REVIEW PAPER

NORMED DIVISION ALGEBRAS APPLICATION TO THE MONOPOLE PHYSICS

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Abstract. *We review some normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ applications to the monopole physics and MICZ-Kepler problems. More specifically, we will briefly review some results in applying the normed division algebras to interpret the existence of Dirac, Yang, and $SO(8)$ monopoles. These monopoles also appear during the examination of the duality between isotropic harmonic oscillators and the MICZ-Kepler problems. We also revisit some of our newest results in the nine-dimensional MICZ-Kepler problem using the generalized Hurwitz transformation.*

Keywords: normed division algebras; Hopf maps; Kusstanheimo-Stiefel transformation; Hurwitz transformations; Levi-Civita transformation; harmonic oscillator; MICZ-Kepler problem; duality; Dirac, Yang, and $SO(8)$ monopoles; nine-dimensional space, octonion.

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I. INTRODUCTION

More than one and a half-century of being discovered, the family of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$ known as normed division algebras are no longer abstract mathematical objects. However, they gradually become the popular language for various physics aspects, from the low energy regime of condensed matter physics to the high energy regime of particle physics.

The first appearance of normed division algebra in physics is on the aspect of monopole physics. Dirac proposed a gauge field of the magnetic monopole and led it to the quantization condition between magnetic charge and electric charge in 1931 [1]. In gauge field theory, people also prefer Dirac monopole as the U(1) monopole. Almost the same time, a mathematician named Hopf discovered a map from one three-dimensional unit sphere S^3 onto another two-dimensional unit sphere S^2, at that the 3-sphere becomes the 2-sphere whose each point is a circle S^1 ([2]. He then also generalized it for other cases of dimensions in the next two years [3]. In modern language, Hopf maps (also known as Hopf fibrations/bundles) are the consequence of the normed division algebras in differential topology [4]. Four decades after these works, in 1978, C. N. Yang noticed that the Dirac monopole discovery was an accidental appearance of the Hopf map. Hence, he used the next Hopf map to propose a SU(2) monopole in five-dimensional space, named Yang monopole [5–7]. In 1984, Grossman *et al.* used the last Hopf map belonging to octonions $\mathbb{O}$ to describe an $SO(8)$ monopole field [8]. This gauge field also appeared in the work of Zhang *et al.* in 2001 on the 8D Hall effect [9, 10]. Then, Le *et al.* constructed the SO(8) monopole's explicit form in 2009 using the generalized Hurwitz transformation [11, 12].

In 1983, Laughlin investigated the 2D quantum Hall effect and showed its relation to the first Hopf map $S^1 \hookrightarrow S^3 \longrightarrow S^2$, which relates to the algebra of complex numbers $\mathbb{C}$ [13]. This result is the first step of differential topology application to condensed matter physics. After that, Zhang et al. used the next two generations of Hopf maps ($S^3 \hookrightarrow S^7 \longrightarrow S^4$ and $S^7 \hookrightarrow S^{15} \longrightarrow S^8$) relating to the quaternion and octonion algebras $\mathbb{H}, \mathbb{O}$ to investigate the 4D and 8D quantum Hall effects at the beginning of this century [9, 10].

One can also find the application of normed division algebras in the high-energy regime of particle physics. As well known, the Standard model leaves so many unsolved questions for physicists to understand the nature of our world. In the scientific endeavor of solving these mysteries, two recent publications [14, 15] showed the possibility of exploring and extending the Standard Model by using octonion algebra $\mathbb{O}$.

One of the fascinating problems in mathematical physics is the duality between the two most basic systems: a harmonic oscillator and a hydrogen atom [16]. This duality led to the birth of MICZ-Kepler problems in three-, five-, and nine-dimensional spaces [11, 17–19]. Indeed, the harmonic oscillator is equivalent to the Kepler problem adding the monopole field in three cases of dimension. They are Dirac U(1), Yang SU(2), and SO(8) monopoles corresponding to the three-, five-, and nine-dimensional MICZ-Kepler problems [20]. To establish the connection between the harmonic oscillator and the Kepler-Coulomb problem, scientists adopted the Hurwitz transformations [21–24] directly related to the normed division algebras. Interestingly, only four generations of the oscillator-Coulomb problem duality correspond to four normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

The above observations encourage us to summarize fundamental physics, where applied the normed division algebra and its cousins (Hopf maps and Hurwitz transformations) as the basic mathematical language. We focus on the normed division algebras applications to monopole physics and the MICZ-Kepler problems.

II. MATHEMATICAL BACKGROUND: NORMED DIVISION ALGEBRAS, HOPF MAPS, AND HURWITZ TRANSFORMATIONS

For further discussions, we will review some background knowledge of the normed division algebras, Hopf maps, and Hurwitz transformations and also settle a relationship between these mathematical concepts.

