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OCTONIONIC EXTENSION OF $U(1) \times SU(2)$ GAUGE ANALYTICITY OF DYONS

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Abstract. *Starting with the generalized electromagnetic (and gravi-magnetic) fields of dyons (and grvito-dyons), the octonion generalization of $U(1) \times SU(2)$ gauge theories of dyons (and grvito-dyons) has been developed in a consistent way. The resemblance between octonion covariant derivative and the gauge covariant derivative has also been established . Expressing the generalized four - potential, current and fields of dyons (and gravito - dyons) in terms of split octonion variables, we have discussed the consistent forms of $U(1)$ abelian and $SU(2)$ non - Abelian gauge theories for dyons (and gravito - dyons). It is shown that the present formalism reproduces the theory of electric (gravitational) charge (mass) in the absence of magnetic (Heavisidean) charge (mass) on dyons (gravito - dyons) or vice versa.*

I. INTRODUCTION

According to the celebrated Hurwitz theorem [1] there exists four - division algebra's [2] consisting of $\mathbb{R}$(real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{Q}$ (quaternion) and $\mathcal{O}$ (octonions). All four algebra's are alternative with totally anti symmetric associators. A detailed introduction on the various aspects and applications of Exceptional, Jordan, Division, Clifford and noncommutative as well as non associative algebras has recently been discussed by Castro [3] by extending the octonionic geometry (gravity) developed long ago by Marques-Oliveira [4]. On the other hand, some authors [4–12] have discussed the possibility of octonion Dirac equation, and octonion wave equation. Supersymmetric Yang-Mills and Superstring theories in critical spacetime dimensions d = 3, 4, 6,and 10 attracted [13–15] much attention regarding their relevance to four division algebras. Their respective dimensions 1, 2, 4 and 8 equal $d-2$ physical modes are associated respectively with four - division algebra's consisting of $\mathbb{R}$(real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{Q}$ (quaternion) and $\mathcal{O}$ (octonions) corresponding to various transverse degrees of freedoms in critical space-time dimensions.

Now, the question of existence of monopole [16–19] and dyons [20–22] has become a challenging new frontier and the object of more interest in high energy physics. Dirac showed [16] that the quantum mechanics of an electrically charged particle of charge e and a magnetically charged particle of charge g is consistent only if $eg = 2\pi n$, n being an integer. Schwinger-Zwanziger [20,21] generalized this condition to allow for the possibility of particles (dyons) that carry both electric and magnetic charge. A quantum mechanical theory can have two particles of electric and magnetic charges (e_1, g_1) and (e_2, g_2) only if

$e_1 g_2 - e_2 g_1 = 2\pi\, n$. The fresh interests in this subject have been enhanced by 't Hooft - Polyakov [17,18] with the idea that the classical solutions having the properties of magnetic monopoles may be found in Yang - Mills gauge theories. Julia and Zee [22] extended the 't Hooft-Polyakov theory [17, 18] of monopoles and constructed the theory of non Abelian dyons. The quantum mechanical excitation of fundamental monopoles include dyons which are automatically arisen from the semi-classical quantization of global charge rotation degree of freedom of monopoles. In view of the explanation of CP-violation in terms of non-zero vacuum angle of world [23], the monopoles are necessary dyons and Dirac quantization condition permits dyons to have analogous electric charge. Accordingly, a self consistent and manifestly covariant theory has alredy been discussed [24, 25] for the generalized electromagnetic fields of dyons and accordingly the the quaternionic [26, 27] and octonionic [8–10] forms of generalized fields of dyons are developed in unique, simple, compact and consistent manner.

Likewise, the postulation of Heavisidian an monopole [28, 29] immediately follows the structural symmetry [30, 31] between generalized gravito-Heavisidian and generalized electromagnetic fields of dyons. Avoiding the use of arbitrary string variable, the manifestly covariant and consistent theory of gravito - dyons has also been developed [32–35] in terms of two four-potentials [36] leading to the structural symmetry between generalized electromagnetic fields of dyons and generalized gravito-Heavisidian fields of gravito-dyons. Extending this recently, a consistent theory for the dynamics of four charges (masses) (namely electric, magnetic, gravitational, Heavisidian) have also been formulated [37] in simple, compact and consistent manner.

Keeping all these facts in mind and starting with the generalized electromagnetic (and gravi-magnetic) fields of dyons (and gravito-dyons), in the present paper, we have made an attempt to develop the octonion generalization of $U(1) \times SU(2)$ gauge theories of dyons (and gravito-dyons) in a consistent way. The resemblance between octonion covariant derivative and the gauge covariant derivative has also been established. Octonion gauge formalism describes the $U(1)$ Abelian and $SU(2)$ non - Abelian gauge structure of dyons (and gravito-dyons). Expressing the generalized four - potential, current and fields of dyons (and gravito - dyons) in terms of split octonion variables, we have discussed the consistent forms of $U(1)$ Abelian and $SU(2)$ non - Abelian gauge theories for dyons (and gravito - dyons).The existence of gravitational analogue of magnetic monopoles describes the extension of octonion gauge formalism to gravito-dyons whereas imaginary units are shown to be responsible to incorporate the curvature in gravitational fields. It is also discussed that, in octonion gauge formalism, the $SL(2, C)$ gauge group of gravitation and $SU(2)$ gauge groups of Yang - Mill's gauge theory play the similar role in a symmetrical manner. Finally, it is concluded that the octonion gauge formalism reproduces the theory of electric (gravitational) charge (mass) in the absence of magnetic (Heavisidian) charge (mass) on dyons (gravito - dyons) or vice versa.

