Communications in Physics, Vol. 32, No. 2 (2022), pp. 141-156
DOI: https://doi.org/10.15625/0868-3166/16508

HIGHER-ORDER NONCLASSICALITY IN SUPERPOSITION OF THREE-MODE PHOTON-ADDED TRIO COHERENT STATES

TRAN QUANG DAT[1], HO SY CHUONG[2,3], DAO VAN HUNG[2,4]
AND TRUONG MINH DUC[2†]

[1]*University of Transport and Communications, No.3 Cau Giay Street, Lang Thuong Ward, Dong Da District, Hanoi, Vietnam*

[2]*Center for Theoretical and Computational Physics, University of Education, Hue University, Hue, Vietnam*

[3]*Dong Nai University, 4 Le Quy Don, Tan Hiep, Bien Hoa City, Dongnai, Vietnam*

[4]*Naval Academy, 30 Tran Phu Street, Nha Trang City, Vietnam*

E-mail: [†]tminhduc@hueuni.edu.vn

Received 4 September 2021; *Accepted for publication 28 September 2021*
Published 27 March 2022

Abstract. *In this paper, we study some higher-order nonclassical properties of the superposition of three-mode photon-added trio coherent states such as antibunching, squeezing, and entanglement. The results show that, for a fixed order, when the number of added photons is increased, the three-mode sum squeezing is more obvious, but the degrees of antibunching and three-mode entanglement are less manifested. Besides, for a fixed total number of photon added to the superposition of three-mode photon-added trio coherent states, the higher the order the stronger the antibunching and entanglement, but the weaker the three-mode sum squeezing.*

Keywords: Higher-order nonclassicality; antibunching; squeezing; entanglement; photon-added trio coherent state.

Classification numbers: 42.50.-p; 03.65.Ud.

I. INTRODUCTION

The nonclassical properties in nonclassical states play a heart role in many quantum tasks. For example, antibunching is used to produce the single-photon resources [1], squeezing is exploited to detect the gravitational wave in the famous LIGO experiment [2], and entanglement is applied to the protocols of quantum computation and quantum information [3], etc. Because the efficiency of the quantum tasks strongly depends on the strength of these nonclassical properties, researchers try to construct states with strong nonclassical features. Therefore, many new nonclassical states have been proposed and studied such as single-mode squeezed vacuum state [4], two-mode squeezed vacuum state [5], pair coherent state [6], even and odd charge coherent states [7], trio coherent state [8], three-mode squeezed vacuum state [9], and so on. The degree of nonclassicality in these nonclassical states can be adjusted by changing the amplitude of fields or the degenerated parameters [9–11]. Recommending new nonclassical states with high nonclassical features is still welcome in the study of quantum optics.

Given the available nonclassical states, we can enhance their degree of nonclassicality by using non-Gaussian operations such as adding or subtracting photons. For example, the degree of entanglement and squeezing can be enhanced by adding photons on the two-mode squeezed vacuum state [12], the two-mode squeezed thermal state [13], the two-mode displaced squeezed state [14], the pair coherent state [15], the trio coherent state [16, 17], etc. More importantly, quantum tasks including quantum key distribution, quantum teleportation, quantum illumination as well as quantum state engineering can be improved by the photon addition and subtraction operations [15, 18–20]. The interesting thing is that these operations are no longer on paper, they were done early in the laboratory [21]. Therefore, adding and subtracting photons to nonclassical states have a lot of potential applications and need to be studied further.

In the multimode regime, we aim to add photons on three-mode states. These states can be used to implement network quantum tasks, for example, controlled teleportation [22], quantum secret sharing [23], joint remote state preparation [24], and so on. A nonclassical state which we are interested in is the trio coherent state $|\Psi_{p,q}\rangle_{abc}$ of three modes a, b and c [8]. This state is defined as the simultaneous eigenstate of three operators $\hat{a}\hat{b}\hat{c}$, $\hat{b}^\dagger\hat{b}-\hat{a}^\dagger\hat{a}$ and $\hat{c}^\dagger\hat{c}-\hat{b}^\dagger\hat{b}$ with respective eigenvalues ξ, p and q, i.e.,

$$\hat{a}\hat{b}\hat{c}|\Psi_{p,q}\rangle_{abc} = \xi|\Psi_{p,q}\rangle_{abc}, \tag{1}$$

$$(\hat{b}^\dagger\hat{b}-\hat{a}^\dagger\hat{a})|\Psi_{p,q}\rangle_{abc} = p|\Psi_{p,q}\rangle_{abc}, \tag{2}$$

$$(\hat{c}^\dagger\hat{c}-\hat{b}^\dagger\hat{b})|\Psi_{p,q}\rangle_{abc} = q|\Psi_{p,q}\rangle_{abc}, \tag{3}$$

where $\hat{x}(\hat{x}^\dagger)$ is bosonic annihilation (creation) operator of mode $x, x=\{a,b,c\}$, $\xi = re^{i\varphi}$, r and φ are real, and degenerated parameters p and q are nonnegative integers. In terms of Fock states, the trio coherent state is given by

$$|\Psi_{p,q}\rangle_{abc} = \sum_{n=0}^{\infty}\frac{N_{p,q}(r)\xi^n}{\sqrt{n!(n+p)!(n+p+q)!}}|n, n+p, n+p+q\rangle_{abc}, \tag{4}$$

where $|n,n+p,n+p+q\rangle_{abc} \equiv |n\rangle_a|n+p\rangle_b|n+p+q\rangle_c$ is a three-mode Fock state, and the normalization factor $N_{p,q}(r)$ is determined as

$$N_{p,q}^{-2}(r) = \sum_{n=0}^{\infty}\frac{r^{2n}}{n!(n+p)!(n+p+q)!}. \tag{5}$$

