i\left(\frac{a}{2f_a}\right)\gamma_5 Q_A}\,, \tag{46}$$

$$\begin{aligned} f_L &\to f'_L = e^{-i\left(\frac{a}{2f_a}\right)Q_A} f_L\,, \quad \bar{f}_L \to \bar{f}'_L = \bar{f}_L e^{i\left(\frac{a}{2f_a}\right)Q_A}\,, \\ f_R &\to f'_R = e^{i\left(\frac{a}{2f_a}\right)Q_A} f_R\,, \quad \bar{f}_R \to \bar{f}'_R = \bar{f}_R e^{-i\left(\frac{a}{2f_a}\right)Q_A}\,, \end{aligned} \tag{47}$$

$$\varphi \to \varphi' = e^{+i\left(\frac{a}{2f_a}\right)Q_A}\varphi\,. \tag{48}$$

It is worth noting that in the realization the Georgi-Kaplan-Randall (GKR) field basis, all fields except the axion, transform by additive constant. It can be proved that the two transformations of f_L and f_R result from the combined transformation of $f = (f_L, f_R)^T$ given in Eq. (46). Consequently, a four-component fermion f and its right-handed component have the same Q_A value and always have a opposite sign with the left-handed component.

It is important to note that the PQ charge of the similar singlet σ in Ref. [62] cannot be fixed (see Table 1 there).

The following remarks are in order.

(1) The formula (45) shows that the difference of electric charges of up and down quarks/leptons is 1, i.e., $\Delta Q = 1$, while for PQ charge $\Delta Q_A = 2$. For fermions, electric charges (Q) of left-handed and right-handed fermions are to be equal, while their PQ (Q_A) charges are opposite. For scalars, only neutral scalars have PQ charges equal to ± 2, where the sign plus and minus corresponding to scalars in top and bottom of doublets. The charged scalar does not have a PQ charge. So the electric charge (Q) and (Q_A) have again opposite property.

(2) It is worth mentioning that in the model under consideration, the relations among PQ charges (Q_A) are quite simple

$$\begin{aligned} Q_A(u) &= -Q_A(v) \\ &= \frac{1}{2}\left[Q_A(\eta) = Q_A(\chi) = -Q_A(\rho) = Q_A(\phi)\right] \\ &= 1\,, \end{aligned} \tag{49}$$

which is completely different from those in [62].

To solve the strong CP problem, the $U(1)_{Q_A}$ symmetry is spontaneously breaking by VEV of the singlet ϕ as follows

$$Q_A\langle\phi\rangle = \sqrt{2}v_\phi \neq 0\,. \tag{50}$$

From Table 3, we see that PQ charge of neutral scalar equals twice of PQ charges of fermions receiving for which scalar provides masses, so all Yukawa couplings are invariant. In addition, exotic quarks carry lepton number 2, while ordinary SM quarks do not, so mass eigenstates of exotic quarks are their original states, while ordinary quarks are with mass mixing.

ii) Axion couplings

- The axion - fermion derivative couplings have the form

$$\mathscr{L}_{(f-a)} = +\left(\frac{1}{f_a}\right)\partial^\mu a \left[\bar{d}\,\mathbf{c}_d\,\gamma_\mu\gamma_5 d + \bar{u}\,\mathbf{c}_u\,\gamma_\mu\gamma_5 u + \bar{T}\,\mathbf{c}_T\,\gamma_\mu\gamma_5 T + \bar{D}_\alpha\,\mathbf{c}_{D_\alpha}\,\gamma_\mu\gamma_5 D_\alpha\right. \tag{51}$$

$$\left. + \bar{l}\,\mathbf{c}_l\gamma_\mu\,\gamma_5 l + I_\nu\,\bar{\nu}_a\,\mathbf{c}_\nu\,\gamma_\mu\gamma_5\nu_a + \frac{1}{2}\bar{N}_a\,\mathbf{c}_{N_a}\,\gamma_\mu P_R N_a\right]. \tag{52}$$

In Eq. (52), for the coefficients ($\mathbf{c}_f, f = d, u, \cdots N_R$), one has to count the number of color, flavor indexes and PQ charge $Q_A(f)$.