II. DYONIC FIELDS

Starting with the idea of Cabibbo and Ferrari [36] of two four - potentials, a gauge invariant and Lorentz covariant quantum field theory of fields associated with dyons has been developed [24–27] in purely group theoretical manner by assuming the generalized charge (q), generalized current $\{J_\mu\}$ and generalized four- potential $\{V_\mu\}$ as complex quantities with their real and imaginary parts as electric and magnetic constituents i.e.

$$q = e - i\,g; \tag{1}$$

$$\{J_\mu\} = \{j_\mu\} - i\,\{k_\mu\}\,; \tag{2}$$

$$\{V_\mu\} = \{A_\mu\} - i\,\{B_\mu\}\,; \tag{3}$$

where $(i = \sqrt{-1})$, e is the electric charge, g is magnetic charge,$\{j_\mu\} = (\rho_e,\, \overrightarrow{j_e})$ is the electric four - current with ρ_e and $\overrightarrow{j_e}$ as its scalar and vector constituents, $\{k_\mu\} = (\rho_m,\, \overrightarrow{j_m})$ is the magnetic four - current with ρ_m and $\overrightarrow{j_m}$ as its scalar and vector constituents, $\{A_\mu\} = (\phi_e,\, \overrightarrow{A})$ is electric four - potential with ϕ_e and $\overrightarrow{A}$ as its scalar and vector constituents, and $\{B_\mu\} = (\phi_g,\, \overrightarrow{B})$ is magnetic four - potential with ϕ_g and $\overrightarrow{B}$ as its scalar and vector counter parts. Two four - potentials are required [24, 25, 36] for the removal of arbitrary string variables in Dirac theory [16]of monopoles (or dyons) to symmetrize the following generalized Dirac Maxwell's (GDM) equations of dyons

$$\begin{aligned}
\overrightarrow{\nabla} \cdot \overrightarrow{E} &= \rho_e;\\
\overrightarrow{\nabla} \cdot \overrightarrow{H} &= \rho_m;\\
\overrightarrow{\nabla} \times \overrightarrow{E} &= -\frac{\partial \overrightarrow{H}}{\partial t} - \overrightarrow{j_m};\\
\overrightarrow{\nabla} \times \overrightarrow{H} &= \overrightarrow{j_e} + \frac{\partial \overrightarrow{E}}{\partial t};
\end{aligned} \tag{4}$$

where for brevity we have used the physical constants as unity with natural units $c = \hbar = 1$ along with the flat metric $(+,-,-,-)$. $\overrightarrow{E}$ and $\overrightarrow{H}$ in equations (4) are respectively described as the generalized electric and magnetic fields of dyons defined in terms of the components of two four - potentials as

$$\begin{aligned}
\overrightarrow{E} &= -\frac{\partial \overrightarrow{A}}{\partial t} - \overrightarrow{\nabla}\phi_e - \overrightarrow{\nabla} \times \overrightarrow{B};\\
\overrightarrow{H} &= -\frac{\partial \overrightarrow{B}}{\partial t} - \overrightarrow{\nabla}\phi_g + \overrightarrow{\nabla} \times \overrightarrow{A}.
\end{aligned} \tag{5}$$

Generalized Dirac Maxwell's (GDM) equations (4) are invariant not only under Lorentz and conformal transformations but are also invariant under the following duality transformations between electric $\mathcal{E}$ and magnetic $\mathcal{H}$ quantities i.e.

$$\begin{aligned}
\mathcal{E} &\Longrightarrow \mathcal{E}\cos\vartheta + \mathcal{H}\sin\vartheta;\\
\mathcal{H} &\Longrightarrow \mathcal{H}\cos\vartheta - \mathcal{E}\sin\vartheta;
\end{aligned} \tag{6}$$

where $\mathcal{E} \equiv \left(e, \overrightarrow{E}, \rho_e, \overrightarrow{j_e}, \phi_e, \overrightarrow{A}\right)$ and $\mathcal{H} \equiv \left(g, \overrightarrow{H}, \rho_m, \overrightarrow{j_m}, \phi_m, \overrightarrow{B}\right)$. For a particular value of $\vartheta = \frac{\pi}{2}$, equations (6) reduces to

$$\mathcal{E} \longmapsto \mathcal{H} \quad \mathcal{H} \longmapsto -\mathcal{E}. \tag{7}$$

Similar to equation (1) the generalized vector field associated with dyons is defined as

$$\overrightarrow{\psi} = \overrightarrow{E} - i\,\overrightarrow{H} \tag{8}$$

which reduces the four different Generalized Dirac Maxwell's (GDM) equations (4) to the following two differential equations as

$$\begin{aligned}\overrightarrow{\nabla}\cdot\overrightarrow{\psi} &= J_0;\\ \overrightarrow{\nabla}\times\overrightarrow{\psi} &= -i\frac{\partial\overrightarrow{\psi}}{\partial t} - i\,\overrightarrow{J};\end{aligned} \tag{9}$$

where J_0 and $\overrightarrow{J}$ are the temporal and spatial parts of generalized four - current $\{J_\mu\} = (J_0,\ \overrightarrow{J})$ (2) of dyons. Similarly the generalized electromagnetic field tensor $G_{\mu\nu}$ of dyons is defined [24–26] as

$$G_{\mu\nu} = F_{\mu\nu} - i\,\tilde{F}_{\mu\nu} \tag{10}$$

where

$$F_{\mu\nu} = E_{\mu\nu} - \tilde{H}_{\mu\nu}; \quad \tilde{F}_{\mu\nu} = H_{\mu\nu} + \tilde{E}_{\mu\nu}; \tag{11}$$

and

$$\begin{aligned}&E_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}; \quad H_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu};\\ &\tilde{E}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\lambda}A^{\sigma\lambda}; \quad \tilde{H}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\lambda}B^{\sigma\lambda};\end{aligned} \tag{12}$$

where the symbol ($\sim$) denotes the dual, $E_{\mu\nu}$ and $H_{\mu\nu}$of equation (12) are the electromagnetic field tensors respectively associated with the dynamics of electric and magnetic charges related to four - potential $\{A_\mu\}$ and $\{B_\mu\}$ and $\epsilon_{\mu\nu\sigma\lambda}$ is the four index Levi-Civita symbol. Generalized electric and magnetic fields (5) may readily be obtained from the electromagnetic field tensors $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ as

$$\begin{aligned}F_{0a} &= E^a \qquad F_{ab} = \varepsilon_{abc}H^c \quad (\forall\, a,b,c = 1,2,3);\\ \tilde{F}_{oa} &= H^a \qquad \tilde{F}_{ab} = \varepsilon_{abc}E^c \quad (\forall\, a,b,c = 1,2,3).\end{aligned} \tag{13}$$