Adding photons to the trio coherent state can enhance the non-Gaussianity, the degree of three-mode sum squeezing, and the degree of entanglement in the photon-added trio coherent state (PATCS) [16] as well as the superposition of three-mode photon-added trio coherent state (STM-PATCS) [17]. Besides, the higher-order antibunching, squeezing, and entanglement are revealed in the PATCS [25]. However, these properties have not yet been studied in the STMPATCS. Therefore, in this paper, we investigate some of the higher-order nonclassical properties in this state including single-mode and two-mode antibunching, three-mode sum squeezing, and three-mode entanglement. The STMPATCS was proposed by adding nonlocal photons to the trio coherent state as follows:

$$|\Phi_{p,q;h,k,l}\rangle_{abc} = N_{p,q;h,k,l}(r)(\epsilon\hat{a}^{\dagger h} + \lambda\hat{b}^{\dagger k} + \sigma\hat{c}^{\dagger l})|\Psi_{p,q}\rangle_{abc}, \tag{6}$$

where h,k,l are positive integers, ϵ,λ,σ are reals and lie in $[-1,1]$, and the normalization factor $N_{p,q;h,k,l}(r)$ is written as

$$N^{-2}_{p,q;h,k,l}(r) = N^2_{p,q}(r)\big[\epsilon^2 A^{h,0,0}_{0,0,0}(r) + \lambda^2 A^{0,k+p,0}_{0,p,0}(r) + \sigma^2 A^{0,0,l+p+q}_{0,0,p+q}(r)\big], \tag{7}$$

therein

$$A^{i,j,m}_{t,u,v}(r) = {}_PF_Q(1+i,1+j,1+m;1+p,1+p+q,1+t,1+u,1+v;r^2)\frac{i!j!m!}{p!(p+q)!t!u!v!}, \tag{8}$$

with ${}_PF_Q$ being the hypergeometric function.

The quantum averages of operators $\hat{a}^i\hat{a}^{\dagger i}\hat{b}^j\hat{b}^{\dagger j}\hat{c}^w\hat{c}^{\dagger w}$ and $(\hat{a}^\dagger\hat{b}^\dagger\hat{c}^\dagger)^u$ in the STMPATCS are given by

$$\begin{aligned} B_{i,j,w} &= \langle\hat{a}^i\hat{a}^{\dagger i}\hat{b}^j\hat{b}^{\dagger j}\hat{c}^w\hat{c}^{\dagger w}\rangle \\ &= N^2_{p,q;h,k,l}(r)N^2_{p,q}(r)\big[\epsilon^2 A^{i+h,j+p,w+p+q}_{0,p,p+q}(r) + \lambda^2 A^{i,j+k+p,w+p+q}_{0,p,p+q}(r) \\ &\quad +\sigma^2 A^{i,j+p,w+l+p+q}_{0,p,p+q}(r)\big], \end{aligned} \tag{9}$$

and

$$\begin{aligned} C_u &= \langle(\hat{a}^\dagger\hat{b}^\dagger\hat{c}^\dagger)^u\rangle \\ &= N^2_{p,q;h,k,l}(r)N^2_{p,q}(r)\xi^{*u}\big[\epsilon^2 A^{u+h,0,0}_{u,0,0}(r) + \lambda^2 A^{0,u+k+p,0}_{0,u+p,0}(r) \\ &\quad +\sigma^2 A^{0,0,u+l+p+q}_{0,0,u+p+q}(r)\big], \end{aligned} \tag{10}$$

with i,j,w,u being the nonnegative integers. In the next section, we investigate the higher-order single-mode and two-mode antibunching of the STMPATCS. We study the higher-order three-mode sum squeezing in Section III, and the higher-order three-mode entanglement of the STMPATCS in Section IV. Finally, we briefly summarize the higher-order nonclassical properties of the STMPATCS in the conclusion.

II. HIGHER-ORDER ANTIBUNCHING

II.1. Single-mode antibunching

Single-mode higher-order antibunching (SMHOA) was first introduced by Lee [26], then developed by several authors [27]. According to An [28], a state reveals the mth-order antibunching in mode x if

$$A_x(m) \equiv \frac{\langle\hat{x}^{\dagger(m+1)}\hat{x}^{m+1}\rangle}{\langle\hat{x}^{\dagger m}\hat{x}^m\rangle\langle\hat{x}^\dagger\hat{x}\rangle} - 1 < 0. \tag{11}$$

The factor $A_x(m)$ also indicates the degree of the SMHOA. The more negative the $A_x(m)$ is, the more obvious the SMHOA becomes. The degree of the SMHOA is maximum when $A_x(m) = -1$. In the STMPATCS, we obtain

$$A_a(m) = \frac{D_{m+1,0,0}}{D_{m,0,0}D_{1,0,0}} - 1 \tag{12}$$

for mode a. In mode b, the degree of the SMHOA becomes

$$A_b(m) = \frac{D_{0,m+1,0}}{D_{0,m,0}D_{0,1,0}} - 1, \tag{13}$$

and in mode c, we have

$$A_c(m) = \frac{D_{0,0,m+1}}{D_{0,0,m}D_{0,0,1}} - 1, \tag{14}$$

where

$$\begin{aligned} D_{i,j,w} &= \langle \hat{a}^{\dagger i}\hat{a}^{i}\hat{b}^{\dagger j}\hat{b}^{j}\hat{c}^{\dagger w}\hat{c}^{w}\rangle \\ &= N^2_{p,q;h,k,l}(r)N^2_{p,q}(r)\Big(\epsilon^2\sum_{n=X}^{\infty}\frac{r^{2n}[(n+h)!]^2}{(n!)^2(n+h-i)!(n+p-j)!(n+p+q-w)!} \\ &\quad+\lambda^2\sum_{n=Y}^{\infty}\frac{r^{2n}[(n+p+k)!]^2}{[(n+p)!]^2(n-i)!(n+p+k-j)!(n+p+q-w)!} \\ &\quad+\sigma^2\sum_{n=Z}^{\infty}\frac{r^{2n}[(n+p+q+l)!]^2}{[(n+p+q)!]^2(n-i)!(n+p-j)!(n+p+q+l-w)!}\Big), \end{aligned} \tag{15}$$

where $X = \max[0, i-h, j-p, w-p-q], Y = \max[0, i, j-p-k, w-p-q]$, and $Z = \max[0, i, j-p, w-p-q-l]$.

The factors $A_x(m), x = \{a, b, c\}$, in Eqs. (12), (13), and (14) allow us to investigate the SMHOA in the STMPATCS. For mode a, in Fig. 1, we plot $A_a(m)$ in Eq. (12) as a function of r when $p = q = 0$ for several values of (h, k, l) and different orders m. The results show that the degree of the SMHOA in mode a is enhanced by increasing the order m. However, it is reduced if the numbers of added photons h, k and l are increased. Besides, when m is small and h, k and l are large, the factors $A_a(m)$ becomes more negative in the high value region of r. More importantly, when m is large, the STMPATCS always exists the SMHOA.