II.1. Octonions as a last normed division algebra

We call $\mathbb{A}$ a division algebra over the real field $\mathbb{R}$ if, for any of its elements a and non-zero element b, there is one element x and one element y in $\mathbb{A}$ such that $a = b \times x$ and $a = y \times b$. Here, the multiplication $\times$ is a bilinear map in the algebra. The division algebra is normed if we can define a norm $||a||$ as a real number, and this norm satisfies the following condition:

$$||a \times b|| = ||a||\,||b||. \tag{1}$$

Interestingly, there exist only four normed division algebras over the real field R by the Hurwitz composition theorem [25]. They are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$. For more details of these algebras, one can see Refs. [4, 26, 27]. Below, we will go quickly through these normed division algebras and focus on the composition law, which is essential for further applications.

Elements of a normed division algebra can be written in the general form

$$x = x_0 e_0 + x_1 e_1 + \ldots + x_{D-1} e_{D-1}, \tag{2}$$

where $x_0, x_1, \ldots, x_{D-1}$ are real numbers; e_0 is a unit element; $e_1, \ldots, e_{D-1}$ are image unit elements. The dimension D equals $1, 2, 4, 8$, corresponding to the real $\mathbb{R}$, complex $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$ algebras. We also define a scalar product of two elements a, b in algebra as a map to a real number $a \cdot b$, where

$$
\begin{aligned}
e_0 \cdot e_0 &= 1, \\
e_j \cdot e_j &= -1, \quad (j = 1, 2, \ldots, D-1), \\
e_j \cdot e_k &= 0, \quad j \neq k \quad (j, k = 0, 1, 2, \ldots, D-1).
\end{aligned} \tag{3}
$$

Thus, the unit elements are perpendicular to each other.

By this definition of the scalar product, we can now define the norm of an element x as

$$||x|| = \sqrt{x \cdot \bar{x}} = \sqrt{x_0^2 + x_1^2 + \ldots + x_{D-1}^2}, \tag{4}$$

where $\bar{x}$ is a conjugate of element x, defined as

$$\bar{x} = x_0 e_0 - x_1 e_1 - \ldots - x_{D-1} e_{D-1}. \tag{5}$$

It is easy to see that the conjugates of the unit elements are

$$\bar{e}_0 = e_0, \quad \bar{e}_j = -e_j \quad (j = 1, 2, \ldots, D-1).$$

From (3), we also have the norms of unit elements as $||e_0|| = ||e_1|| = \ldots = ||e_{D-1}|| = 1$.

We can rewrite equation (4) as

$$||x|| = \sqrt{x_0^2 + x_1^2 + \ldots + x_{D-1}^2}. \tag{6}$$

After applying this definition to any three elements a, b, c where $c = a \times b$, condition (1) now reads as

$$||c|| = ||a \times b|| = ||a|| \, ||b||$$
$$\Longleftrightarrow c_0^2 + c_1^2 + \ldots + c_{D-1}^2 = \left(a_0^2 + a_1^2 + \ldots + a_{D-1}^2\right)\left(b_0^2 + b_1^2 + \ldots + b_{D-1}^2\right). \tag{7}$$

It is the composition law for the normed division algebras. Here, the explicit forms of a, b, c are used as $a = a_0 e_0 + a_1 e_1 + \ldots + a_{D-1} e_{D-1}$, $b = b_0 e_0 + b_1 e_1 + \ldots + b_{D-1} e_{D-1}$, and $c = a_0 e_0 + c_1 e_1 + \ldots + c_{D-1} e_{D-1}$.

In 1898, A. Hurwitz had proved a theorem that the composition law (7) is only valid for four cases of dimensionality $D = 1, 2, 4, 8$, i.e., there are only four normed division algebras whose dimensions are $D = 1$ (real, $\mathbb{R}$), $D = 2$ (complex, $\mathbb{C}$), $D = 3$ (quaternions, $\mathbb{H}$), and $D = 4$ (octonions, $\mathbb{O}$) using Cayley construction [25]. This theorem is known as the Hurwitz theorem of composition algebras.

The multiplication $\times$ needs to define by such a way that satisfies the composition law (7). For this purpose, we first rewrite explicitly the map $c = a \times b$ as follows [27]

$$c_k = \sum_{i=0}^{D-1}\sum_{j=0}^{D-1} C_{ij}^k a_i b_j, \quad k = 0, 1, \ldots D-1, \tag{8}$$

whereas C_{ij}^k is the projection of the multiplication $e_i \times e_j$ on unit vector e_k, i.e., $C_{ij}^k = (e_i \times e_j) \cdot e_k$. The coefficients C_{ij}^k are called structural constants, which define the multiplication $\times$. Replacing c_k in the law (7) by the expression (8), we obtain

$$C_{ij}^k C_{mn}^k = \delta_{im} \delta_{jn}, \tag{9}$$

which is also called the composition law for the structure constants. Here, δ_{jk} is Kronecker delta notation. Note that there is another version of the composition law $||a \times b|| = ||a|| \, ||b||$. It is

$$(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c) \tag{10}$$

which leads to the more general law for structure constants

$$C_{ij}^k C_{mn}^l \delta_{kl} + C_{mj}^k C_{in}^l \delta_{kl} = 2\delta_{im} \delta_{jn}. \tag{11}$$

We provide below specific values for the structure constants of the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$.