Classical Abelian Lorentz invariant generalized Maxwell's equations associated with dyons may then be written as

$$\begin{aligned}F_{\mu\nu,\nu} &= E_{\mu\nu,\nu} = j_\mu;\\ \tilde{F}_{\mu\nu,\nu} &= H_{\mu\nu,\nu} = k_\mu.\end{aligned} \tag{14}$$

Using equations (2,10,11) and (14), we may write the covariant form of Generalized Dirac Maxwell's (GDM) of dyons as

$$G_{\mu\nu,\nu} = J_\mu. \tag{15}$$

The present model of dyon and the the field equations may also be derived [24, 25] for minimum action principle of Lagrangian applied for generalized electromagnetic fields of dyons and accordingly the Lorentz force equation of motion for dyons be obtained as

$$m\frac{d^2x_\mu}{d\tau^2} = \frac{1}{2}(qG^\star_{\mu\nu} + q^\star G_{\mu\nu}) = (eF_{\mu\nu} + g\tilde{F}_{\mu\nu})\,u^\nu \tag{16}$$

where m is the mass of particle, $\{x_\mu\}$ is displacement four vector, τ is proper time, $(\star)$ denotes the complex conjugation and $\{u^\nu\}$ is the particle four velocity. Equation (16) also reduces to the dual invariant Loretz force equation of motion for dyons as

$$m\frac{d^2x}{dt^2} = e\,(\overrightarrow{E} + \overrightarrow{u} \times \overrightarrow{H}\,) + g\,(\overrightarrow{H} - \overrightarrow{u} \times \overrightarrow{E}\,) \tag{17}$$

where $\overrightarrow{u}$ is the velocity of a particle and other symbols are alredy described above.

III. OCTONIONS

An octonion is defined as,

$$\mathcal{O} = \mathcal{O}_0 e_0 + \sum_{j=1}^{j=7} \mathcal{O}_j e_j, \qquad (\mathcal{O}_0, \mathcal{O}_j \ \in \mathbb{R}) \tag{18}$$

where e_j are octonion unit elements and satisfy the following multiplication rules;

$$\begin{aligned} &e_0 e_0 = e_0; \\ &e_j e_0 = e_0 e_j = e_j; \\ &e_j e_k = -\delta_{jk} e_0 + f_{jkl} e_l \ (\forall j, k, l = 1, 2, 3, 4, 5, 6, 7); \end{aligned} \tag{19}$$

where δ_{jk} is the usual Kronecker delta symbol and f_{jkl} (which was regarded as the Levi - Civita tensor for quaternions) is fully anti symmetric tensor with $f_{jkl} = +1$ for $(jkl) = (123), (516), (624), (435), (471), (673), (672)$. The above multiplication table directly follows that the algebra of octonion $\mathcal{O}$ is not associative i.e.

$$e_j(e_k e_l) \quad \neq \quad (e_j e_k) e_l. \tag{20}$$

The commutation rules for octonion basis elements are given by,

$$[e_j, e_k] \quad = \quad 2 f_{jkl} e_l; \ \{e_j, e_k\} = -2\delta_{AB} e_0 \tag{21}$$

and the associator

$$\{e_j, e_k, e_l\} = (e_j e_k) e_l - e_j(e_k e_l) = -\delta_{jk} e_l + \delta_{kl} e_j + (\epsilon_{jkl}\epsilon_{mnp} - \epsilon_{klp}\epsilon_{jpn}) e_n. \tag{22}$$

Octonion conjugate is defined as,

$$\overline{\mathcal{O}} \quad = \quad \mathcal{O}_0 e_0 - \mathcal{O}_j e_j \tag{23}$$

and

$$\overline{\overline{\mathcal{O}}} \ = \mathcal{O}; \qquad \overline{\mathcal{O}\mathcal{P}} = \overline{\mathcal{P}}\,\overline{\mathcal{O}}. \tag{24}$$

The norm N of the octonion is defined as,

$$N(\mathcal{O}) = \mathcal{O}\overline{\mathcal{O}} = \overline{\mathcal{O}}\mathcal{O} = (\mathcal{O}_0^2 + X\mathcal{O}_j^2) e_0 \tag{25}$$

while the inverse is defined as,

$$\mathcal{O}^{-1} = \frac{\overline{\mathcal{O}}}{N(\mathcal{O})}; \qquad \mathcal{O}\mathcal{O}^{-1} = \mathcal{O}^{-1}\mathcal{O} = 1.e_0. \tag{26}$$

The norm given by equation (25) is non - degenerate and positively defined (over $\mathbb{R}$) and therefore every element $X \in \mathcal{O}$ has the unique inverse element $X^{-1} \in \mathcal{O}$. For the split octonion algebra the following new basis is considered [3, 9, 38–40] on the complex field (instead of real field) i.e.

