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LANCZOS GENERATOR IN GÖDEL GEOMETRY

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Abstract. *We exhibit three ways to obtain a Lanczos spintensor for the Gödel cosmological model. In particular, a symmetric tensor of rank two is constructed which satisfies the wave equation and generates a Lanczos potential.*

I. INTRODUCTION

Lanczos [1-6] showed, for any R_4, the existence of a potential K_{abc} with the properties:

$$K_{abc} = -K_{bac}, \quad K_{abc} + K_{bca} + K_{cab} = 0\ , \tag{1}$$

which generates the conformal tensor via the expression [2,7]:

$$C_{abcd} = K_{abc;d} - K_{abd;c} + K_{cda;b} - K_{cdb;a} +$$
$$\tfrac{1}{2}\left[g_{ad}\left(K_{bc}+K_{cb}\right) - g_{ac}\left(K_{bd}+K_{db}\right) + g_{bc}\left(K_{ad}+K_{da}\right) - g_{bd}\left(K_{ac}+K_{ca}\right)\right] +$$
$$\frac{2}{3}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) K^{pq}{}_{p;q}\ , \tag{2}$$

such that

$$K_{bc} \equiv K_{b}{}^{p}{}_{c;p} - K_{b}{}^{p}{}_{p;c} \tag{3}$$

The possible relevance of K_{abc} in general relativity is given in [8]. Besides, the idea of "Lanczos spintensor" is very useful in the analysis [9-19] of the Liénard-Wiechert field [20].

Given the Weyl tensor, it may be very difficult to obtain a Lanczos superpotential by integrating directly (2), but here we shall show three ways to deduce one solution of (2) for Gödel spacetime [21,22], with the interesting structure:

$$K_{abc} = Q_{ca;b} - Q_{cb;a}\ , \quad Q_{ab} = Q_{ba} \tag{4}$$

which matches with conditions (1); therefore the symmetric tensor Q_{ab} is a potential for the Lanczos generator. A relation similar to (4) also appears for weak gravitational fields [1], Kerr geometry [22-27], plane gravitational waves [28] and Kinnersley metrics [29]. The physical meaning of the Lanczos spintensor for a given spacetime is yet unknown

[1,6-8,24,30-32].

II. METHOD OF LOCAL AND ISOMETRIC EMBEDDING

Here we show that the Gauss equation employed in the embedding of R_4 into E_5, permits to obtain a symmetric tensor b_{ij} which generates a Lanczos potential for the Gödel cosmological model.

A spacetime can be embedded into E_5 if and only if there exist the second fundamental form $b_{ac} = b_{ca}$ fulfilling the Gauss-Codazzi equations [22]:

$$R_{acij} = \varepsilon \left(b_{ai} b_{cj} - b_{aj} b_{ci} \right) \ , \tag{5}$$

$$B_{cij} \equiv b_{ji;c} - b_{jc;i} = 0 \ , \tag{6}$$

where $\varepsilon = \pm 1$, R_{acij} is the curvature tensor and $;j$ denotes the covariant derivative. Then we say that such 4-space has class one.

From the Gauss relation (5) it is possible to deduce the identity [33-37]:

$$p b_{ij} = \frac{K_2}{48} g_{ij} - \frac{1}{2} R_{iacj} G^{ac} \ , \tag{7}$$

where $G_{ac} = R_{ac} - \frac{R}{2} g_{ac}$ and $R_{ac} = R^r{}_{acr}$ are the Einstein and Ricci tensors, respectively, and $K_2 = {}^*R^{*ijac} R_{ijac}$ is a Lanczos invariant [5,38] in terms of the double dual [1] of Riemann tensor ${}^*R^{*ij}{}_{ac} = \frac{1}{4} \eta^{ijrm} R_{rm}{}^{nr} \eta_{nrac}$, where η_{ijac} is the Levi-Civita tensor. Besides:

$$p^2 = -\frac{\varepsilon}{6} \left(\frac{R}{24} K_2 + R_{imnj} G^{ij} G^{mn} \right) \geq 0, \tag{8}$$

If $p \neq 0$ then (7) permits to obtain explicitly a b_{ij} verifying (5).

Now we apply (7) to the Gödel metric [21,22] (signature +2):

$$ds^2 = -\left(dx^1\right)^2 - 2 e^{x^4} dx^1 dx^2 - \frac{1}{2} e^{2x^4} \left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^4\right)^2 \ , \tag{9}$$

therefore $\varepsilon = 1$, $p = \frac{\sqrt{2}}{4}$ and:

$$(b_{ij}) = -\frac{\sqrt{2}}{2} \begin{pmatrix} 1 & e^{x^4} & 0 & 0 \\ e^{x^4} & \frac{3}{2} e^{2x^4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{10}$$