(i) For the real numbers $\mathbb{R}$, the structure constant is only one value $C_{00}^0 = 1$.

(ii) For the complex numbers $\mathbb{C}$, there are two unit vectors, $e_0 = 1$ and $e_1 = i$; thus, their structure constants C_{ij}^0, C_{ij}^1 are elements of the two matrices correspondingly

$$C^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(iii) The quaternions $\mathbb{H}$ described first by the Irish mathematician W. R. Hamilton in 1843 contain four unit vectors $e_0 = 1$, $e_1 = i$, $e_2 = j$, and $e_3 = k$. The multiplication rule is as follows

$$\begin{cases} e_0 \times e_1 = e_1 \times e_0 = e_1, \quad e_0 \times e_2 = e_2 \times e_0 = e_2, \quad e_0 \times e_3 = e_3 \times e_0 = e_3, \\ \qquad\qquad e_0 \times e_0 = e_0, \quad e_1 \times e_1 = e_2 \times e_2 = e_3 \times e_3 = -e_0, \\ e_1 \times e_2 = -e_2 \times e_1 = e_3, \quad e_2 \times e_3 = -e_3 \times e_2 = e_1, \quad e_3 \times e_1 = -e_1 \times e_3 = e_2. \end{cases} \quad (12)$$

We can see that, except $e_0 = 1$, the multiplication rule of i, j, k is similar to the cross product of three unit vectors in Cartesian coordinates: $i \times i = j \times j = k \times k = -1$, $i \times j = k$, $j \times k = i$, $k \times i = j$. From the formula $C_{ij}^k = (e_i \times e_j) \cdot e_k$, the structure constants $C_{ij}^0, C_{ij}^1, C_{ij}^2$, and C_{ij}^3 are elements of the four matrices correspondingly

$$C^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad C^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$C^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad C^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(iv) The last normed division algebra, octonions $\mathbb{O}$, was discovered by J. T. Graves in 1843 [4]. Except for $e_0 = 1$, there are seven image unit vectors $e_1, e_2, e_3, \ldots, e_7$ which obey the following multiplication rule

$$\begin{cases} e_0 \times e_i = e_i \times e_0 = e_i \\ \quad e_i \times e_i = -e_0 \\ e_i \times e_j = \sum_{k=1}^{7} \varepsilon_{ijk} e_k \quad i \neq j, \quad (i, j = 1, 2, \ldots, 7). \end{cases} \quad (13)$$

Here, ε_{ijk} is a completely antisymmetric Levi–Civita symbol which equals 1 for (i, j, k) equal to $(1, 2, 3), (5, 1, 6), (6, 2, 4), (4, 3, 5), (1, 7, 4), (3, 7, 6)$, or $(2, 7, 5)$ [27]. The structure constants of octonion are then written as

$$C_{ij}^k = \delta_{0j} \delta_{ik} + \delta_{0i} \delta_{jk} - \delta_{0k} \delta_{ij} + \varepsilon_{ijk}. \quad (14)$$

It should be noted that $u \times \bar{u} = u \cdot \bar{u} = ||u||$ for all $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$. Further, we will omit the multiplication and scalar product notations for simplicity. For example, we use notation ab instead of $a \times b$; $u\bar{u}$ instead of $u \cdot \bar{u}$.

II.2. Hopf maps

Hopf maps (also called Hopf fibrations or Hopf bundles) are important topological objects applied in various aspects of physics [28]. These were named after H. Hopf for his discovery of these bundles. The first Hopf map discovered in 1931 is a map from a unit 3-sphere S^3 to the unit 2-sphere S^2 such that each point of the S^2 is mapped by the distinct circle S^1 of the unit 3-sphere [2]. Mathematically, this Hopf map is denoted as

$$S^1 \hookrightarrow S^3 \to S^2. \quad (15)$$

In differential topology language, the fiber space S^1 is embedded in the sphere S^3 when projecting to the sphere S^2. H. Hopf, in his works [2, 3], showed that there are possible only 4 such these maps

$$S^{n-1} \hookrightarrow S^{2n-1} \to S^n$$

for $n = 1, 2, 4, 8$. The four families of Hopf maps correspond to the four types of normed division algebras introduced in the previous Subsection. To obtain this relation, we first need to construct the map from $S^{2n-1} \to S^n$. We will demonstrate for the case of $n = 2$. The other cases are obtained analogically.