$$\begin{aligned} u_1 &= \frac{1}{2}(e_1 + ie_4); \quad u_1^\star = \frac{1}{2}(e_1 - ie_4); \\ u_2 &= \frac{1}{2}(e_2 + ie_5), \quad u_2^\star = \frac{1}{2}(e_2 - ie_5); \\ u_3 &= \frac{1}{2}(e_3 + ie_6); \quad u_3^\star = \frac{1}{2}(e_3 - ie_6); \\ u_0 &= \frac{1}{2}(e_0 + ie_7); \quad u_0^\star = \frac{1}{2}(e_0 - ie_7); \end{aligned} \tag{27}$$

where $i = \sqrt{-1}$ is assumed to commute with $e_A (A = 1, 2, 3, ...7)$ octonion units. The split octonion basis elements satisfy the following multiplication rules;

$$\begin{aligned} u_i u_j &= \epsilon_{ijk} u_k^\star; \quad u_i^\star u_j^\star = -\epsilon_{ijk} u_k (i, j, k = 1, 2, 3) \\ u_i u_j^\star &= -\delta_{ij} u_0; \quad u_i u_0 = 0; \quad u_i^\star u_0 = u_i^\star \\ u_i^\star u_j &= -\delta_{ij} u_0; \quad u_i u_0^\star = u_0; \quad u_i^\star u_0^\star = 0 \\ u_0 u_i &= u_i; \quad u_0^\star u_i = 0; \quad u_0 u_i^\star = 0 \\ u_0^\star u_i^\star &= u_i; u_0^2 = u_0; \quad u_0^{\star 2} = u_0^\star; \quad u_0 u_0^\star = u_0^\star u_0 = 0. \end{aligned} \tag{28}$$

Günaydin and Gürsey [38, 39] pointed out that the automorphism group of octonion algebra is the 14− dimensional exceptional G_2 group that admits a $SU(3)$ subgroup leaves imaginary octonion unit e_7 invariant (or equivalently the idempotents u_0 and $u_0^\star$). This $SU(3)_C$was identified as color group acting on quark and anti-quark triplets. From split octonian algebra multiplication rule (28), it is clear that the octonion units u_i and $u_i^\star (i = 1, 2, 3)$ transform respectively like a triplet and anti triplet accordingly associated with colour and anti colour triplets of $SU(3)$ group. Let us introduce a convenient realization for the basis elements $(u_0, u_i, u_0^\star, u_i^\star)$ in terms of Pauli spin matrices [3, 9, 38–40] as

$$\begin{aligned} u_0 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad u_0^\star = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \\ u_i &= \begin{bmatrix} 0 & 0 \\ e_i & 0 \end{bmatrix}; \quad u_i^\star = \begin{bmatrix} 0 & -e_i \\ 0 & 0 \end{bmatrix} (i = 1, 2, 3) \end{aligned} \tag{29}$$

where 1, e_1, e_2, e_3 are quaternion units satisfying the multiplication rule $e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k$. As such, for an arbitrary split octonion $\mathcal{O}$ we have [3, 9, 38–40],

$$\mathcal{O} = a u_0^\star + b u_0 + x_i u_i^\star + y_i u_i = \begin{bmatrix} a & -\overrightarrow{x} \\ \overrightarrow{y} & b \end{bmatrix} \tag{30}$$

which is a realization via the 2×2 Zorn's vector matrices $\begin{bmatrix} a & \overrightarrow{x} \\ \overrightarrow{y} & b \end{bmatrix}$ where a and b are scalars and $\overrightarrow{x}$ and $\overrightarrow{y}$ are three vectors. The product of two octonions in terms of 2×2 Zorn's vector matrix realization is then defined as

$$\begin{bmatrix} a & \overrightarrow{x} \\ \overrightarrow{y} & b \end{bmatrix} \begin{bmatrix} c & \overrightarrow{u} \\ \overrightarrow{v} & d \end{bmatrix} = \begin{bmatrix} ac + \overrightarrow{x} \cdot \overrightarrow{v} & a\overrightarrow{u} + d\overrightarrow{x} - \overrightarrow{y} \times \overrightarrow{v} \\ c\overrightarrow{y} + b\overrightarrow{u} - \overrightarrow{x} \times \overrightarrow{u} & bd + \overrightarrow{y} \cdot \overrightarrow{u} \end{bmatrix} \tag{31}$$

and $(\times)$ denotes the usual vector product, $e_i(i=1,2,3)$ with $e_i \times e_j = \varepsilon_{ijk} e_k$ and $e_i e_j = -\delta_{ij}$. Then we can relate the split octonions to the vector matrices given by equation (29). Octonion conjugation of equation (30) is defined as,

$$\overline{\mathcal{O}} \;\; = b u_0^\star + a u_0 - x_i u_i^\star - y_i u_i = \begin{bmatrix} b & \overrightarrow{x} \\ -\overrightarrow{y} & a \end{bmatrix}. \tag{32}$$

The norm of an octonion A is thus defined as,

$$\overline{\mathcal{O}} \quad \mathcal{O} \;\; = \mathcal{O}\,\overline{\mathcal{O}} = (ab + \overrightarrow{x} \cdot \overrightarrow{y}) \cdot \hat{1} \tag{33}$$

where $\hat{1}$ is the identity element of the algebra given by $\hat{1} = \hat{1} u_0^\star + \hat{1} u_0$. Despite the fact that the octonions are non associative the Zorn vector matrix product (31) satisfies the cyclic trace property [3]

$$\begin{aligned} Trace\;[\mathsf{P}\cdot\mathsf{Q}] &= Trace\,[\mathsf{Q}\cdot\mathsf{P}]\,; \\ Trace\;[(\mathsf{P}\cdot\mathsf{Q})\cdot\mathsf{R}] &= Trace\,[\mathsf{P}\cdot(\mathsf{Q}\cdot\mathsf{R})] = Trace\;[\mathsf{P}\cdot\mathsf{Q}\cdot\mathsf{R}] \\ &= Trace\;[\mathsf{R}\cdot\mathsf{P}\cdot\mathsf{Q}] == Trace\;[\mathsf{Q}\cdot\mathsf{R}\cdot\mathsf{P}] \end{aligned} \tag{34}$$

where P, Q, and R are three octonions written in terms of Zorn vector matrix realizations.