- Regarding the scalar sector, Lagrangian for covariant kinetic terms of a complex scalar φ is

$$\begin{aligned} \mathscr{L}_\varphi &= (D^\nu\varphi)^\dagger D_\nu\varphi \\ &= [(\partial_\nu - iP^\varphi_\nu)\varphi]^\dagger(\partial^\nu - iP^{\varphi\nu})\varphi \\ &= \partial^\nu\varphi^\dagger\partial_\nu\varphi - i\partial_\nu\varphi^\dagger P^{\varphi\nu}\varphi + i\varphi^\dagger P^{\varphi\nu}\partial_\nu\varphi + \varphi^\dagger P^\varphi_\nu P^{\varphi\nu}\varphi. \end{aligned} \tag{53}$$

The total part concerned to axion is given below

$$
\begin{aligned}
\mathscr{L}_a &= \frac{1}{2}\partial_\nu a \partial^\nu a - \frac{1}{2}m_{ao}^2 a^2 + c_{GG}\frac{\alpha_s}{4\pi}\frac{a}{2f_a}G\tilde{G} + c_{WW}\frac{\alpha_2}{4\pi}\frac{a}{2f_a}W\tilde{W} + c_{BB}\frac{\alpha_1}{4\pi}\frac{a}{2f_a}B\tilde{B} && (54)\\
&+ \frac{\partial^\alpha a}{2f_a}\left(\sum_{f=u,d,T,D}^{l,\nu}\bar{\psi}_f c_f \gamma_\alpha\gamma_5\psi + \frac{1}{2}\bar{N}_a \mathbf{c}_{N_a}\gamma_\alpha P_R N_a\right) - (\bar{q}_L M_q q_R + \text{H.c.}) && (55)\\
&+ \left(\frac{1}{f_a}\right)^2 \partial_\alpha a \partial^\mu a \left(\sum_{H=\eta_1^0,\eta_3^0,\rho_2^0}^{\chi_1^0,\chi_3^0,\phi} H^* H\right) \quad ; && (56)\\
&- i\left(\frac{c_\varphi}{2f_a}\right)\partial^\alpha a \sum_{D=\eta,\chi,\phi}^{K=\rho^0}\left[D^* \overset{\leftrightarrow}{\partial_\alpha} D - K^* \overset{\leftrightarrow}{\partial_\alpha} K\right] && (57)\\
&+ 2\left(\frac{c_\varphi}{2f_a}\right)\partial^\alpha a \sum_{H=\eta_1^0,\eta_3^0,\rho_2^0}^{\chi_1^0,\chi_3^0,\phi} H^\dagger P_\alpha^H H\,, && (58)
\end{aligned}
$$

where $H^*H = \frac{1}{2}[(v_H+R_H)^2 + I_H^2]$. It is worth emphasizing that in (55), the matrices M_q are diagonal.

iii) Axion mass

In the chiral perturbative theory, the axion mass arises from the part [72,73]

$$
\begin{aligned}
\mathscr{L}_{amass} &= \frac{f_\pi^2}{4}2B_0 2\frac{1}{3}(m_u+m_d+m_s)\cos\frac{\mathscr{K}}{f_\pi}\\
&= \frac{1}{6}f_\pi^2(m_\pi^2 + m_{K^0}^2 + m_{K^+}^2)\cos\frac{\mathscr{K}}{f_\pi} - \frac{m_a^2}{2}\,. && (59)
\end{aligned}
$$

Here [65]

$$
\begin{aligned}
m_a^2 &= \frac{1}{2}B_0\left(\frac{f_\pi}{f_a}\right)^2\left(\frac{m_u m_d m_s}{m_u m_d + m_u m_s + m_d m_s}\right)\\
&= \frac{1}{2}m_\pi^2\left(\frac{f_\pi}{f_a}\right)^2\frac{1}{(m_u+m_d)}\left(\frac{m_u m_d m_s}{m_u m_d + m_u m_s + m_d m_s}\right)
\end{aligned}
$$

This result is similar to that in [74]

$$
\begin{aligned}
m_a &= 4\frac{f_\pi m_\pi}{f_a/N}\left[\frac{m_u m_d m_s}{(m_u m_d + m_u m_s + m_d m_s)(m_u+m_d)}\right]^{\frac{1}{2}}\\
&\simeq (1.2\times 10^{-5}\,\text{eV})\left(\frac{10^{12}\text{ GeV}}{f_a/N}\right), && (60)
\end{aligned}
$$

6. Phenomenology

In this section, I just mention on main consequences of the models.

6.1. Collider physics

The doubly charged beleptons $X^{\pm\pm}$ of the minimal version are highly attached and their production at LHC was considered in [75]. Production of the Higgs boson in the M331 model at the e^+e^- Next Linear Collider, and in CERN Linear Collider (CLIC) has been considered in [76]. The bilepton gauge boson masses are usually fixed to be in the range of 600 GeV, while mass of the extra neutral gauge boson Z' is limited in the region $[1 \div 1.2]$ TeV [77].

The limits on Z' mass in the 331RN model is as follows: $M_{Z'} > 4.1$ TeV [78].

