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Effect of time-dependent inter-particle interaction on the dynamics of a dissipative double-well Bose-Einstein condensate

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Abstract. *In this paper we study the dynamics of a dissipative double-well Bose-Einstein condensate driven by sinusoidally oscillating time-dependent inter-particle interaction. The phase-space of system is illustrated physically by its population imbalance and phase-difference. The macroscopic dynamics of the model is generated within the mean-field limit. Noise in the system is neglected since physical parameters are calculated using single moment averages only. However, noise-noise correlation and dissipation persist in the system. We found phase-space dynamics is sensitive to infinitesimal changes of the initial conditions. Dissipation in concurrent with large driving frequency or amplitude of the time-dependence inter-particle interaction enhances route to chaos.*

Keywords: Double-well BEC; dissipation; Quantum noise; Markovian; Chaos.
Classification numbers: 03.75kk, 42.50Gg, 03.75Gg, 42.50Lc.

1. Introduction

Bose-Einstein condensate (BEC) in a double-well system attracted the interest of researchers for its close physical analogy with Boson-Josephson Junction (BJJ) models in superconductor theory. Mimicking the ideas in superconductor physics many interesting quantum features such as tunneling transport, macroscopic coherence, modulated collapse and revival were observed in a double-well BEC trap setting, see for instance in [1, 2]. Manipulating inter-particle interaction strengths and varying input of atomic initial numbers in the double-well yield fascinating new phenomena such as Macroscopic quantum coherence or novel state of atom localization within respective trapping wells coined as the macroscopic Quantum Self Trapping (MQST), for instance see [3, 4]. Many experimental measurements followed thence such as Josephson tunneling, thermally induced phase fluctuations on the double-well BEC system were reported by the researchers of [5–7] . Measurements on Josephson's AC and DC effects on the BJJ were made by Levy et al. [8] and the interference-fringe experiments were made by Schmiedmayer et al. [9]. Enormous progress has been made in the field shaping out from these fundamental studies.

In general, system interacting with environment (reservoirs) within experimental conditions cannot be fully explained by purely quantum notions alone but requires also out of equilibrium semi-classical physics. The persistence of coherence needs to be sustained when system is subjected to dissipation and decoherence effect. Several groups have successfully addressed this issue and suggested how dissipation and decoherence could be tamed in the system, interested readers may refer to these references [10–18]. This was supported by later experimental work of [19, 20]. We have explored similar line of research employing Generalized Langevin Equation (GLE) model to study dissipation dynamics in our system of interest. Our earlier works, see for instance in [21–23], elaborated the detail about double-well BEC-reservoirs system described within mean-field approximation subjecting to Markovian and non-Markovian operational dynamics.

We modified the optical tweezer model in [24, 25] simplifying it into a generic double-well reservoir coupled system. The mentioned authors' models were actually inspired by Wolfgang Ketterlee MIT groups' atom-laser experiment (Chikkatur et al. [26]). In the said experiment, each Bose-condensate was clamped by optical tweezers and brought closer to establish Bose-Josephson Junction. BECs are coherently coupled and atom can tunnel between the traps. A condition for small separation between the BEC components is established to yield coherence. This is in agreement with condition generally assumed in the double-well BEC literature.

Earlier works based on external time-dependent inter-particle interaction potential coupled with damping can be referred to works for instance in [27,28]. Effect on the one-dimensional BEC dynamics due to time-dependent scattering length were studied by Abdullaev et al. [29]. Coherent atomic oscillation and resonance between coupled BEC using time-dependent trapping potential were also studied earlier by Abdullaev et al. [30]. Analysis on double-well BEC system subjected to both constant inter-particle interaction and dissipation has been reported by one of us in [23] where the results are consistent, for instance, with theoretical work in [14, 22, 31] or experimental observation in [7].