IV. OCTONION GAUGE ANALYSIS

For octonion gauge theory, we may consider octonionic or $\mathbb{O}-$ transformation like equation (30). Let us define an unitary octonion [4] as

$$\mathbb{U} = p_0 u_0^\star + q_0 u_0 + p_i u_i^\star + q_i u_i = \begin{bmatrix} p_0 & -\overrightarrow{p} \\ \overrightarrow{q} & q_0 \end{bmatrix} \tag{35}$$

and its conjugate octonion as

$$\overline{\mathbb{U}} = q_0 u_0^\star + p_0 u_0 - p_i u_i^\star - q_i u_i = \begin{bmatrix} q_0 & \overrightarrow{p} \\ -\overrightarrow{q} & p_0 \end{bmatrix}. \tag{36}$$

These are the bi-modular representation of a quaternion transformations subjected by two fold quaternionic $U(1) \times SU(2)$ transformations so that it satisfies the unitarity condition $\mathbb{U}^{-1} = \overline{\mathbb{U}}$ and

$$\overline{\mathbb{U}}\,\mathbb{U} = \mathbb{U}\overline{\mathbb{U}} = \mathbb{U}^{-1}\mathbb{U} = \mathbb{U}\mathbb{U}^{-1} = (p_0 q_0 + \sum_{i=1}^{i=3} p_i q_i) = \hat{\mathbf{1}} \tag{37}$$

where $\hat{\mathbf{1}}$ is 4×4 identiy (unit) matrix and quaternion basis elements $e_0 \equiv \widehat{1}$ and $e_j \equiv -\sqrt{-1}\sigma_j$ $(\forall\, j = 1,2,3)$ are described in terms of 2×2 unit and Pauli-spin $\sigma-$ matrices. Thus, we may write the octonion unitary gauge transformation as

$$\mathbb{U} = \exp(-\alpha_j u_j - \beta_j u_j^\star) \tag{38}$$

where $\alpha_j$and β_j are real parameters and $\sqrt{-1}$ is included in u_j and $u_j^\star$. Equation (38) thus follows the $SU(2) \times SU(2)$ connection of octonion gauge transformation. The $\mathbb{O}$-

transformation of an octonion Ψ is defined [4] as

$$\Psi' = \frac{1}{2}[(\mathbb{U}\Psi)\mathbb{U}^{-1} + \mathbb{U}(\Psi\mathbb{U}^{-1})] = \mathbb{U}\Psi\mathbb{U}^{-1} \tag{39}$$

which reduces to following unitary transformations of a matter field spinor ψ as

$$\psi' = \mathbb{U}\psi. \tag{40}$$

As such, the covariant derivative or $\mathbb{O}$-derivative of an octonion K is defined [4] as

$$K_{\|\mu} = K_{,\mu} + [\Im_\mu, K] \tag{41}$$

where$\Im_\mu$ is the octonion affinity namely it is the object that makes $K_{\|\mu}$transform like an octonion under$\mathbb{O}-$ transformations i.e.

$$\begin{aligned} K' &= \mathbb{U}\,K\,\mathbb{U}^{-1} \\ K'_{\|\mu} &= \mathbb{U}K_{\|\mu}\mathbb{U}^{-1}; \\ \Im'_\mu &= \mathbb{U}\,\Im_\mu\mathbb{U}^{-1} - \frac{\partial\mathbb{U}}{\partial x^\mu}\,\mathbb{U}^{-1}. \end{aligned} \tag{42}$$

Thus it describes $SU(2)$ like octonion transformations. As such , we may write $\Im_\mu$ as the trace free Zorn matrix supposedly an Yang - Mill's type field i.e.,

$$\Im_\mu = -L_{\mu i}u_i^\star - K_{\mu i}u_i = \begin{bmatrix} 0_2 & L_{\mu i}e_i \\ -K_{\mu i}e_i & 0_2 \end{bmatrix} (i = 1, 2, 3) \tag{43}$$

where $L_{\mu i}$and $K_{\mu i}$are two different gauge fields associated with two $SU(2)$ type gauge fields.

V. OCTONION GAUGE ANALYTICITY OF DYONS

Let us define the electric and magnetic gauge potential respectively as

$$\begin{aligned} L_{\mu i} &= eA_{\mu i}; \\ K_{\mu i} &= g\,B_{\mu i}. \end{aligned} \tag{44}$$

In order to reformulate the quantum equations of dyons by means of split octonion realization, we write the O - derivative as follows,

$$\mathcal{O}_{\|\mu} = \mathcal{O}_{,\mu} + [\Im_\mu, \mathcal{O}] \tag{45}$$

where we have used

$$\begin{aligned} \Im_\mu &= -eA^a_\mu u^\star_a - gB^a_{\mu a}u_a = \begin{bmatrix} 0_2 & eA_{\mu i}e_i \\ -gB_{\mu i}e_i & 0_2 \end{bmatrix} \\ &= e_a(eA^a_\mu + gB^a_\mu) + ie_{a+3}(eA^a_\mu - gB^a_\mu) = -e_a Re(q^\star V^a_\mu) + ie_{a+3}Re(qV^a_\mu). \end{aligned} \tag{46}$$

Then we get

$$\begin{aligned} \mathcal{O}_{\|\mu} &= \mathcal{O}_{,\mu} + \left[-e_a Re(q^\star V^a_\mu), \mathcal{O}\right] + ie_{a+3}\left[Re(qV^a_\mu), \mathcal{O}\right] \\ &= \mathcal{O}_{,\mu} - e_a \left[Re(q^\star V^a_\mu), \mathcal{O}\right] + i \quad e_{a+3}\left[Re(qV^a_\mu), \mathcal{O}\right] \end{aligned} \tag{47}$$

where $q^\star$ being the complex conjugate of generalized charge q of dyons and $\left\{V^a_\mu\right\}$plays the role of gauge potential in internal $SU(2)$ non - Abelian gauge space. The gauge potential