When $h+k+l$ is fixed, i.e. the total number of photons added is a constant, in Fig. 2, we plot the factor $A_a(m)$ as a function of r when $p = q = 0$ and $m = 2$ with different values of h, k and l. We can see that the degree of the SMHOA is the largest when the number of photons added to mode a is the smallest. From another perspective, when $p = q = 0, h = k = l = \lambda = 1$ and $r = 4$, the negativeness of $A_a(m)$ is highest if $\lambda = \sigma = 0$, i.e. photons are added only to the mode a (see Fig. 3). Note that the above expressions are also true for modes b and c.

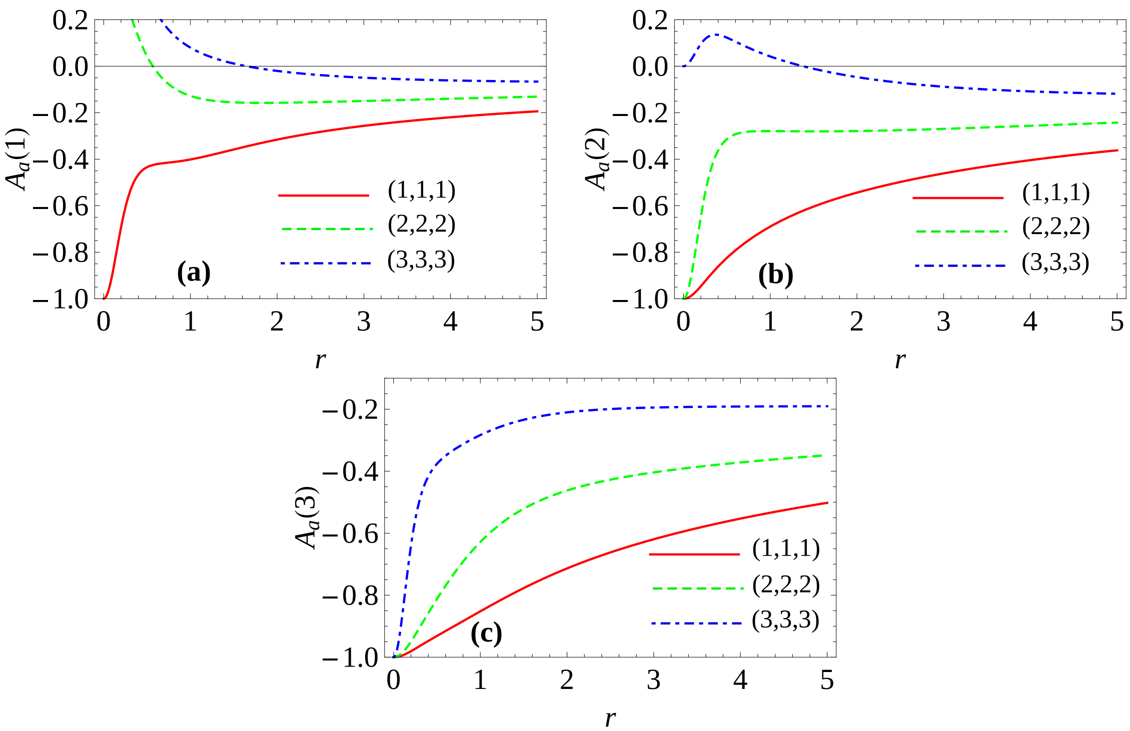

Fig. 1. The factor $A_a(m)$ as a function of r when $p=q=0$ and $\epsilon=\lambda=\sigma=1$ for $h=k=l=1$ (the red solid line), $h=k=l=2$ (the green dashed curve), and $h=k=l=3$ (the blue dot-dashed curve) with in (a) $m=1$, in (b) $m=2$, and in (c) $m=3$.

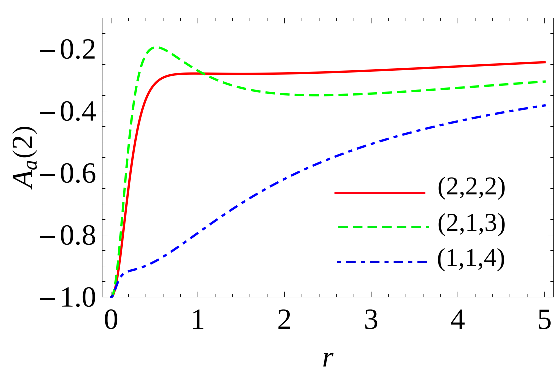

Fig. 2. The factor $A_a(m)$ as a function of r when $p=q=0, \epsilon=\lambda=\sigma=1$ and $m=2$ for $h=k=l=2$ (the red solid line), $h=2, k=1, l=3$ (the green dashed curve), and $h=k=1, l=4$ (the blue dot-dashed curve).

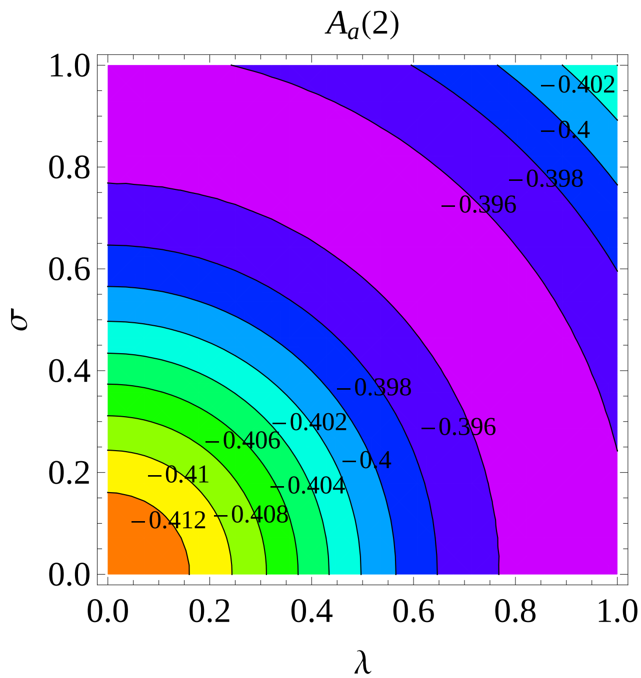

Fig. 3. The factor $A_a(m)$ as a function of λ and σ when $p=q=0, h=k=l=\epsilon=1, m=2$ and $r=4$.