Implementing time-dependent inter-particle interactions into the real experimental setup can be a challenging feat. It can be implemented by small frequency and amplitude oscillation of external field on the double-well barrier, see detail for instance in Saha et. al. [32]. Modification on the optical tweezers employing pulsating lasers could provide an oscillation response on atoms in the double-well trap. We conceive this may generate oscillating inter-particle interaction effect on the double well BEC-reservoir system which is the main focus of this work. Our model is shown in Fig. 1.

This paper is organized in the following way. Section 2 illustrates the model of our double-well BEC-reservoir system. We show the detailed mechanism involved in getting the phase-space dynamical equation for our model. Later in Section 3, we describe the system by mean-field description where physical quantities such as population imbalance, phase difference can be illustrated in terms of phase portraits. In Section 4, we detail the concept of Lyapunov exponents which is a quantitative measure to detect chaos. Lyapunov exponents were computed numerically to characterize stability of the system in deciding whether they are periodic or non-periodic (become chaotic). Finally, we summarize and conclude our results in Section 5.

2. Double-well BEC out-coupled to reservoirs

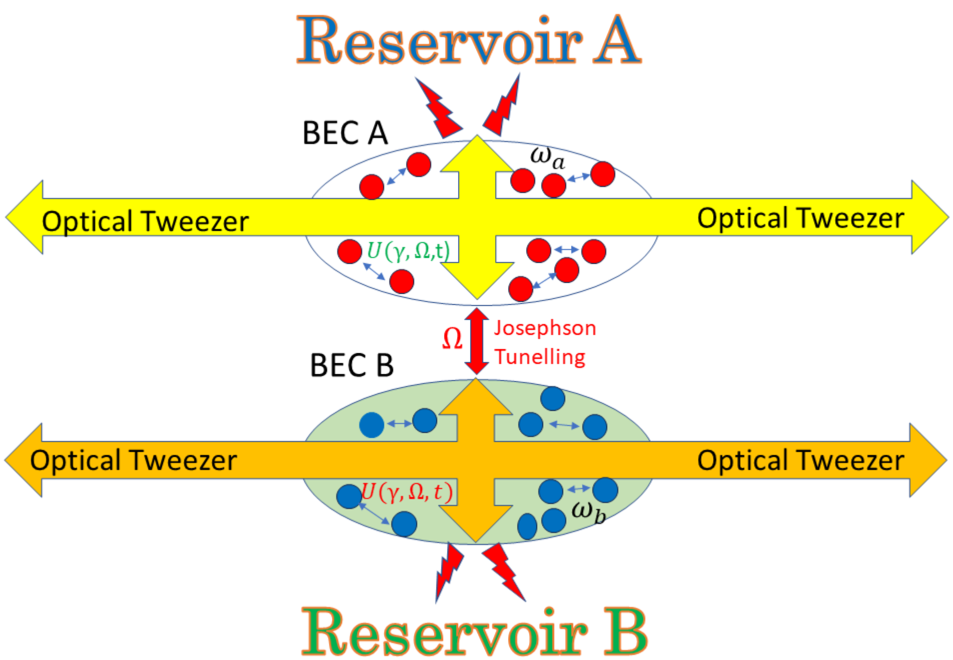

Fig. 1. Schematic model of a double-well BEC-dual reservoir system.

A double-well loaded with Bose-Einstein condensate atoms can be well-described by the following Hamiltonian:

$$H_{sys} = \hbar\omega\,\hat{a}^\dagger\hat{a} + \hbar\omega\,\hat{b}^\dagger\hat{b} + \hbar\Omega\,(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}) + \frac{\hbar U(t)}{2}(\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}) \tag{1}$$

where sets $(\hat{a}^\dagger, \hat{a})$ and $(\hat{b}^\dagger, \hat{b})$ are the creation and annihilation operators of the boson at traps A and B, respectively. The first two terms describe free Hamiltonian with frequencies ω, $\hbar\Omega$ is the tunnelling splitting. We have introduced a time-dependent inter-particle interactions U(t) in the above equation whose detail is given in the following section.