The tensor (10) not satisfies (6) because we know [39-43] that (9) has not class one, in fact, $b_{12;4} \neq b_{14;2}$. Thus for this Gödel solution we have a B_{cij} whose only non-zero independent components are:

$$B_{124} = B_{412} = -\frac{\sqrt{2}}{2} e^{x^4} \ , \qquad B_{242} = \frac{3}{2} \sqrt{2} e^{2x^4} \ , \tag{11}$$

with the same symmetries as Lanczos potential K_{cij} [6,44,45]:

$$B_{ijr} = -B_{jir} \ , \quad B_{ijr} + B_{jri} + B_{rij} = 0 \ ,$$

$$B_i{}^r{}_r = 0 \qquad \text{Lanczos algebraic gauge,} \tag{12}$$

$$B_{ij}{}^{r}{}_{;r} = 0 \qquad \text{Lanczos differential gauge.}$$

Then the following "ansatz" is very natural:

$$K_{ijr} = QB_{ijr}, \qquad \text{Q= constant,} \tag{13}$$

which must generate the Weyl tensor via the relation (2) with the simplification [24,46]:

$$C_{aijr} = K_{aij;r} - K_{air;j} + K_{jra;i} - K_{jri;a} + g_{ar}K_{ji} - g_{aj}K_{ri} + g_{ij}K_{ra} - g_{ir}K_{ja} \ , \tag{14}$$

where $K_{ij} = K_{i}{}^{r}{}_{j;r} = K_{ji}$, because $K_{i}{}^{r}{}_{r} = 0$. Using (11) and (13) we find that (14) implies correctly all components of the conformal tensor if $Q = \frac{\sqrt{2}}{18}$, which means that (10) produces a Lanczos potential for Gödel geometry:

$$K_{ijr} = \frac{\sqrt{2}}{18}\left(b_{rj;i} - b_{ri;j}\right) \ , \tag{15}$$

equivalent to (4) with $Q_{ac} = -\frac{\sqrt{2}}{18}b_{ac}$.

We know that (9) not accepts embedding into E_5, however, the study of the Gauss-Codazzi equations is important because it permits to construct the Lanczos generator (15) for the Gödel spacetime. Then, if a metric has not class one, perhaps a b_{ij} verifying (5) may have a relationship similar to (15) with a Lanczos potential for this metric. Our work calls the attention towards an interesting connection between the embedding of Riemann 4-spaces and the Lanczos generator, which must be studied carefully. For example, it is still ignored if (9) admits embedding into E_6 [41-43,47], then the Lanczos potential gives us a new approach to this open problem.

III. METHOD OF THE LOVELOCK'S THEOREM

The Lanczos algebraic gauge $K_{a}{}^{b}{}_{b} = 0$ may be satisfied if in (4) we ask the conditions:

$$Q^{r}{}_{r} = \text{constant}, \tag{16}$$

$$Q_{a}{}^{b}{}_{;b} = 0 \ ; . \tag{17}$$

Besides, if now we suppose that Q_{ab} depends locally on the intrinsic geometry of R_4, that is:

$$Q_{ir} = Q_{ir}\left(g_{ab}; g_{ab,c}; g_{ab,cd}\right) \ , \tag{18}$$

then a Lovelock's theorem [34,48-50] affirms that in four dimensions a tensor of second rank with the properties (17) and (18) must have the form:

$$Q_{ab} = \alpha G_{ab} + \beta g_{ab}, \quad \alpha, \beta \quad = \quad \text{constant,} \tag{19}$$

Thus $Q_{r}{}^{r} = 4\beta - \alpha R$ satisfies (16) because $R = 1$ for the Gödel metric (9); if we put (19) into (4) it results:

$$K_{abc} = \alpha\left(R_{ca;b} - R_{cb;a}\right) \ . \tag{20}$$

Finally with the help of (9), (14) and (20) we conclude that $\alpha = -\frac{1}{9}$, that is [51-53]:

$$K_{ijr} = \frac{1}{9}\left(R_{rj;i} - R_{ri;j}\right) \ , \tag{21}$$

which is the same as (15) because b_{ac} is connected with R_{ac} by the expression:

$$b_{ij} = \sqrt{2}\left[R_{ij} + \frac{1}{2}\left(B_i B_j - g_{ij}\right)\right] \ , \tag{22}$$

where B_j is a constant spacelike vector:

$$(B_r) = (0,0,1,0) \ , \quad B_{r;c} = 0 \ . \tag{23}$$

It is interesting to observe that $K_{abc}v^c \propto \omega_{ab}$, where $(v^c) = (1,0,0,0)$ is the velocity of the fluid and ω_{ij} is the spin [54] of the matter rotating in this Gödel model, in accordance with the proposition [10] of that K_{ijr} represents some type of angular momentum into the spacetime.

IV. METHOD OF THE WAVE EQUATION

Here we consider symmetric tensors of second order verifying the wave equation:

$$W_{ab}{}^{;r}{}_{;r} = 0 \ , \tag{24}$$

in the Gödel spacetime, and we exhibit an attractive solution of (24) which generates a Lanczos potential for (9) with the structure (4).

In fact, (24) admits the solution:

$$(W_{ab}) = \frac{1}{27}\begin{pmatrix} 4 & 4e^{x^4} & 0 & 0 \\ 4e^{x^4} & \frac{7}{2}e^{2x^4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ , \qquad W_a{}^b{}_{;b} = 0, \tag{25}$$

then

$$K_{ijr} = W_{ri;j} - W_{rj;i}, \tag{26}$$

implies a Lanczos generator whose only non-zero independent components are:

$$K_{124} = K_{412} = -\frac{e^{x^4}}{18} \ , \qquad K_{242} = \frac{e^{2x^4}}{6} \ , \tag{27}$$

with all properties (12), which reproduces –via (14)– correctly the Weyl tensor. The Lanczos potential (26) is equivalent to (15) due the connection:

$$W_{ri} = \frac{1}{18}\left[-\sqrt{2}b_{ri} + \frac{5}{3}\left(B_r B_i - g_{ri}\right)\right] \ . \tag{28}$$

Our work shows that it is important to study the wave equation (24) for the construction of K_{ijr} in Gödel geometry, but it is clear that this situation may appear in other spacetimes.

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