There are some geometric interpretations of the Hopf map. However, we will show below the direct analytical construction of the Hopf map (15). Considering the four-dimensional real space $\mathbb{R}^4$ with coordinates (u_1, u_2, v_1, v_2), we can identify it with the two-dimensional complex space $\mathbb{C}^2$ with coordinates $(u, v) = (u_1 + iu_2, v_1 + iv_2)$. The unit 3-sphere S^3 can be defined as a subset of all (u, v) in $\mathbb{C}^2$ so that

$$||u||^2 + ||v||^2 = 1. \tag{16}$$

Now let us consider a point (u, v) of S^3, i.e., satisfying the condition (16), and map it to the space $\mathbb{R} \times \mathbb{C}$ by the rule

$$\begin{cases} x_3 &= u\bar{u} - v\bar{v}, \\ x &= 2u\bar{v}. \end{cases} \tag{17}$$

Here, x is a complex number while x_3 is a real number because of the identity $u\bar{u} - v\bar{v} = ||u|| - ||v||$. We will prove that if (u, v) belongs to the unit 3-sphere S^3, i.e., satisfies the condition (16), its mapped point (x_3, x) belongs to the unit 2-sphere S^2. Indeed,

$$\begin{aligned} x_n^2 + ||x||^2 &= (u\bar{u} - v\bar{v})^2 + 4||u\bar{v}||^2 \\ &= \left(||u||^2 - ||v||^2\right)^2 + 4||u||^2||v||^2 \\ &= \left(||u||^2 + ||v||^2\right)^2 = 1. \end{aligned} \tag{18}$$

Here, to prove (18), we use the composition law (1) for the complex numbers $\mathbb{C}$: $||u\bar{v}|| = ||u|| \, ||v||$.

More than one point of the S^3 connects to one point (x_3, x) of the S^2 by the map (17). Indeed, it is easy to verify that all the points $(gu, \bar{g}v)$ with the complex number g satisfy the map (17). If we choose g so that $||g|| = 1$, the points $(gu, \bar{g}v)$ belong to S^3, i.e., satisfy the identity (16). Moreover, for any given point (u, v) of the S^3, all the points $(gu, \bar{g}v)$ are the subset of the complex numbers $\mathbb{C}$, and because $||g|| = 1$, these points belong to a cycle S^1. Conversely, we can also prove that any point in S^3 which maps to the point (x_3, x) of S^2 belongs to the cycle S^1, i.e., has the coordinate $(gu, \bar{g}v)$. Indeed, from the equations (16) and (17), we can obtain

$$||u||^2 = \frac{1 + x_3}{2}, \qquad ||v||^2 = \frac{1}{2(1 + x_3)} ||x||$$

that leads to the map from (x_3, x) to the points (u, v) of S^3 as follows

$$u = \sqrt{\frac{1 + x_3}{2}} \, g, \quad v = \sqrt{\frac{1}{2(1 + x_3)}} \bar{x} g, \tag{19}$$

where $||g|| = 1$. It means any points of S^3 projected to S^2 belong to the cycle S^1.

In conclusion, we have provided the concrete analytical construction of the Hopf map $S^1 \hookrightarrow S^3 \to S^2$. For proof, we use the composition law of the complex numbers $\mathbb{C}$. This law is available for the real numbers $\mathbb{R}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$; therefore, the constructions of the other Hopf maps are analogic. Because of only four normed division algebras, there are only four Hopf maps as follows

$$S^0 \hookrightarrow S^1 \to S^1,$$
$$S^1 \hookrightarrow S^3 \to S^2,$$
$$S^3 \hookrightarrow S^7 \to S^4,$$
$$S^7 \hookrightarrow S^{15} \to S^8. \tag{20}$$

For more details about the proof of Hopf maps, one can see Refs. [10, 28, 29] .

II.3. Hurwitz transformations

Hurwitz transformations is a class of bilinear transformations [11, 22–24, 30] connecting two Euclidean spaces, $\mathbb{R}^{n+1}$ and $\mathbb{R}^{2n}$, that satisfy the Euler identity

$$\left(u_1^2 + u_2^2 + \ldots + u_n^2 + v_1^2 + v_2^2 + \ldots + v_n^2\right)^2 = x_1^2 + x_2^2 + \ldots + x_n^2 + x_{n+1}^2. \tag{21}$$

This equation can be rewritten as $r = \rho^2$, a relation between the radial distances ρ and r of the points $(u_1, u_2, \ldots u_n, v_1, v_2, \ldots v_n) \in \mathbb{R}^{2n}$ and $(x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$, respectively. Here, we use the notations:

$$\rho = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2 + v_1^2 + v_2^2 + \ldots + v_n^2}, \tag{22}$$

$$r = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2 + x_{n+1}^2}. \tag{23}$$

These transformations have a long history of discovery and highly relate to the fundamental problems in physics. Like the normed division algebras and Hopf maps, only four Hurwitz transformations correspond to $n = 1, 2, 4, 8$.