The two separate reservoirs (or multi-mode fields) attached at each trap (A,B) are represented by $H_A = \sum_k \omega_k \hat{A}_k^\dagger \hat{A}_k$ and $H_B = \sum_k \omega_k \hat{B}_k^\dagger \hat{B}_k$. In this model, we assume the reservoirs are composed of closely spaced oscillators with frequencies ω_k. The corresponding creation and annihilation operators are $(\hat{A}_k^\dagger, \hat{A}_k)$ and $(\hat{B}_k^\dagger, \hat{B}_k)$, for which we suppose the reservoirs are in thermal equilibrium satisfying $\langle \hat{X}^\dagger(0)\rangle = \langle \hat{X}(0)\rangle = 0$ while $\langle \hat{X}_k^\dagger(0)\hat{X}_{k'}(0)\rangle = \delta_{kk'} n_j(\omega_{k'})$ and $\langle \hat{X}_k(0)\hat{X}_{k'}(0)\rangle = 0$

for $\hat{X} = \{\hat{A}, \hat{B}\}$. Here $n_j(\omega_k) = 1/[\exp(\omega_k/k_B T_j) - 1]$ for $j = A, B$ are the thermal average boson numbers, with k_B Boltzmann constant and (T_A, T_B) the temperatures of reservoirs A and B respectively. We set $T_A = T_B$ to avoid temperature gradient between the traps that causes heat flow in certain direction. Interaction between the system and reservoirs is denoted by the following sub-Hamiltonians

$$H_{sys-A} = \sum_k g_k(\hat{a}\hat{A}_k^\dagger + \hat{a}^\dagger \hat{A}_k)$$
$$H_{sys-B} = \sum_k f_k(\hat{b}\hat{B}_k^\dagger + \hat{b}^\dagger \hat{B}_k)\,, \tag{2}$$

where g_k or f_k are the bi-linear out-coupling function of traps A or B respectively. With the provided information above, the total Hamiltonian of the system (interacting double-well BEC) coupled to a dual-reservoirs lucidly denoted by the relation $H_{Total} = H_{sys} + H_A + H_B + H_{sys-A} + H_{sys-B}$. The dynamical property of this system can be studied by solving the Heisenberg equation of motion $\frac{d\hat{O}}{dt} = -\frac{i}{\hbar}[\hat{O}, \hat{H}_{Total}]$ as known from the text-books [33, 34]:

$$\frac{d\hat{a}}{dt} = -i\,\omega\hat{a} - i\,\Omega\hat{b} - iU(t)\hat{a}^\dagger\hat{a}\hat{a} + F_A(t) - \int_0^t dt' K(t-t')a(t') \tag{3}$$
$$\frac{d\hat{b}}{dt} = -i\,\omega\hat{b} - i\,\Omega\hat{a} - iU(t)\hat{b}^\dagger\hat{b}\hat{b} + F_B(t) - \int_0^t dt' M(t-t')b(t')\,, \tag{4}$$

where, $F_A(t) = -i\sum_k g_k \hat{A}_k(0)\exp(-i\omega_k t)$ and $F_B(t) = -i\sum_k f_k \hat{B}_k(0)\exp(-i\omega_k t)$ correspond to the noise operators with reservoir variables whereas the last terms are the dissipation part with memory kernels $K(t) = \sum_k g_k^2 \exp(-i\omega_k t)$ and $M(t) = \sum_k f_k^2 \exp(-i\omega_k t)$. The choices of memory kernels will determine whether the system runs under Markovian or non-Markovian operational dynamics. For instance, memory kernel in the form $constant \times \delta(t - t')$ or $constant \times \exp(\frac{t'-t}{\tau})$ with τ as memory correlation time, yields Markovian or non-Markovian dynamics [33, 34]. We extensively used the latter two types of memory kernels in our earlier works [21, 22]. The role of Markovian type dissipation is to recover the system at drastic rate their equilibrium state been perturbed strongly by reservoir fluctuations. In other word, system can drastically recede from out of equilibrium. In this work we choose the memory-less dissipation kernels $K_\delta(t) = Q_1\delta(t)$ and $M_\delta(t) = Q_2\delta(t)$ corresponding to out-coupling dissipation strengths (Q_1,Q_2) of trap A and B respectively. With these forms of memory kernels, we obtained the following Markovian dynamical equations:

$$\frac{d\hat{a}}{dt} = -i\,\omega\hat{a} - i\,\Omega\hat{b} - iU(t)\hat{a}^\dagger\hat{a}\hat{a} + F_A(t) - Q_1\,\hat{a}(t) \tag{5}$$
$$\frac{d\hat{b}}{dt} = -i\,\omega\hat{b} - i\,\Omega\hat{a} - iU(t)\hat{b}^\dagger\hat{b}\hat{b} + F_B(t) - Q_2\,\hat{b}(t), \tag{6}$$

The following section details how the above coupled equations with operator variables are transformed into computable complex variables which gives physical meaning.

3. Markovian dynamics in the mean-field limit

In the present work, we average out equations (5)-(6). The third-order moment appearing on the right-hand side of the equations is decorrelated by approximation $\langle\hat{d}^\dagger\hat{d}\hat{d}\rangle \approx \langle\hat{d}^\dagger\rangle\langle\hat{d}\rangle\langle\hat{d}\rangle$ where $d = \{a, b\}$. This truncation is valid in the macroscopic limit and when the many-particle quantum state is close to pure BEC [16]. In other word, the macroscopic dynamics of BEC with large number of atoms ($N \to \infty$) is valid and well prescribed by the mean-field approximation [2–4, 14].

Defining $\alpha = \langle\hat{a}\rangle$, $\alpha^* = \langle\hat{a}^\dagger\rangle$, $\beta = \langle\hat{b}\rangle$ and $\beta^* = \langle\hat{b}^\dagger\rangle$, at the same time ignoring average fluctuations effect due to the large reservoirs each sides, i.e., both $\langle F_A\rangle$ and $\langle F_B\rangle$ vanish, equations (5)-(6) are then transformed into the following set of mean-field coupled differential equations:

$$\frac{d\alpha}{dt} = (-i\omega - Q_1)\alpha - i\Omega\beta - iU(t)|\alpha|^2\alpha\,, \tag{7}$$

$$\frac{d\beta}{dt} = (-i\omega - Q_2)\beta - i\Omega\alpha - iU(t)|\beta|^2\beta\,. \tag{8}$$

Total number of atoms at any instance can be computed using relation $n(t) = |\alpha|^2 + |\beta|^2 = n_A(t) + n_B(t)$. Though, single order moment for noise vanishes in the system, second-order moments $\langle F_A^\dagger(t)F_A(t')\rangle$ and $\langle F_B^\dagger(t)F_B(t')\rangle$ are not negligible. Computation of physical parameters such as population imbalance z(t) and phase difference $\theta(t)$ only requires first order moments. Re-writing the complex number variables $(\alpha,\ \beta)$ as $\alpha = |\alpha|\exp(i\theta_a)$ and $\beta = |\beta|\exp(i\theta_b)$, the population imbalance and relative phase are defined by:

$$z(t) = \frac{|\alpha|^2 - |\beta|^2}{n(t)}\,, \tag{9}$$

$$\theta(t) = \theta_a(t) - \theta_b(t)\,, \tag{10}$$

where, (θ_a, θ_b) are the phases of each trap (A, B) respectively. Performing some algebra using the above re-definitions on the equations above, we can illustrate phase portrait (z(t),$\theta(t)$) of the system that satisfies the following set of coupled equations [22, 31]:

$$\frac{dz}{dt} = -2\sqrt{1-z^2}\,\sin\theta + \zeta(1-z^2)\,, \tag{11}$$

$$\frac{d\theta}{dt} = \frac{2z\cos\theta}{\sqrt{1-z^2}} - U(t)\,z\,, \tag{12}$$

where $U(t)$ is the inter-particle interaction function explicitly depending on time t and dissipation bias constant is $\zeta = Q_1 - Q_2$. Time has been scaled in unit Ω. We consider a function $U(t) = \chi(1 + \gamma\sin\Omega_D t)$ where χ is the inter-particle interaction strength while γ and Ω_D are the driving amplitude and oscillating frequency. The said inter-particle interaction function is similar to the one used by Saha et al. [32]. The considered function has explicit time dependence making the non-linear ODE equations (11)-(12) non-autonomous which is cumbersome to solve. Introducing new parameter $\phi = \Omega_D t$, converts back the equation into following solvable autonomous non-linear ODE:

$$\frac{dz}{dt} = -2\sqrt{1-z^2}\sin\theta + \zeta(1-z^2), \tag{13}$$

$$\frac{d\theta}{dt} = \frac{2z\cos\theta}{\sqrt{1-z^2}} - \chi z \sin\phi, \tag{14}$$

$$\frac{d\phi}{dt} = \Omega_D. \tag{15}$$

Control parameters of the above equations are the inter-particle interaction amplitude χ, dissipation bias strength ζ and driving frequency Ω_D respectively. Further, the two-dimensional phase-space (z, θ) described by equations (11)-(12) has been expanded into three-dimensional phase-space (z, θ, ϕ) described by equations (13)-(15). The latter set of equation satisfy the criteria for possible chaos:

- Poincare-Bendixson theorem [35] ruled out the existence of chaos in a two-dimensional system. We have three variables now.
- Equations describing the system has to be non-linear [35–37]. The above set of equations are obviously non-linear.

As an initial test, we check the phase portrait of population imbalance z(t) versus phase difference $\theta(t)$ as a function of inter-particle interaction, driving amplitude and frequencies. The trajectories initiated by close-by selected initial conditions generated by the set of dynamical equations (13)-(15) is illustrated in Figures 2-3.

Qualitatively, we observe an evolution of phase from Quantum coherent state (elliptical trajectory orbits) to localized state called the Macroscopic self-trapping state (hyperbolic trajectories) as reported for the effect due to constant inter-particle interaction dissipation pair on the system [22,23] or in references [14,31].

An extra information revealed by the illustrated portrait is the sensitivity of trajectories to the initial condition. Quantitative measure on the divergence due to small increment in initial condition is analysed in the next section.

Non-regular dynamics is clearly noticeable as driving frequency is approaching $\Omega = \pi/3$ in tandem with increase of inter-particle interaction. Their effect is amplified by the increase in driving amplitude as one compares Fig. 2 and Fig. 3.

4. Lyapunov exponents to test chaos

Let us consider an n-dimensional non-linear ordinary differential equation system with constant coefficients of the form:

$$\dot{\mathbf{Z}} = \mathbf{A}\mathbf{Z} + \mathbf{F}(t). \tag{16}$$

Stability of the system can be analysed by studying the divergence or convergence of trajectories generated by infinitesimally separated initial conditions. In other word, dynamics of the system is sensitive to changes in initial conditions. For instance, let $\mathbf{Z}$(t) and $\mathbf{Z}_o(t)$ be two trajectories generated by closely separated assumed initial conditions. The time-dependent separation between them is $\delta\mathbf{Z}(t) = \mathbf{Z}(t) - \mathbf{Z}_o(t)$. Then the evolution of small increment between the