$\{V_\mu\}$ and gauge field strength $\{G_{\mu\nu}\}$ of dyons may be expressed in terms of split octonion realization as,

$$V_\mu = \quad V_\mu u_0 + V_\mu^a u^0 + V_\mu u_0^\star + V_\mu^a u_a^\star \quad = \begin{bmatrix} V_\mu e^0 & -V_\mu^a e^a \\ V_\mu^a e^a & V_\mu e^0 \end{bmatrix}; \tag{48}$$

$$G_{\mu\nu} \quad = g_{\mu\nu} u_0 + g_{\mu\nu} u_0^\star + E_{\mu\nu}^a u^a + H_{\mu\nu}^a u^a \quad = \begin{bmatrix} g_{\mu\nu} e^0 & -E_{\mu\nu}^a e^a \\ H_{\mu\nu}^a e^a & g_{\mu\nu} e^0 \end{bmatrix}; \tag{49}$$

where $G_{\mu\nu} = V_{\mu,\nu} - V_{\nu,\mu}$. In equation (47), we have expressed the generalized four-potential $\{V_\mu\}$ and generalized field tensor $G_{\mu\nu}$of dyons in terms of Abelian $U(1)$ and non - Abelian $SU(2)$ gauge coupling strengths where the real quaternion unit e_0 is associated with $U(1)$ Abelian gauge group and the pure imaginary unit quaternion $e_a (a = 1, 2, 3)$ is related with the $SU(2)$ non - Abelian Yang - Mill's field and it may readily be seen that the gauge field strength $G_{\mu\nu}$ is invariant under local and global phase transformations. The non - Abelian $SU(2)$ Yang - Mill's gauge field strength may then be expressed as,

$$G_{\mu\nu}^a \quad = \partial_\nu V_\mu^a - \partial_\mu V_\nu^a \quad + q^\star \epsilon_{abc} V_\mu^b V_\nu^c \tag{50}$$

which yields the correct field equation for non - Abelian gauge theory of dyons and gives rise its extended structure. Hence the $\mathcal{O}$- derivative of generalized field tensor $G_{\mu\nu}$given by (47) of dyons leads to the following field equation;

$$G_{\mu\nu\|\nu} = G_{\mu\nu,\nu} + [\Im_\nu, G_{\mu\nu}] = J_\mu (u_0 + u_0^\star) + J_\mu^a (u_a + u_a^\star) = J_\mu; \tag{51}$$

where

$$J_\mu^a = G_{\mu\nu\,,\nu}^a + i q^\star \epsilon_{abc} V_\mu^b G_{\mu\nu}^c \tag{52}$$

is the non - Abelian $SU(2)$ generalized four - current of dyons. In deriving Eq. (36) we have used Eq. (32), (34) and (35). Thus the generalized four current of dyons in split octonion realization leads to the Abelian and non - Abelian nature of current in this theory.

The $\mathcal{O}$- derivative of generalized four - current

$$J_\mu = J_\mu (u_0 + u_0^\star) + J_\mu^a (u_a + u_a^\star) = \begin{bmatrix} J_\mu e^0 & -J_\mu^a e^a \\ J_\mu^a e^a & J_\mu e^0 \end{bmatrix} \tag{53}$$

gives rise to

$$J_{\mu\|\mu} = 0 \tag{54}$$

which shows the resemblance with Noetherian current and describes analogues continuity equation in Abelian gauge theory.

VI. OCTONIONIC EXTENSION FOR GRAVITO - DYONS

Following the idea of Dowker and Roche [19] of dual mass playing the role of magnetic charge (Heavisidian monopole), the gravito - dyons may also be defined as the particle carrying gravitational mass m and dual mass (Heavisidian mass) h having the generalized mass Q given [30] by,

$$Q = m - i\, h. \tag{55}$$

Analogous to the theory of electromagnetic (dyons), the generalized four - potential of gravito - dyons may then be expressed as,

$$\{\mathfrak{V}_\mu\} = \{\mathfrak{A}_\mu\} - i\,\{\mathfrak{B}_\mu\} \tag{56}$$

where $\{\mathfrak{A}_\mu\}$ is the usual space - time gravitational four-vector potential due to the presence of gravitational charge (mass) and $\{\mathfrak{B}_\mu\}$ is the gravito-Heavisidian (gravi-magnetic) four potential due to the existence of gravito-magnetic monopole. The effective mass of gravito - dyons is defined [30] as

$$M_{eff} \;=\; m + h. \tag{57}$$

Thus we can express the generalized four - potential of gravito - dyons by means of octonions gauge formalism given by equation (49). Similarly, one can construct the generalized field tensor of gravito - dyons in terms of split octonion as,

$$\mathfrak{G}_{\mu\nu} \;=\; \mathfrak{g}_{\mu\nu}(u_0+u_0^\star)+\varepsilon^a_{\mu\nu}u^a+\mathfrak{h}^a_{\mu\nu}u^a=\begin{bmatrix}\mathfrak{g}_{\mu\nu}e^0 & -\varepsilon^a_{\mu\nu}e^a\\ \mathfrak{h}^a_{\mu\nu}e^a & \mathfrak{g}_{\mu\nu}e^0\end{bmatrix} \tag{58}$$

where $\mathfrak{G}_{\mu\nu} = \mathfrak{V}_{\mu,\nu} - \mathfrak{V}_{\nu,\mu}$and $\mathfrak{G}^a_{\mu\nu} = \partial_\nu\mathfrak{V}^a_\mu - \partial_\mu\mathfrak{V}^a_\nu + q^\star\epsilon_{abc}\mathfrak{V}^b_\mu\mathfrak{V}^c_\nu$ have the same form of equation (48-49) for the case of non - Abelian gauge theory of dyons associated with generalized electromagnetic fields. Defining the generalized current of gravito - dyons as