Although these transformations were discovered independently later, they were similar to the maps given by Hurwitz while proving the composition theorem. For this reason, in several references, these were named the Hurwitz transformations [11, 22–24, 30]. The first $(2-2)$ Hurwitz transformation was constructed by Levi-Civita in 1904 [21]. It is also the only solution of Pythagorean triangles problem $x^2 + y^2 = z^2$ in positive integer variables. In 1965, Kustaanheimo and Stiefel built the $(3-4)$ Hurwitz transformation to examine the three-dimensional classical Kepler problem [22], known as the Kustaahnheimo-Stiefel transformation. The $(5-8)$ Hurwitz transformation was constructed first in 1986 by Kibler et al [23] and independently used and examined in some other works of Davtyan et $al.$ and Van-Hoang Le et $al.$ [30,31]. In 1988, Kibler and Lambert had found the relation between the Hurwitz transformation construction and the Cayley – Dickson algebras, whose first four cases are the real numbers $\mathbb{R}$, complex numbers, $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$. Kibler et $al.$ also mentioned about generalized Hurwitz problem in the $(9-16)$ case [32]. However, the explicit form of the $(9-16)$ Hurwitz transformation was constructed in 1993 by Van-Hoang Le and Komarov utilizing a simple graphic method. They called it the generalized Hurwitz transformation [24]. Historically, higher dimensional Hurwitz transformation appeared in mathematical square identities: Brahmagupta–Fibonacci identity, Euler's

four-square identity (1748, letter to Goldbach), and Degen's eight-square identity (1818). Therefore, the Hurwitz transformations naturally relate to the normed division algebras because both originally arise from problems of quadratic forms [26].

Once we know about the normed division algebras and Hopf maps, we may easily construct the Hurwitz transformations. Indeed, the Hopf map (17) from S^3 to S^2 can be used for the map from the $\mathbb{C} \times \mathbb{C}$ space with coordinates (u,v) onto the $\mathbb{C} \times \mathbb{R}$ space with coordinates (x,x_3) as

$$
\begin{cases}
x_3 & = u\bar{u} - v\bar{v}, \\
x & = 2u\bar{v}.
\end{cases}
\tag{24}
$$

We can verify that this transformation leads to

$$
x_3^2 + ||x||^2 = \left(||u||^2 + ||v||^2 \right)^2,
\tag{25}
$$

i.e., the Euler identity for the three-dimensional real space (x_2, x_2, x_3) and four-dimensional real space (u_1, u_2, v_1, v_2) as

$$
x_1^2 + x_2^2 + x_3^2 = \left(u_1^2 + u_2^2 + v_1^2 + v_2^2 \right)^2.
\tag{26}
$$

Considering the complex numbers $x = x_1 + ix_2$, and $u = u_1 + iu_2$, $v = v_1 + iv_2$, we rewrite equation (24) more explicitly as

$$
\begin{aligned}
x_1 &= 2u_1 v_1 + 2u_2 v_2, \\
x_2 &= -2u_1 v_2 + 2u_2 v_1, \\
x_3 &= u_1^2 + u_2^2 - v_1^2 - v_2^2.
\end{aligned}
\tag{27}
$$

This is the so-called Kustaanheimo-Stiefel transformation [22].

Analogically, the transformation (24) can be considered for mapping from the $\mathbb{R} \times \mathbb{R}$ space with coordinates (u,v) onto the $\mathbb{R} \times \mathbb{R}$ space with coordinates (x_1, x_2). As a result, we have the Levi-Civita transformation [21] as

$$
x_1 = 2uv, \quad x_2 = u^2 - v^2,
\tag{28}
$$

which also satisfies the Euler identity $x_1^2 + x_2^2 = \left(u^2 + v^2 \right)^2$. This transformation is also equivalent to the zeroth Hopf map $S^0 \hookrightarrow S^1 \to S^1$.

For the map from $\mathbb{H} \times \mathbb{H}$ space with coordinates (u,v) onto the $\mathbb{H} \times \mathbb{R}$ space with coordinates (x,x_5), the transformation (24) can be rewritten as $x_5 = u\bar{u} - v\bar{v}$, $x = 2u\bar{v}$. Plugging the explicit form of quaternion coordiantes x and u,v into this equation leads to the Kibler transformation [23]:

$$
\begin{aligned}
x_1 &= 2u_1 v_1 + 2u_2 v_2 + 2u_3 v_3 + 2u_4 v_4, \\
x_2 &= -2u_1 v_2 + 2u_2 v_1 - 2u_3 v_4 + 2u_4 v_3, \\
x_3 &= -2u_1 v_3 + 2u_2 v_4 + 2u_3 v_1 - 2u_4 v_2, \\
x_4 &= -2u_1 v_4 - 2u_2 v_3 + 2u_3 v_2 + 2u_4 v_1, \\
x_5 &= u_1^2 + u_2^2 + u_3^2 + u_4^2 - v_1^2 - v_2^2 - v_3^2 - v_4^2.
\end{aligned}
\tag{29}
$$

Here, to obtain (29), we use the coordinates $x = x_1 + x_2 i + x_3 j + x_4 k$ and $u = u_1 + u_2 i + u_3 j + u_4 k$, $v = v_1 + v_2 i + v_3 j + v_4 k$ and the multiplication rules (12) for quaternions. We can verify the Euler identity for this case as

$$
x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = \left(u_1^2 + u_2^2 + u_3^2 + u_4^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2 \right)^2.
$$