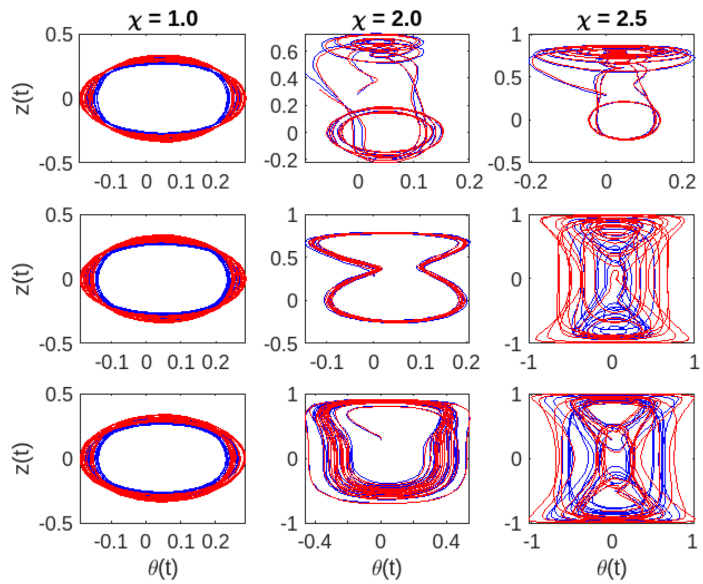

Fig. 2. Phase portrait (θ(t), z(t)) for driven dissipative dynamics as a function of inter-particle interaction χ (panels left to right with increasing order $\chi = \{1.0, 2.0, 2.5\}$) and driving frequencies (top to bottom panels with increasing order $\Omega_D = \{0, \frac{\pi}{6}, \frac{\pi}{3}\}$). Driving amplitude is fixed $\gamma = 0.35$. The initial conditions used in the simulation is (z(0),θ(0), ϕ(0))=$\{(0.28, 0., 0,), (0.3, 0., 0.)\}$. Phase difference θ is taken in unit radian.

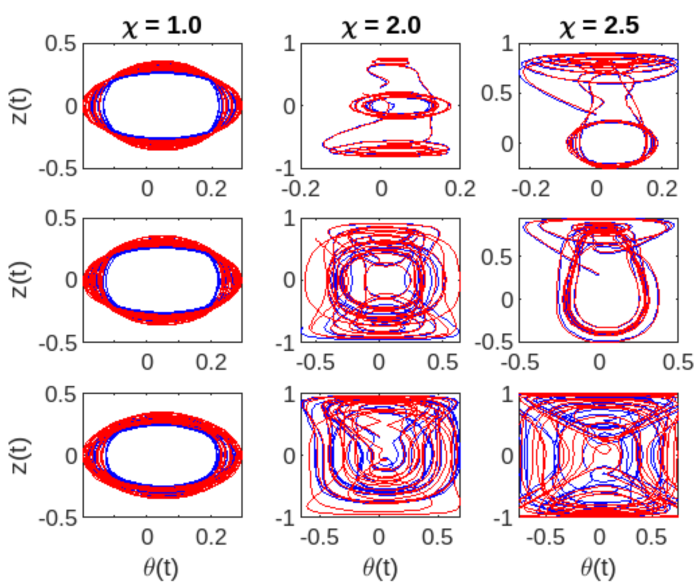

Fig. 3. The same as in Fig. 2. for driving amplitude $\gamma = 0.5$.

trajectories can be obtained from the linearized solution of the equation:

$$\delta\dot{\mathbf{Z}}(t) = \mathbf{J}(x_o)\cdot\delta\mathbf{Z}(t)\,, \tag{17}$$

where, $\mathbf{J}(x_o)$ is the Jacobian matrix whose elements $J_{ij}(t) = \frac{f_i(t)}{dx_i}|_{x_o}$ are evaluated at initial values x_o. Here f_i are the elements of vector $\mathbf{F}$ in Eq.(16). The real parts of the n-different eigenvalues $\lambda_1\,,\lambda_2\,,\ldots,\lambda_n$ of the Jacobian matrix $\mathbf{J}$ is naturally the Lyapunov exponents, where the largest value of them is defined by [38, 39]:

$$\lambda_{max}(t) = \lim_{t\to\infty}\lim_{|\delta\mathbf{Z}_o(t)|\to 0}\frac{1}{t}\ln\frac{|\delta\mathbf{Z}(t)|}{|\delta\mathbf{Z}_o(t)|}\,. \tag{18}$$