II.2. Two-mode antibunching

According to Lee and An [26,28], the factor for measuring the degree of two-mode higher-order antibunching (TMHOA) of the two mode x and y can be written as

$$A_{x,y}(m)=\frac{\langle\hat{x}^{\dagger(m+1)}\hat{x}^{m+1}\rangle+\langle\hat{y}^{\dagger(m+1)}\hat{y}^{m+1}\rangle}{\langle\hat{x}^{\dagger m}\hat{x}^{m}\hat{y}^{\dagger}\hat{y}\rangle+\langle\hat{y}^{\dagger m}\hat{y}^{m}\hat{x}^{\dagger}\hat{x}\rangle}-1. \tag{16}$$

In a given multimode state, it reveals mth-order two-mode antibunching when $A_{x,y}(m)<0$. This criterion was used to study the TMHOA in several nonclassical states [28,29]. In the STMPATCS, we derive the mth-order antibunching in two modes a and b as

$$A_{a,b}(m)=\frac{D_{m+1,0,0}+D_{0,m+1,0}}{D_{m,1,0}+D_{1,m,0}}-1. \tag{17}$$

Similaly, we obtain the factor $A_{a,c}(m)$ for two modes a and c as

$$A_{a,c}(m)=\frac{D_{m+1,0,0}+D_{0,0,m+1}}{D_{m,0,1}+D_{1,0,m}}-1, \tag{18}$$

For two modes b and c, we have

$$A_{b,c}(m)=\frac{D_{0,m+1,0}+D_{0,0,m+1}}{D_{0,m,1}+D_{0,1,m}}-1, \tag{19}$$

where $D_{i,j,w}$ is also given in Eq. (15).

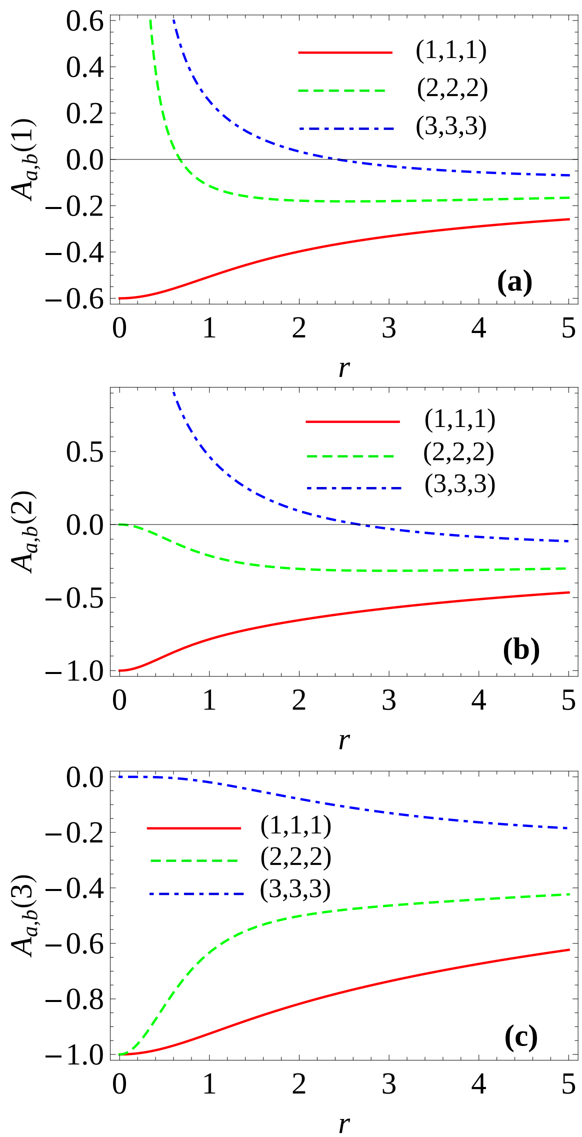

Fig. 4. The factor $A_{a,b}(m)$ as a function of r when $p = q = 0$ and $\epsilon = \lambda = \sigma = 1$ for $h = k = l = 1$ (the red solid line), $h = k = l = 2$ (the green dashed curve), and $h = k = l = 3$ (the blue dot-dashed curve) with in (a) $m = 1$, in (b) $m = 2$, and in (c) $m = 3$.

We use the analytical expressions in Eqs. (17), (18) and (19) to study the TMHOA in the STMPATCS. In Fig. 4, we plot the factor $A_{a,b}(m)$ as a function of r when $p = q = 0$ for several values of h, k, l and m. It is not difficult to recognize that the degree of the TMHOA is enhanced by increasing the order m. It is also true for a large number of added photons, and high r. Similar to the SMHOA, the degree of the TMHOA in the STMPATCS is reduced when increasing the number of added photons.

If we fix the sum of h,k and l but change the number of photons added to each mode, as in Fig. 5, $h+k+l=6, p=q=0, \epsilon=\lambda=\sigma=1$ and $m=2$, the results show that the degree of the TMHOA is the highest when the numbers of photons added to the two modes a and b are the smallest. This can be explained as the effect of adding photons to reduce the degree of antibunching. In the other hand, in the fixed case $p=q=0, h=k=l=\epsilon=1, r=4$ and $m=2$, in Fig. 6, we plot $A_{a,b}(m)$ as a function λ and σ. We can easily see that the antibunching degree is improved in the small $|\lambda|$ and high $|\sigma|$ regions. The above results are also true for the two modes a and c or b and c.

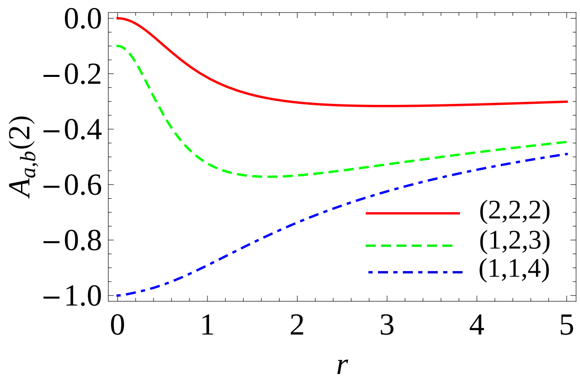

Fig. 5. The factor $A_{a,b}(m)$ as a function of r when $p=q=0, \epsilon=\lambda=\sigma=1$ and $m=2$ for $h=k=l=2$ (the red solid line), $h=1, k=2, l=3$ (the green dashed curve), and $h=k=1, l=4$ (the blue dot-dashed curve).