At the LHC, the cross section of $pp \to Z' \to XY$ is

$$\sigma_{(pp\to V'\to \bar{f}f')} = \sum_{\{ij\}} \int_{\tau_0}^{1} \frac{d\tau}{\tau} \cdot \frac{1}{s} \frac{d\mathscr{L}_{ij}}{d\tau} \cdot [\hat{s}\,\hat{\sigma}_{(ij\to V'\to \bar{f}f')}(\hat{s})] , \tag{61}$$

where X and Y denote the decay products of the Z' boson, $\sqrt{s}$ is the total energy of the incoming proton-proton beam, $\sqrt{\hat{s}}$ is the partonic center-of-mass energy and $\tau \equiv \hat{s}/s$ and $\frac{1}{s}\frac{d\mathscr{L}_{ij}}{d\tau}$ is the parton lumonosity [79].

It is worth noting that the single Z' can be produced in the hadron collision. The total cross-section of the scattering process $pp \to Z'$ is given by [77]

$$\begin{aligned} \sigma(pp \to Z') &= 2\sum_{i=1}^{5} \int_0^1 dx_1 \int_0^1 dx_2 f(i,x_1,Q) f(-i,x_2,Q) \hat{\sigma}(q_i\bar{q}_i \to Z') \\ &= \frac{2\pi g^2}{3c_W^2} \frac{1}{s} \sum_{i=1}^{5} [(g_{2V}^{qi})^2 + (g_{2A}^{qi})^2] \int_{M_{Z'}^2/s}^{1} f(i,x_1,M_{Z'}) f\left(-i, \frac{M_{Z'}^2}{sx_1}, M_{Z'}\right) \frac{dx_1}{x_1}, \end{aligned} \tag{62}$$

where

$$\hat{\sigma}(q\bar{q} \to Z') = \frac{1}{3}\frac{\pi g^2}{c_W^2} [(g_{2V}^{q})^2 + (g_{2A}^{q})^2] \delta(\hat{s} - M_{Z'}^2) , \quad s = x_1 x_2 s. \tag{63}$$

This subprocess in the M331 model was considered in [80].

For hadron collider CERN LHC the suitable references are [77, 81, 82], for FCC [83–85].

6.2. Neutrino mass and mixing

In this subsection, the mechanisms for generating light neutrinos in the frameworks of two pioneer 3-3-1 gauge models (M331 model and 331RH) are briefly reviewed. It is worth noting that, at the tree level, neutrinos are massless in both versions.

a) *Generation of neutrino mass in the minimal 3-3-1 model*

Due to the Landau pole around 5 TeV, the effective dimension five operator: $h(\bar{f}_L^C \eta^*)(\eta^\dagger f_L)/\Lambda$ provides mass to the neutrinos $m_\nu = h v_\eta^2/\Lambda$. Then, for $v_\eta \approx 10^2$ GeV and $\Lambda = 5$TeV, we get $m_\nu = 10\,h$ GeV As a consequence, the effective dimension-5 operator cannot provide an adaptable small neutrino mass.

Concerning the problem of small neutrino masses in the M331 model, to escape this trouble there are ways. The first one is combination of Z_3 discrete symmetries and effective dimension-11 operator. As a result, neutrinos get mass 5 to get neutrino at eV scale [86]. The second outlet requires the addition of singlet RH neutrinos to the leptonic particle content of the M331 model and then combination of type I with type II seesaw mechanisms leads to tiny neutrino masses [87].

b) *Generation of neutrino mass in the 3-3-1 model with RH neutrinos*

In the original version of the model, neutrinos are massless. Addition of the following effective dimension-five operator,

$$\mathscr{L}_{M_L} = \frac{f^{ab}}{\Lambda}\left(\overline{L^C_a}\eta^*\right)\left(\eta^\dagger L_b\right) + \text{H.c.} \tag{64}$$

leads to the left-handed neutrinos develop Majorana mass terms $(M_L)_{ab} = f_{ab}v_\eta^2/\Lambda$.

The Majorana masses for the right-handed neutrinos are followed from coupling

$$\mathscr{L}_{M_R} \quad = \frac{h^{ab}}{\Lambda}\left(\overline{L^C_a}\chi^*\right)\left(\chi^\dagger L_b\right) + \text{H.c.} \tag{65}$$

leading to term $(M_R)_{ab}\overline{(\nu_{aR})^C}\nu_{bR}$ with $(M_R)_{ab} = h_{ab}v_{\chi'}^2/\Lambda$.

The type II seesaw mechanism can be realized by adding scalar sextet $S \sim (1,6,-\frac{2}{3})$ leading to effective coupling $\bar{L}SL^C$ [88].

It is well known that the seesaw mechanism is the best way for tiny neutrino mass. For this purpose, type II seesaw mechanism has been imposed in the minimal version [87]. For the version with $\beta = -\frac{1}{\sqrt{3}}$, the tiny neutrino mass is realized by inverse seesaw mechanism [89] (For more references see Refs. [90–95])

c) *Inverse Seesaw Mechanism in the 3-3-1 model with RH neutrinos*

At present, the inverse seesaw (ISS) mechanism is attractive since it works at TeV scale. In the 331RN model three singlet neutral fermions have been added : $N_{a_L} \sim (1,0)$ [30, 96].