The above Lyapunov characteristic exponents are computed numerically and depicted in Figures 4 and 5. To analyse the results, we follow criterion of reference [39], also can be found on Dynamical System and Chaos text-books such as [35–37, 40]). The criterion states that the system attractors reduce to

(1) stable fixed points if all the Lyapunov exponents are negative,
(2) a limit cycle if one Lyapunov exponent is zero and the remaining exponents are all negative,
(3) k-dimensional torus if the first k Lyapunov exponents vanish while the remaining ones are negative,
(4) chaotic (strange attractor) if at least one Lyapunov exponent is positive.

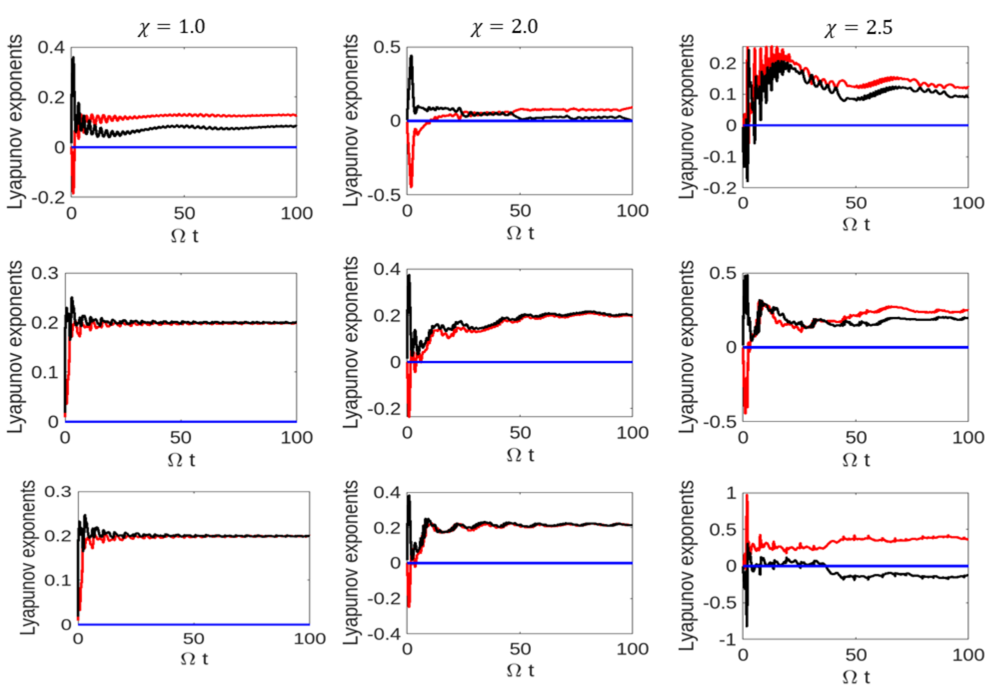

Fig. 4. Evolution of Lyapunov exponents for the system corresponding to Fig. 2.