$$\mathfrak{J}_\mu \;=\; \mathfrak{j}^{(G)}_\mu - i\,\mathfrak{k}^{(H)}_\mu \tag{59}$$

where $\mathfrak{j}^{(G)}_\mu$ is the four - current associated with gravitational mass and $\mathfrak{k}^{(G)}_\mu$ is the four - current associated with Heavisidian monopole. The generalized vector field $\overrightarrow{\psi_G}$ of gravito - dyons may be expressed as,

$$\overrightarrow{\psi_{\mathfrak{G}}} \;=\; \overrightarrow{\mathfrak{G}} - i\,\overrightarrow{\mathfrak{H}} \tag{60}$$

where $\overrightarrow{\mathfrak{G}}$ is the gravitational field and $\overrightarrow{\mathfrak{H}}$ is the Heavisidian field strength defined as follows in terms of two four - potential components as,

$$\begin{aligned}\overrightarrow{\mathfrak{G}} \;&=\; \frac{\partial\overrightarrow{\mathfrak{A}}}{\partial t}+\overrightarrow{\nabla}\phi_G+\overrightarrow{\nabla}\times\overrightarrow{\mathfrak{B}};\\ \overrightarrow{\mathcal{H}} \;&=\; \frac{\partial\overrightarrow{\mathfrak{B}}}{\partial t}+\overrightarrow{\nabla}\phi_h-\overrightarrow{\nabla}\times\overrightarrow{\mathfrak{A}}.\end{aligned} \tag{61}$$

Thus we express the generalized field $\overrightarrow{\psi_G}$ in terms of generalized four - current J_μ as,

$$\begin{aligned}\overrightarrow{\nabla}\cdot\overrightarrow{\psi_G} \;&=\; -\mathfrak{J}_{\mathfrak{o}}\\ \overrightarrow{\nabla}\times\overrightarrow{\psi_G} \;&=\; -i\frac{\partial\overrightarrow{\psi_G}}{\partial t}-i\,\overrightarrow{J}\end{aligned} \tag{62}$$

which is the field equation (generalized Maxwell - Dirac equation) for linear gravito - Heavisidian fields.The $\mathcal{O}$- forms of different gauge potentials associated with gravito -

dyons may be expressed as

$$\mathfrak{V}_{\mu}= \quad \mathfrak{V}_{\mu}u_{0}+\mathfrak{V}_{\mu}^{a}u^{0}+\mathfrak{V}_{\mu}u_{0}^{\star}+\mathfrak{V}_{\mu}^{a}u_{a}^{\star} \quad =\begin{bmatrix} \mathfrak{V}_{\mu}e^{0} & -\mathfrak{V}_{\mu}^{a}e^{a}\\ \mathfrak{V}_{\mu}^{a}e^{a} & \mathfrak{V}_{\mu}e^{0}\end{bmatrix};$$
$$\mathfrak{A}_{\mu}= \quad \mathfrak{A}_{\mu}u_{0}+\mathfrak{A}_{\mu}^{a}u^{0}+\mathfrak{A}_{\mu}u_{0}^{\star}+\mathfrak{A}_{\mu}^{a}u_{a}^{\star} \quad =\begin{bmatrix} \mathfrak{A}_{\mu}e^{0} & -\mathfrak{A}_{\mu}^{a}e^{a}\\ \mathfrak{A}_{\mu}^{a}e^{a} & \mathfrak{A}_{\mu}e^{0}\end{bmatrix};$$
$$\mathfrak{B}_{\mu}= \quad \mathfrak{B}_{\mu}u_{0}+\mathfrak{B}_{\mu}^{a}u^{0}+\mathfrak{B}_{\mu}u_{0}^{\star}+\mathfrak{B}_{\mu}^{a}u_{a}^{\star} \quad =\begin{bmatrix} \mathfrak{B}_{\mu}e^{0} & -\mathfrak{B}_{\mu}^{a}e^{a}\\ \mathfrak{B}_{\mu}^{a}e^{a} & \mathfrak{B}_{\mu}e^{0}\end{bmatrix}; \tag{63}$$

As such the $\mathcal{O}$- derivative of generalized field tensor $\mathfrak{G}_{\mu\nu}$leads to,

$$\mathfrak{G}_{\mu\nu\|\nu}=\mathfrak{G}_{\mu\nu,\nu}+[\Im_{\nu},\mathfrak{G}_{\mu\nu}]=-\mathfrak{J}_{\mu}(u_{0}+u_{0}^{\star})-\mathfrak{J}_{\mu}^{a}(u_{a}+u_{a}^{\star})=-\mathfrak{J}_{\mu} \tag{64}$$

where $\mathfrak{J}_{\mu}$ is the generalized split octonion current associated with gravito - dyons similar to that of dyons in generalized electromagnetic fields given by equation (53). Hence like previous case, here the $\mathcal{O}$ - derivative of the generalized current vanishes and the conservation of generalized four - current follows the continuity equation with Abelian and non - Abelian gauge structures.