Then, Yukawa Lagrangian is composed by summation of terms,

$$L^Y_{\text{ISS}} = g^{ab}\epsilon^{ijk}\bar{L}^C_{a_i}\rho^*_j L_{b_k} + G^{ab}\bar{L}_a\chi(N_{b_L})^C + \frac{1}{2}\bar{N}^C_L\mu N_L + H.c. \tag{66}$$

Then, the mass matrix M_ν has the form

$$M_\nu = \begin{pmatrix} 0 & m_D^T & 0 \\ m_D & 0 & M^T \\ 0 & M & \mu \end{pmatrix}. \tag{67}$$

Here the 3×3 matrices are defined as

$$M_{ab} = G_{ab}\frac{v_{\chi_3^0}}{\sqrt{2}}, \quad m_{Dab} = g_{ab}\frac{v_\rho}{\sqrt{2}}, \tag{68}$$

with M_{ab} and $m_{D_{ab}}$ are Dirac mass matrices, with the last term has anti-symmetric form. The matrix in Eq. (67) is typical for the ISS mechanism. Note that there are two energy scales corresponding to $v_{\chi_3^0}$ and v_ρ. The third scale of energy, μ, is a energy scale specific for the ISS mechanism lying at KeV scale.

The above mass matrix above leads to three light Majorana neutrinos given by,

$$m_{light} = m_D^T M^{-1}\mu(M^T)^{-1}m_D, \tag{69}$$

and six heavy new neutrinos with masses around TeV scale. For $v_\rho \simeq 10^2$ GeV, $v_{\chi_3^0} \simeq 10^3$ GeV and $\mu \simeq 10^{-7}$ GeV , one obtains two light neutrinos with masses around eV.

In Refs. [97, 98], generation of neutrinos mass at two-loop radiative mechanism for the minimal version was presented, while for version with $\beta = -\frac{1}{\sqrt{3}}$ in Ref. [99].

6.3. Lepton flavor violation

Since leptons and antileptons lie in the lepton triplet, the lepton flavor violation (LFV) arises in the 3-3-1 models. However, the generalized lepton number $\mathscr{L}$ [35, 36] is conserved. Based on general formula given in Ref. [100], the LFV of charged leptons in the 3-3-1 models have been presented in Refs. [101, 102].

6.4. $(g-2)_\mu$ *puzzle*

In the frameworks of the 3-3-1 models, the muon $(g-2)$ was studied in [103]. Recent $(g-2)$ experimental value has been updated in Ref. [104], and the deviation from the SM prediction is given by 5.1 σ [104, 105]. Moreover, the 331β model consisting of 6 new inverse seesaw (ISS) neutrinos, named the 331βISS model, and a singly charged Higgs boson can explain successfully both the $(g-2)_{e,\mu}$ data and the neutrino oscillation data [106]. For combination of $(g-2)$ with other aspects, the reader is referred to Refs. [107–109].

6.5. Dark matter

Self-interacting dark matter (SIDM) were considered in the pioneer versions: it has been discussed in the frameworks of the M331 model by D. Fregolente and M. D. Tonasse [110] and for the 331RH model by Long and Lan in [111]. By adding inert scalar singlets, the reader is referred to Dong's works (for example, see Ref. [112]).

6.6. Early Universe

It is well known that the baryon asymmetry of universe (BAU) is one of the most long-standing puzzles in Particle Physics. To solve this puzzle, the key ingredients are three Sakharov conditions, which are *B* violation, *C* and *CP* violations, and deviation from thermal equilibrium

The comm acceptable opinion for the BAU is that in Early Universe, there exists cosmological inflation in the period from 10^{-36} to 10^{-34} s after Big Bang. In the frameworks of the 3-3-1 models, the inflation was considered in Refs. [113, 114]. The electroweak phase transition (EWPT) was studied in Refs. [115–117]. In addition, some related problems are leptogenesis and sphalerons, for which in the frameworks of the 3-3-1 models were condisered in Refs. [118, 119] and [120], respectively.

7. Conclusions

In conclusion, the models based on the $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)_L \otimes \mathrm{U}(1)_X$ gauge group contain intriguing features. The models can give explanations on number of fermion generation, PQ mechanism, neutrino mass and mixing as well as dark matter and baryon asymmetry of the Universe.

It is worth mentioning that in the frameworks of the model with $\beta = -\frac{1}{\sqrt{3}}$, the PQ charge operator is constructed. The advancements continue to unfold, we encourage staying engaged and keeping theoretical interest.

Acknowledgements

This research has received funding from National Foundation for Science and Technology Development (NAFOSTED) under grant number 103.01-2023.45. The author thanks P. V. Dong, L. T. Hue and V. V. Vien for reading draft and corrections.

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