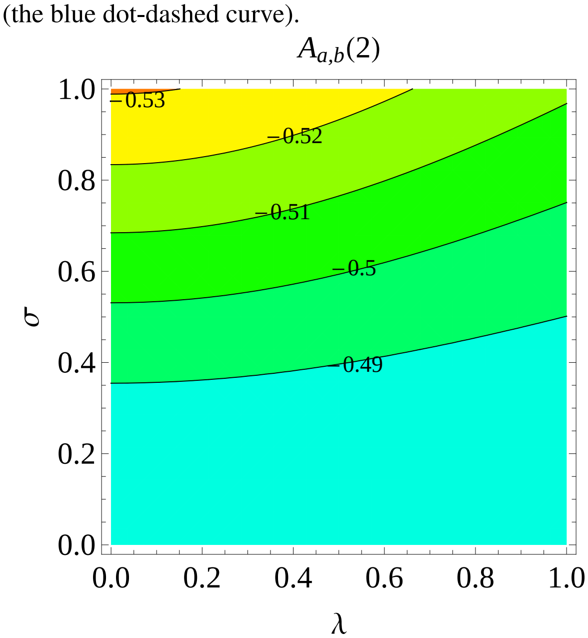

Fig. 6. The factor $A_{a,b}(m)$ as a function of λ and σ when $p=q=0, h=k=l=\epsilon=1, r=4$, and $m=2$.

III. HIGHER-ORDER THREE-MODE SUM SQUEEZING

The condition for first-order three-mode sum squeezing was first introduced by Kumar and Gupta [30]. Using this criterion, several authors investigated the three-mode sum squeezing property in some states [16, 17]. The criterion for higher-order three-mode sum squeezing (HOTMSS) was introduced by An [28]. For this criterion, we consider the two operators in higher-order as follow:

$$\hat{U} = \frac{\hat{a}^{\dagger m}\hat{b}^{\dagger m}\hat{c}^{\dagger m} + \hat{a}^{m}\hat{b}^{m}\hat{c}^{m}}{2}, \tag{20}$$

$$\hat{V} = \frac{i(\hat{a}^{\dagger m}\hat{b}^{\dagger m}\hat{c}^{\dagger m} - \hat{a}^{m}\hat{b}^{m}\hat{c}^{m})}{2}, \tag{21}$$

with m being the nonnegative integer. These two operators obey the commutative relation

$$[\hat{U},\hat{V}] = \frac{i}{2}\hat{W}, \tag{22}$$

in which

$$\hat{W} = \hat{a}^{m}\hat{b}^{m}\hat{c}^{m}\hat{a}^{\dagger m}\hat{b}^{\dagger m}\hat{c}^{\dagger m} - \hat{a}^{\dagger m}\hat{b}^{\dagger m}\hat{c}^{\dagger m}\hat{a}^{m}\hat{b}^{m}\hat{c}^{m}. \tag{23}$$

A state exists the HOTMSS if it is satisfied

$$S_U(m) \equiv \frac{4(\Delta\hat{X})^2 - \langle\hat{W}\rangle}{|\langle\hat{W}\rangle|} < 0, \tag{24}$$

or

$$S_V(m) \equiv \frac{4(\Delta\hat{V})^2 - \langle\hat{W}\rangle}{|\langle\hat{W}\rangle|} < 0, \tag{25}$$

where $(\Delta\hat{f})^2 = \langle\hat{f}^2\rangle - \langle\hat{f}\rangle^2, f = \{U,V\}$. The factors $S_U(m)$ and $S_V(m)$ also manifest the degree of the HOTMSS. The more negative the factor $S_U(m)(S_V(m))$ is, the higher the squeezed degree becomes.

In the STMPATCS, we obtain the explicit expressions of $S_U(m)$ and $S_V(m)$ when $\xi = r$ as follows:

$$S_U(m) = \frac{2C_{2m} - 4C_m^2 + 2D_{m,m,m}}{|B_{m,m,m} - D_{m,m,m}|}, \tag{26}$$

and

$$S_V(m) = \frac{-2C_{2m} + 2D_{m,m,m}}{|B_{m,m,m} - D_{m,m,m}|}, \tag{27}$$

where $B_{i,j,w}, C_u$ and $D_{i,j,w}$ are given by Eqs. (9), (10) and (15). Analytical expressions in Eqs. (26) and (27) allow us to study the HOTMSS in the STMPATCS. In Fig. 7, we graph the factor $S_U(m)$ as a function of r when $p = q = 0$ and $\epsilon = \lambda = \sigma = 1$ with several values of h, k, l and m. From this figure, we can see that the squeezing degree is enhanced by increasing h, k, l as well as r with any order m. However, the negative degree of $S_U(m)$ is reduced if m gets a high value. Note that the trio coherent state does not exist the three-mode sum squeezing. Therefore, the appearance of the HOTMSS in the STMPATCS demonstrates the important role of photon addition.

We set $p = q = 0, \epsilon = \lambda = \sigma = 1$ and $m = 2$, in the case of $h + k + l =$ constant, Fig. 8 plots the dependance of $S_U(m)$ on r but h, k and l are changed. The degree of HOTMSS is the greatest when the numbers of photons added to the two modes are the smallest. In Fig. 9, we also graph the factor $S_U(m)$ as a function of λ and σ with $p = q = 0, h = k = l = \epsilon = 1, r = 4$ and $m = 2$. In this case, the degree of HOTMSS is the highest when both $|\lambda|$ and $|\sigma|$ are lowest.