VII. CONCLUSION

In Yang Mill's theory, the gravito-dyons in gravito - Heavisidian fields plays the same role as that of dyons in electromagnetic fields. As such, the split octonion gauge formalism demonstrates the structural symmetry between generalized electromagnetic fields of dyons and that of generalized gravito - Heavisidian fields of gravito - dyons. In case of split octonions, the automorphism group is described as G_2 (an exceptional Lie group). Thus octonion transformations are isomorphic to the rotation group O_3. Under the $SU(3)$ subgroup of split G_2 leaving u_0 and $u_0^{\star}$ invariant, the three split octonions (u_1, u_2, u_3) transform like a isospin triplet (quarks) and the complex conjugate octonions transform like a unitary anti - triplet (anti quarks) [38, 39]. The Abelian and non - Abelian gauge structures of dyons and gravito - dyons are discussed in terms of split octonion variables in simple, compact and consistent way. The field equations derived here are invariant under octonion gauge transformations. From the foregoing analysis one can obtain the independent theories of electromagnetism and gravitation in the absence of each other. The justification behind the use of octonions is to obtain the simultaneous structure of $SU(2)\times U(1)$ gauge theory of dyons and gravito - dyons in simple and compact manner. As such, the well - known $SU(2)$ non - Abelian and $U(1)$ Abelian gauge structure of dyons and gravito - dyons are reformulated in terms of compact gauge formulation. The $\mathcal{O}$-derivative may be considered as the partial derivative if we do not incorporate the split octonion variable. Split octonions are described here in terms of $U(1)\times SU(2)$ gauge group simultaneously to give rise the Abelian (point like) and non - Abelian (extended structure) of dyons. This gauge group plays the role of $U(1)\times SU(2)$ Salam Weinberg theory of electro - weak interaction in the absence of Heavisidian monopoles. Our enlarged gauge group $SU(2)\times U(1)$ explains the built in duality to reproduce Abelian and non - Abelian gauge structure of dyons and those for gravito - dyons.

REFERENCES

[1] L. E. Dickson, *Ann. Math.* **20** (1919) 155.
[2] J. Baez, *Bull. Amer. Math. Sci.* **39** (2002) 145: *eprint* arXiv: math. RA/0105155.
[3] Carlos Castro, *J. Math. Phys.* **48** (2007), 073517
[4] S. Marques and C. G. Oliveira, *J. Math. Phys.* **26** (1985) 3131; *Phys. Rev.* **D36** (1987) 1716.
[5] R. Penny, *Amer. J. Phys.* **36** (1968) 871; *Nuovo Cimento* **B3** (1971) 95.
[6] K. Imaeda, II. Tachibana and M. Imaeda, *Nuovo Cimento* **B100** (1987) 53.
[7] T. Tachibana and K. Imaeda, *Nuovo Cimento* **B105** (1990) 1203.
[8] P. S. Bisht and O. P. S. Negi, *PRAMANA-Journal of Phys.* **73**(2009) 605.
[9] P. S. Bisht, Bimal Pandey, and O. P. S. Negi, *FIZIKA* **B17** (2008) 405.
[10] B. C. Chanyal, P. S. Bisht and O. P. S. Negi, *Int. J. Theor. Phys.* **49** (2010), 1333.
[11] M. Gogberashvili, *J. Phys. A: Math. Gen.* **39** (2006) 7099.
[12] T. Tolan, K. Özdas, and M.Tanisli, *Nuovo Cimento* **121** (2006) 43.
[13] T. Kugo and P. Townsend, *Nucl. Phys.* **B221** (1983), 357.
[14] L. Brink and J. H. Schwarz, *Nucl. Phys.* **B121** (1977), 77.
[15] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory", Vol. I & II, *Cambridge University Press*, (1987).
[16] P. A. M. Dirac, *Proc. Roy. Soc.* **A133** (1930) 60.
[17] G. t' Hooft, *Nucl. Phys.* **B79** (1974), 276.
[18] A. M. Polyakov, *JETP Lett.* **20** (1974) 194.
[19] N. Craigie, *Theory and Detection of Magnetic Monopoles in Gauge Theories*, World Scientific, (1986).
[20] J. Schwinger, *Science* **165** (1969) 757.
[21] D. Zwanziger, *Phys. Rev.* **176** (1968) 1489.
[22] B. Julia and A. Zee, *Phys Rev.* **D11** (1975) 2227
[23] E. Witten, *Phys. Lett.* **B86** (1979) 283.
[24] B. S. Rajput and D.C. Joshi, *Pramana* **13** (1995) 637.
[25] B. S. Rajput and D. C. Joshi, *Hadronic J.* **4** (1981) 1805.
[26] P. S . Bisht, O. P. S. Negi and B. S. Rajput, *Prog. Theo. Phys.* **85** (1991) 157.
[27] Shalini Bisht, P. S. Bisht and O. P. S. Negi, *Nuovo Cimento* **113B** (1998), 1449.
[28] J. S. Dowker and R.A. Roche, *Proc. Phys. Soc.* **92** (1967) 1.
[29] D. D. Cantani, *Nuovo Cimento* **60** (1980) 67.
[30] B. S. Rajput, *J. Math. Phys.* **25** (1984) 351.
[31] A. Singh, Lett. NuovoCimento, **32** (1981) 281.
[32] P. S . Bisht, O. P.S. Negi and B. S. Rajput, *Ind. J. Pure and Appl. Phys.* **28** (1990) 157.
[33] P. S. Bisht and O. P. S. Negi, *Ind. J. Pure and Appl. Phys.* **32** (1994) 183.
[34] P. S. Bisht, O. P. S. Negi and B. S. Rajput, *Ind. J. Pure and Appl. Maths.* **24** (1993) 543.
[35] P. S. Bisht, O. P. S. Negi and B. S. Rajput, *Inter. J. Theor. Phys.* **32** (1993) 2099.
[36] N. Cabibbo and E. Ferrari, *Nuovo Cimento* **23** (1962) 1147.
[37] Gaurav Karnatak, P. S. Bisht and O. P. S. Negi, *Inter. J. Theor. Phys.* **49** (2010) 1344.
[38] M. Günaydin and F. Gürsey, *Phys. Rev.* **D9** (1974) 3387.
[39] M. Gunünaydin and F. Gürsey, *J. Math. Phys.* **14** (1973) 1651.
[40] S. Catto, *Exceptional Projective Geometries and Internal Symmetries*, *eprint* arXiv: hep-th/0302079.

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