The last $(9-16)$ Hurwitz transformation can be constructed using the transformation (24) mapping the $\mathbb{O} \times \mathbb{O}$ space with coordinates (u,v) onto the $\mathbb{O} \times \mathbb{R}$ space with coordinates (x,x_9). Plugging the explicit octonion coordinates $x = x_1 + x_2 e_1 + x_3 e_2 + \ldots + x_8 e_7$, $u = u_1 + u_2 e_1 + u_3 e_2 + \ldots + u_8 e_7$, and $v = v_1 + v_2 e_1 + v_3 e_2 + \ldots + v_8 e_7$ into equation $x_9 = u\bar{u} - v\bar{v}$, $x = 2u\bar{v}$ and then using the multiplication rule (13) for octonions, we obtain

$$
\begin{aligned}
x_1 &= 2u_1 v_1 + 2u_2 v_2 + 2u_3 v_3 + 2u_4 v_4 + 2u_5 v_5 + 2u_6 v_6 + 2u_7 v_7 + 2u_8 v_8, \\
x_2 &= -2u_1 v_2 + 2u_2 v_1 - 2u_3 v_4 + 2u_4 v_3 + 2u_5 v_8 - 2u_8 v_5 + 2u_6 v_7 - 2u_7 v_6, \\
x_3 &= -2u_1 v_3 + 2u_3 v_1 + 2u_2 v_4 - 2u_4 v_2 - 2u_5 v_7 + 2u_7 v_5 + 2u_6 v_8 - 2u_8 v_6, \\
x_4 &= -2u_1 v_4 + 2u_4 v_1 - 2u_2 v_3 + 2u_3 v_2 + 2u_5 v_6 - 2u_6 v_5 + 2u_7 v_8 - 2u_8 v_7, \\
x_5 &= -2u_1 v_5 + 2u_5 v_1 - 2u_2 v_8 + 2u_8 v_2 + 2u_3 v_7 - 2u_7 v_3 - 2u_4 v_6 + 2u_6 v_4, \\
x_6 &= -2u_1 v_6 + 2u_6 v_1 - 2u_2 v_7 + 2u_7 v_2 - 2u_3 v_8 + 2u_8 v_3 + 2u_4 v_5 - 2u_5 v_4, \\
x_7 &= -2u_1 v_7 + 2u_7 v_1 - 2u_3 v_5 + 2u_5 v_3 + 2u_2 v_6 - 2u_6 v_2 - 2u_4 v_8 + 2u_8 v_4, \\
x_8 &= -2u_1 v_8 + 2u_8 v_1 + 2u_2 v_5 - 2u_5 v_2 + 2u_3 v_6 - 2u_6 v_3 + 2u_4 v_7 - 2u_7 v_4, \\
x_9 &= u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2 + u_8^2 - v_1^2 - v_2^2 - v_3^2 - v_4^2 - v_5^2 - v_6^2 - v_7^2 - v_8^2.
\end{aligned}
\tag{30}
$$

The transformation (30), satisfying the Euler identity

$$
x_1^2 + x_2^2 + \ldots + x_9^2 = \left(u_1^2 + u_2^2 + \ldots + u_8^2 + v_1^2 + v_2^2 + \ldots + v_8^2 \right)^2,
$$

was constructed in 1993 by Van-Hoang Le and Komarov [24] and named the generalized Hurwitz transformation.

Because of only four normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$, we can construct only four Hurwitz transformations corresponding to $(2-2)$, $(3-4)$, $(5-8)$, $(9-16)$. Besides, we can write the four Hurwitz transformations in one general form. Let us consider two spaces, $\mathbb{R}^{2n}$ and $\mathbb{R}^{n+1}$, with the coordinates $(u_1, u_2, \ldots u_n, v_1, v_2, \ldots v_n)$ and $(x_1, x_2, \ldots, x_n, x_{n+1})$. Scaling these spaces by the transforms

$$
U_j = \frac{1}{\rho} u_j, \quad V_j = \frac{1}{\rho} v_j, \quad X_j = \frac{1}{r} v_j,
$$

leads to two unit-spheres $(U_1, U_2, \ldots U_n, V_1, V_2, \ldots V_n) \in S^{2n-1}$ and $(X_1, X_2, \ldots, X_{n+1}) \in S^n$. Following the Hopf maps construction $S^{n-1} \hookrightarrow S^{2n-1} \longrightarrow S^n$, we can represent the Hurwitz transformations as

$$
\begin{cases}
x_k &= 2 \sum_{1 \le i, j \le n} \Gamma_{ij}^k u_i v_j, \quad k = 1, 2, \ldots, n, \\
x_n &= u_1^2 + u_1^2 + \ldots + u_n^2 - v_1^2 - v_2^2 - \ldots - v_n^2,
\end{cases}
\tag{31}
$$

with the tensor Γ_{ij}^k determined via the structural constants as

$$
\Gamma_{ij}^k = \sum_{0 \le m \le n-1} C_{i-1,m}^{k-1} C_{m,j-1}^0.
\tag{32}
$$

In the pioneer works on Hurwitz transformations of Levi-Civita [21], Kustaanheimo and Stiefel [22], Kibler *et al.* [23], and Van-Hoang Le and Komarov [24], the tensors Γ_{ij}^k were constructed independently to the structure constants of the normed division algebras. It is not a big deal since there are various solutions for Γ_{ij}^k to obey the Euler identity (21). These solutions are just permutation or changing the sign of non-zero elements from Γ_{ij}^k.