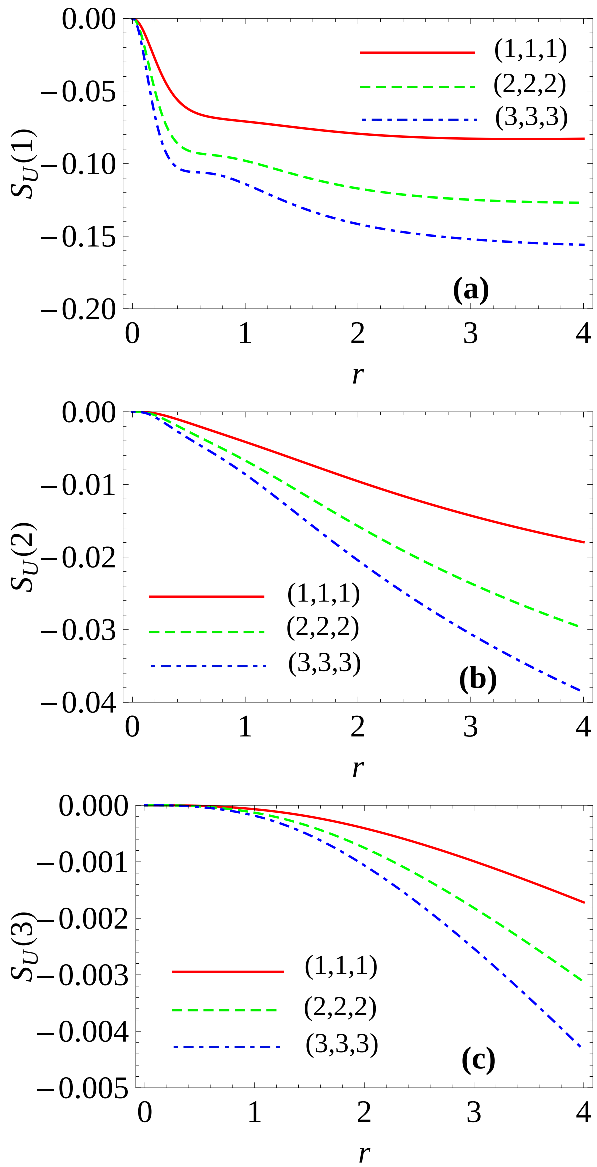

Fig. 7. The factor $S_U(m)$ as a function of r when $p = q = 0$ and $\epsilon = \lambda = \sigma = 1$ for $h = k = l = 1$ (the red solid line), $h = k = l = 2$ (the green dashed curve), and $h = k = l = 3$ (the blue dot-dashed curve) with in (a) $m = 1$, in (b) $m = 2$, and in (c) $m = 3$.

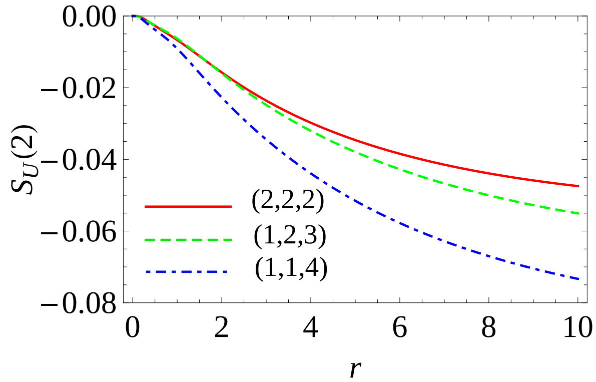

Fig. 8. The factor $S_U(m)$ as a function of r when $p = q = 0, \epsilon = \lambda = \sigma = 1$ and $m = 2$ for $h = k = l = 2$ (the red solid line), $h = 1, k = 2, l = 3$ (the green dashed curve), and $h = k = 1, l = 4$ (the blue dot-dashed curve).

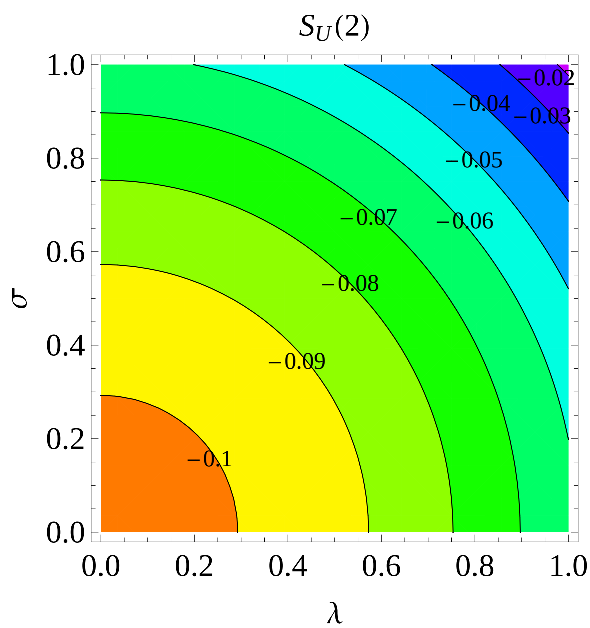

Fig. 9. The factor $S_U(m)$ as a function of λ and σ when $p = q = 0, h = k = l, \epsilon = 1, r = 4$ and $m = 2$.

With respect to $S_V(m)$, in Fig. 10, we plot this factor as a function or r when $p = q = 0$ and $m = 2$ for several values of h, k and l. It is easy to see that the STMPATCS does not reveal the HOTMSS in $\hat{V}$ because the values of $S_V(m)$ are always nonnegative. This is explained because such state already exists the STMPATCS in $\hat{U}$.

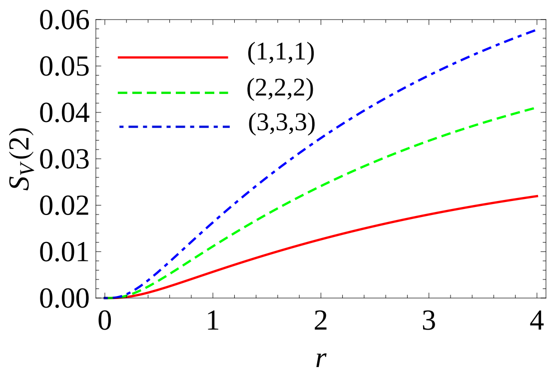

Fig. 10. The factor $S_V(m)$ as a function of r when $p = q = 0$ and $\epsilon = \lambda = \sigma = 1$ for $h = k = l = 1$ (the red solid line), $h = k = l = 2$ (the green dashed curve), and $h = k = l = 3$ (the blue dot-dashed curve).

IV. HIGHER-ORDER THREE-MODE ENTANGLEMENT

There are many criteria for detecting entanglement in multimode states [31–34]. These criteria were also applied to investigate the entangled property in many two-mode and three-mode states [11, 25, 29]. To verify the higher-order entangled characteristic in the STMPATCS, we use the criterion introduced by Duc et al. [11]. It is given in the form of the inequality as follows:

$$|\langle \hat{a}^m \hat{b}^m \hat{c}^m \rangle| > \left(\langle \hat{a}^{\dagger m} \hat{a}^m \rangle \langle \hat{b}^{\dagger m} \hat{b}^m \rangle \langle \hat{c}^{\dagger m} \hat{c}^m \rangle \right)^{1/2}. \tag{28}$$

A state is entangled if it satisfies the inequality in Eq. (28). By setting

$$E(m) = \left(\langle \hat{a}^{\dagger m} \hat{a}^m \rangle \langle \hat{b}^{\dagger m} \hat{b}^m \rangle \langle \hat{c}^{\dagger m} \hat{c}^m \rangle \right)^{1/2} - |\langle \hat{a}^m \hat{b}^m \hat{c}^m \rangle|, \tag{29}$$

we can see that the condition $E(m) < 0$ confirms the entanglement in a given state. The more negative $E(m)$ is, the more entangled the state becomes. Using Eqs. (10) and (15), we write the factor $E(m)$ in the case $\xi = r$ as follows:

$$E(m) = \left(D_{m,0,0} D_{0,m,0} D_{0,0,m} \right)^{1/2} - C_m. \tag{30}$$

The factor $E(m)$ in Eq. (30) allows us to study the entangled property in the STMPATCS. In Fig. 11, we plot the dependance of $E(m)$ on r with $p = q = 0$ and $\epsilon = \lambda = \sigma = 1$ for several values of $(h, k, l$. It indicates that the negative degree of the factor $E(m)$ is increased (decreased) with raising $m(h, k, l)$. In addition, the higher the parameter r becomes, the more negative the factor $E(m)$ is.