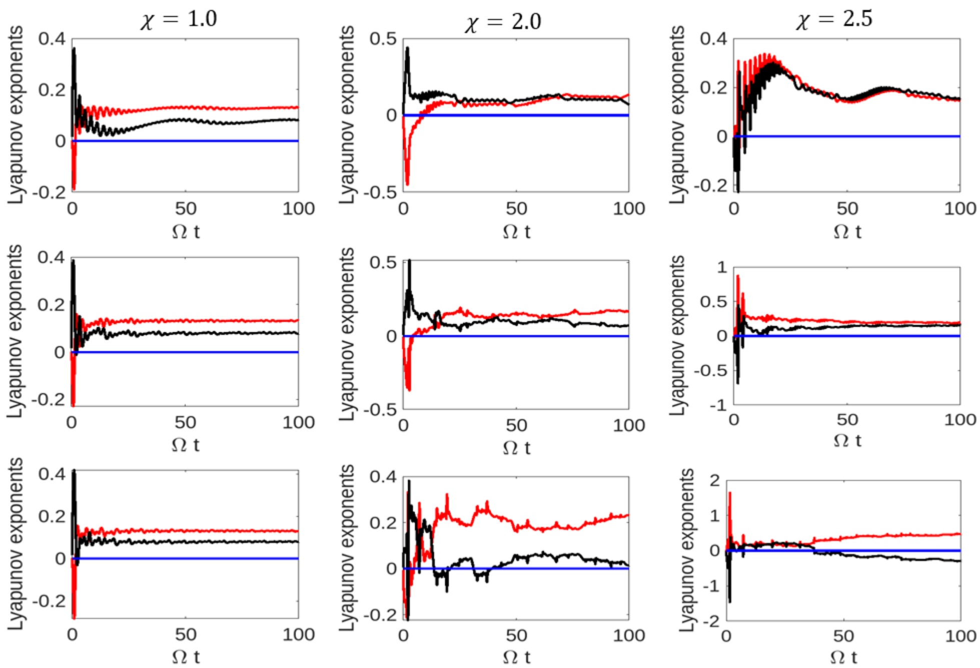

Fig. 5. Evolution of Lyapunov exponents for the system corresponding to Fig. 3.

Note that Figures 4 and 5 correspond to Figures 2 and 3 respectively. The Lyapunov exponents give precise quantitative picture compared to the phase portrait results. For example the far right top panels of Figures 4-5 look chaotic but they are not reflected by the Lyapunov exponents counterparts. Based on the said criterion and result of Figures 4 and 5, we can infer that the onset of chaos occurs when inter-particle is large (here our maximum is $\chi = 2.5$) coupled with larger driving frequency (here our maximum is $\Omega_D = \pi/3$).

5. Conclusion

We have studied the dynamics of a double-well BEC dual reservoir subjected to time-dependent inter-particle interaction and dissipation. The system considered here works under Markovian operational dynamics culminated from the delta-correlated memory dissipation kernel. The macroscopic dynamics of the model is described within mean-field approximation, which means noise in the system is averaged out by taking single moment. However, the two point noise-noise correlation term is not negligible. They are simply not visible since single moment physical parameters depicted in this work do not require the cumbersome two-point correlations calculations.

Physical parameters of the system were obtained by solving equations (11)-(12) numerically using MATLAB ODE-45 solver which employs in-build fourth order Runge-Kutta (RK4) algorithm. The solver is tested reliable in solving non-stiff ordinary or non-linear differential equations such as Van der Pol or Duffing equations. Small parameter values were used to yield converging results. We found the following results:

(1) the trajectories solution of the dynamical equation is highly sensitive to the small changes in the initial conditions as illustrated by the phase portraits,
(2) the onset of chaos is induced by the increase in driving frequency Ω_D,
(3) presence of dissipation, stronger inter-particle interaction strength coupled with the driving amplitude enhances the route to chaos. The region to predict chaos was located near attractor (repeller) state of the non-driven case.

This work has its own limitations. Calculating physical parameter via mean-field approximations neglects noise and the rich physics that should culminated from the noise-noise correlations effect. However, dissipation still persists in the system. The use of delta-correlated memory kernel in the GLE (Generalized Langevin Equation) generates Markovian operational dynamics. The work can be extended to further involving non-Markovian system by using Ornstein-Uhlenbeck (OU) or step-functional memory kernels in the GLE. Findings of this study can be used in the application of quantum information processing or precision measurements.

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Compliance with Ethical standards

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