As the above observations, even though coming from different aspects of mathematics, the normed division algebras, Hopf maps, and Hurwitz transformations are closely related. Table 1 illustrates the relationship between these mathematical objects for four cases $n = 1, 2, 4, 8$.

Table 1. Four generations of the normed division algebras, Hopf maps and Hurwitz transformations.

Normed division algebras	Hopf maps	Hurwitz transformations
Real numbers $\mathbb{R}$	$S^0 \hookrightarrow S^1 \longrightarrow S^1$	Levi-Civita transformation [21]
Complex numbers $\mathbb{C}$	$S^1 \hookrightarrow S^3 \longrightarrow S^2$	Kustanheimo-Steifel transformation [22]
Quaternions $\mathbb{H}$	$S^3 \hookrightarrow S^7 \longrightarrow S^4$	Hurwitz transformation [23]
Octonions $\mathbb{O}$	$S^7 \hookrightarrow S^{15} \longrightarrow S^8$	Generalized Hurwitz transformation [30]

III. The normed division algebras and Dirac, Yang and $SO(8)$ monopoles

The story of applying normed division algebras or Hopf maps on physics was started in 1931 by P. A. M. Dirac. It is well-known to the physics community about the proposal of Dirac on magnetic monopole and Dirac quantization condition between electric charge and magnetic charge of a particle [1]. The work of Dirac on constructing the $U(1)$ gauge field for Dirac monopole may be the discovery of fiber even before the works of H. Hopf and other mathematicians [7]. Both works of Dirac and Hopf were done in 1931.

Here, we outline some basic knowledge about the connection between the Hopf map $S^1 \hookrightarrow S^3 \longrightarrow S^2$ and the $U(1)$ Dirac monopole. For more details, one can see Refs. [5, 6, 8, 33, 34]. Dirac monopole with magnetic charge g placed on the origin of $\mathbb{R}^3$ Euclidean space induces the following magnetic field

$$\vec{B} = \frac{\mu_0 g}{4\pi} \frac{\vec{r}}{r^3}, \tag{33}$$

corresponding to the vector potential

$$\vec{A} = \frac{\mu_0 g}{4\pi} \frac{1}{r(r+x_3)} \left[-x_2 \vec{i} + x_1 \vec{j} \right], \tag{34}$$

where $\vec{r} = x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k}$. Notably, the vector potential has a pole at $x_3 = -r$; however, the Yang overlap technique can avoid this singularity [5, 6]:

$$\vec{A}_\pm = \frac{\mu_0 g}{4\pi} \frac{1}{r(x_3 \pm r)} \left(-x_2 \vec{i} + x_1 \vec{j} \right), \quad + \text{ when } x_3 > 0. \tag{35}$$

Trivially, the vector potential is not uniquely defined under the gauge transformation

$$\vec{A} \rightarrow \vec{A}' = \vec{A} - \vec{\nabla}\chi. \tag{36}$$

Then integrating the magnetic field strength over the cross-section at the equator $x_3 = 0$ of the sphere of radius r, we obtain

$$g = \frac{1}{\mu_0} B r^2 \Omega = \int_{x_2=0^+} \vec{B}_+ \cdot d\vec{S} + \int_{x_2=0^-} \vec{B}_- \cdot d\vec{S} = \oint_{x_2=0} \left[\vec{A}_+ - \vec{A}_- \right] \cdot d\vec{l} = \oint \vec{\nabla}\chi \cdot d\vec{l}. \tag{37}$$

When an electric charge q moves under the presence of the Dirac monopole, its momentum is influenced by the equation

$$\vec{p} \to \vec{p} - q\vec{A}. \tag{38}$$

Under the $U(1)$ gauge transformation of the wavefunction

$$\psi \to \psi' = \exp\left(\frac{\iota q \chi}{\hbar}\right)\psi, \tag{39}$$

the momentum becomes $\vec{p} - \vec{A}'$, where the vector potential changes via the gauge transformation (35) as $\vec{A}' = \vec{A} - \vec{\nabla}\chi$. The invariance of observation under this gauge transformation suggests that

$$\frac{q}{\hbar}\oint_C \vec{\nabla}\chi \cdot d\vec{l} = n2\pi, \quad n \in \mathbb{Z}. \tag{40}$$

Comparing (37) and (40), we obtain the Dirac quantization rule

$$gq = nh, \quad n \in \mathbb{Z} \tag{41}$$

with the Planck constant h.