When fixed $h + k + l =$constant, as shown in Fig. 12, $h + k + l = 6$, and fixed some parameters as $p = q = 0$ and $\epsilon = \lambda = \sigma = 1$, the results show the value of $E(m)$ is the highest if both h and k are the smallest. Furthermore, we investigate the dependance of the factor $E(m)$ on λ and σ in the case $p = q = 0, h = k = l = \epsilon = 1, r = 4$, and $m = 2$ (see Fig. 13). It is not surprising that the factor $E(m)$ peaks at $\lambda = \sigma = 0$, i.e. adding only photons to one mode of the trio coherent state.

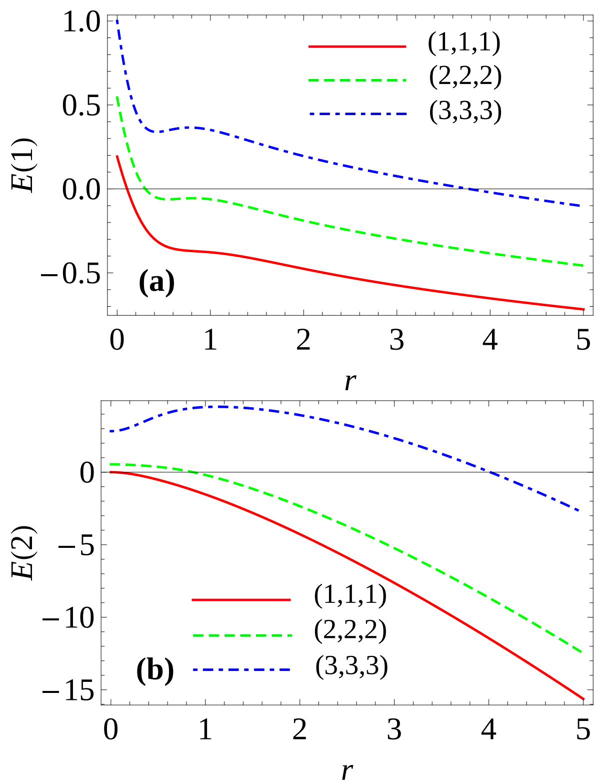

Fig. 11. The factor $E(m)$ as a function of r when $p = q = 0$ and $\epsilon = \lambda = \sigma = 1$ for $h = k = l = 1$ (the red solid line), $h = k = l = 2$ (the green dashed curve), and $h = k = l = 3$ (the blue dot-dashed curve).

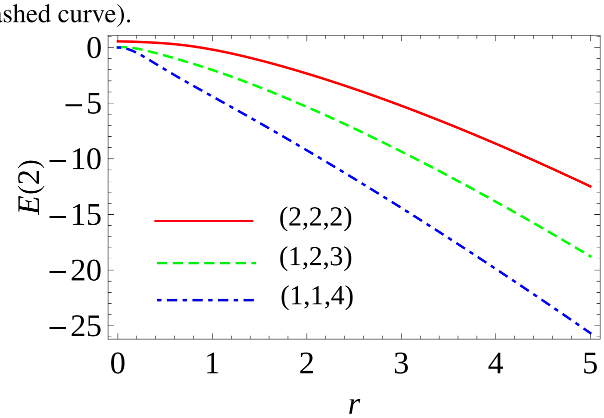

Fig. 12. The factor $E(m)$ as a function of r when $p = q = 0$ and $\epsilon = \lambda = \sigma = 1$ for $h = k = l = 2$ (the red solid line), $h = 1, k = 2, l = 3$ (the green dashed curve), and $h = k = 1, l = 4$ (the blue dot-dashed curve).

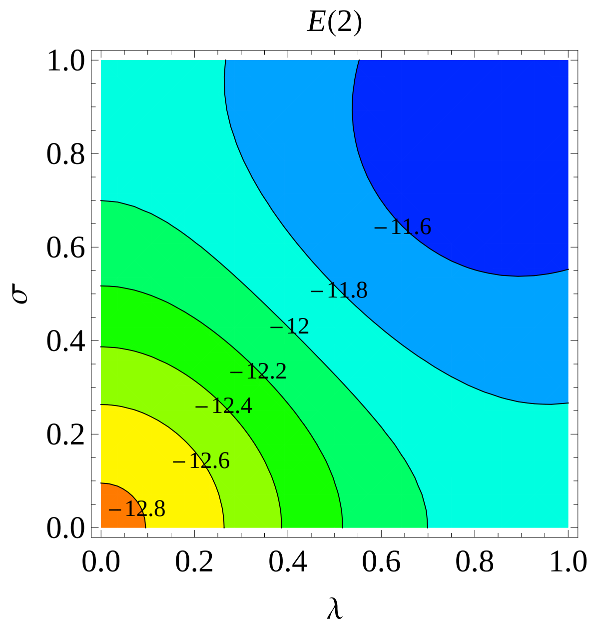

Fig. 13. The factor $E(m)$ as a function of λ and σ when $p=q=0, h=k=l=\epsilon=1, r=4$, and $m=2$.

V. CONCLUSIONS

In this paper, we have studied the higher-order single-mode and two-mode antibunching, higher-order three-mode sum squeezing, and higher-order three-mode entanglement in the STMPATCS. The investigated results have indicated that the nonclassical degree is improved in higher-order. On the other hand, the depth of the factors of higher-order antibunching, higher-order squeezing, and entanglement is enhanced by increasing the parameter r, i.e. increasing the amplitude of the field of the trio coherent state. Besides, the nonlocal photon addition plays an important role in raising the higher-order three-mode sum squeezing in the STMPATCS, while the trio coherent state does not exist the three-mode sum squeezing. The results have also shown that this operation reduces the intensity of nonclassicality including the higher-order single-mode and two-mode antibunching, and the higher-order three-mode entanglement. It also explains why the parameters p, q, h, k, l, ϵ and r are fixed, the nonclassical intensity peaks at $\lambda = \sigma = 0$. Finally, by fixing the total of h, k and l but varying h, k or l, we have found that all the antibunching, squeezing, and entanglement degrees in higher-order are the highest when the number of added photons to the two modes of the trio coherent state is the smallest.

ACKNOWLEDGMENT

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 103.01-2018.361.

REFERENCES

[1] A. Pathak and M. Garcia, *Control of higher order antibunching*, *Appl. Phys. B* **84** (2006) 479.

[2] J. Aasi *et al*, *Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light*, *Nature Photonics* **7** (2013) 613.

[3] S. L. Braunstein and P. V. Loock, *Quantum information with continuous variables*, *Rev. Mod. Phys.* **77** (2005) 513.

[4] B. C. Sanders, *Superposition of two squeezed vacuum states and interference effects*, *Phys. Rev. A* **39** (1989) 4284.

[5] T. Gantsog and R. Tanas, *Phase properties of the two-mode squeezed vacuum states*, *Phys. Lett. A* **152** (1991) 251.

[6] G. S. Agarwal, *Nonclassical statistics of fields in pair coherent states*, *J. Opt. Soc. Am. B* **5** (1988) 1940.

[7] X. M. Liu, *Even and odd charge coherent states and their non-classical properties*, *Phys. Lett. A* **279** (2001) 123.

[8] N. B. An and T. M. Duc, *Trio coherent states*, *J. Opt. B: Quan. Semi. Opt.* **4** (2002) 80.

[9] H. Fan and G. Yu, *Three-mode squeezed vacuum state in Fock space as an entangled state*, *Phys. Rev. A* **65** (2002) 033829.

[10] G. S. Agarwal and A. Biswas, *Quantitative measures of entanglement in pair-coherent states*, *J. Opt. B: Quantum Semiclass. Opt.* **7** (2005) 350.

[11] T. M. Duc, J. Noh and K. Kim, *Entanglement criteria in inequality for pair and trio coherent states*, *Adv. Nat. Sci.* **9** (2008) 123.

[12] T. M. Duc, T. Q. Dat, and H. S. Chuong, *Quantum entanglement and teleportation in superposition of multiple-photon-added two-mode squeezed vacuum state*, *Int. J. Mod. Phys. B* **34** (2020) 2050223.

[13] L. Y. Hu, F. Jia, and Z. M. Zhang, *Entanglement and nonclassicality of photon-added two-mode squeezed thermal state*, *J. Opt. Soc. Am. B* **29** (2012) 1456.

[14] T. M. Truong, H. T. X. Nguyen and A. B. Nguyen, *Sum squeezing, difference squeezing, higher-order antibunching and entanglement of two-mode photon-added displaced squeezed states*, *Int. J. Theor. Phys.* **53** (2013) 899.

[15] D. M. Truong, C. S. Ho, and D. Q. Tran, *Detecting nonclassicality and non-Gaussianity by the Wigner function and quantum teleportation in photon-added-and-subtracted two modes pair coherent state*, *J. Comput. Electron.* (2021).

[16] T. M. Duc and T. Q. Dat, *Enhancing nonclassical and entanglement properties of trio coherent states by photon-addition*, *Optik* **210** (2020) 164479.

[17] T. Q. Dat and T. M. Duc, *Nonclassical properties of the superposition of three-mode photon-added trio coherent state*, *Int. J. Theor. Phys.* **59** (2020) 3206.

[18] Y. Wang, W. S. Bao, H. Z. Bao, C. Zhou, M. S. Jiang and H. W. Li, *High-dimensional quantum key distribution with the entangled single-photon-added coherent state*, *Phys. Lett. A* **381** (2017) 1393.

[19] L. Fan and M. S. Zubairy, *Quantum illumination using non-Gaussian states generated by photon subtraction and photon addition*, *Phys. Rev. A* **98** (2018) 012319.

[20] S. Y. Lee and H. Nha, *Quantum state engineering by a coherent superposition of photon subtraction and addition*, *Phys. Rev. A* **82** (2010) 053812.

[21] A. Zavatta *et al*, *Quantum-to-classical transition with single-photon-added coherent states of light*, *Science* **306** (2004) 660.

[22] A. Karlsson and M. Bourennane, *Quantum teleportation using three-particle entanglement*, *Phys. Rev. A* **58** (1998) 4394.

[23] M. Hillery and V. Buzek, and A. Berthiaume, *Quantum secret sharing*, *Phys. Rev. A* **59** (1999) 1829.

[24] N. B. An and J. Kim, *Joint remote state preparation*, *J. Phys. B: At. Mol. Opt. Phys.* **41** (2008) 095501.

[25] T. Q. Dat and T. M. Duc, *Higher-order nonclassical and entanglement properties in photon-added trio coherent state*, *Hue Uni. J. Sci.: Nat. Sci.* **129** (2020) 49.

[26] C. T. Lee, *Many-photon antibunching in generalized pair coherent states*, *Phys. Rev. A* **41** (1990) 1569.

[27] P. Gupta, P. N. Pandey and A Pathak, *Higher order antibunching is not a rare phenomenon*, *J. Phys. B: At. Mol. Opt. Phys.* **39** (2006) 1137.

[28] N. B. An, *Multimode higher-order antibunching and squeezing in trio coherent states*, *J. Opt. B: Quantum Semiclass. Opt.* **4** (2002) 222.

[29] T. M. Duc, D. H. Dinh and T. Q. Dat, *Higher-order nonclassical properties of nonlinear charge pair cat states*, *J. Phys. B: At. Mol. Opt. Phys.* **53** (2020) 025402.

[30] A. Kumar and P. S. Gupta, *Sum squeezing in four-wave sum frequency generation*, *Opt. Commun.* **136** (1997) 441.

[31] L. M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, *Inseparability Criterion for Continuous Variable Systems*, *Phys. Rev. Lett.* **84** (2000) 2722.

[32] E. Shchukin E and W. Vogel, *Publisher's note: Inseparability criteria for continuous bipartite quantum states [Phys. Rev. Lett. 95, 230502 (2005)]*, *Phys. Rev. Lett.* **95** (2005) 230502.

[33] M. Hillery and M. S. Zubairy, *Entanglement conditions for two-mode states*, *Phys. Rev. Lett.* **96** (2006) 050503.

[34] T. M. Duc and N. T. X. Hoai, *Entanglement criterion for bipartite quantum states: applications*, *Comm. Phys.* **20** (2010) 233.