Now let us consider the first Hopf map $S^1 \hookrightarrow S^3 \longrightarrow S^2$. As mention in the previous Subsection, each point $(X_1, X_2, X_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ in the unit sphere S^2 corresponds to a set of points in the unit sphere S^3 as

$$\begin{cases} U_1 + \iota U_2 &= \cos(\theta/2)\,e^{\iota\alpha} \\ V_1 + \iota V_2 &= \sin(\theta/2)\,e^{\iota(-\phi+\alpha)} \end{cases}. \tag{42}$$

All the points with the same θ and ϕ but different α form a unit circle S^1 on the unit sphere S^3. They have the same projection on the unit sphere S^2. Therefore, the geodesic line element on the unit-sphere S^3 now is

$$dl_{S^3}^2 = d\theta^2 + \sin^2\theta d\phi^2 + \left[d\alpha - \sin^2\left(\frac{\theta}{2}\right)d\phi\right]^2 = ds_{S^2}^2 + \left[d\alpha - \sin^2\left(\frac{\theta}{2}\right)d\phi\right]^2. \tag{43}$$

The appearance of the last term in the above equation shows the connection between the unit sphere S^3 and the basic space, the unit sphere S^2, via the fiber circle S^1. Note that the circle S^1 has a nontrivial fundamental group $U(1)$. Invariance of the form $\left[d\alpha - \sin^2\left(\frac{\theta}{2}\right)d\phi\right]$ under the $U(1)$ gauge transformation of S^3 leads to the gauge potential $A_\phi = C\sin^{-2}\left(\frac{\theta}{2}\right) = C(1+\cos\theta)/\sin\theta$, identical to the Dirac monopole gauge field (36). It is amazing because the Dirac monopole may come naturally from the differential topology. For more details about this discussion, see Refs. [33, 34] and references therein.

In 1975, Wu and Yang utilized the second Hopf map $S^3 \hookrightarrow S^7 \longrightarrow S^4$ to generalize Dirac monopole in five-dimensional space [5, 6]. Since the bundle S^3 is isomorphic to the Lie group $SU(2)$, the Yang monopole is a $SU(2)$ monopole. A decade later, Grossman *et al.* used the last Hopf map $S^7 \hookrightarrow S^{15} \longrightarrow S^8$ to find the eight-dimensional Yang-Mills equation solutions. The gauge field and the $SO(8)$ structure of octonion monopole was also obtained [8]. The $SO(8)$ monopole was also discovered again in the 8*D* quantum Hall effect by Bernevig *et al.* in 2003 [10] and in the 9*D* Kepler problem by Van-Hoang Le *et al.* in 2009 [11, 12]. Table 2 below shows the relationship between the normed division algebras, Hopf maps, Lie groups, and magnetic monopoles.

Table 2. The relationship between the normed division algebras, Hopf maps, Lie groups, and magnetic monopoles.

Algebras	Hopf maps	Lie groups	Mangetic monopoles
Real numbers $\mathbb{R}$	$S^0 \hookrightarrow S^1 \longrightarrow S^1$	$\mathbb{Z}_2$	–
Complex numbers $\mathbb{C}$	$S^1 \hookrightarrow S^3 \longrightarrow S^2$	$U(1)$	Dirac monopole [1]
Quaternions $\mathbb{H}$	$S^3 \hookrightarrow S^7 \longrightarrow S^4$	$SU(2)$	Yang monopole [5,6]
Octonions $\mathbb{O}$	$S^7 \hookrightarrow S^{15} \longrightarrow S^8$	$SO(8)$	$SO(8)$ monopole [8, 10–12]

IV. Duality between isotropic harmonic oscillator and Kepler Coulomb problems

Isotropic harmonic oscillator (HO) and Kepler Coulomb (KC) problem are two of the most fundamental physics problems, from macroscopic to microscopic scale. The former was widely used to describe the macroscopic motion of a pendulum to the microscopic motion of a free electron in a solid lattice or even the motion of nucleon in spherical nuclei; while the latter corresponds to the motion of planets in the solar system as well as an electron in a hydrogen atom. When establishing the basic equation for quantum mechanics, Schrödinger immediately examined these two problems. Furthermore, both HO and KC problems are in a few quantum-mechanical problems whose wavefunctions are exactly expressed by the classical polynomial. Amazingly, these two problems have a dual connection between each other. To the best of our knowledge, the work [16] by Bergmann and Frishman in 1965 was the first study noticing the connection between these two problems' wavefunctions.

The Schrödinger equation of N-dimensional isotropic harmonic oscillator (ND-HO) is given in dimensionless units ($\hbar = m = e = 1$) as follows

$$\left\{ -\frac{1}{2} \triangle_N + \frac{1}{2} \omega^2 \rho^2 \right\} \Psi = \mathscr{E} \Psi, \tag{44}$$

where $\triangle_N$ is the N-dimensional Laplace operator, and ρ is the radius from the origin in $\mathbb{R}^N$ space. Thus, the radius part of ND-HO wavefunctions $\mathscr{R}(\rho)$ is governed by the equation

$$\left\{ -\frac{1}{2} \frac{d^2}{d\rho^2} - \frac{N-1}{2\rho} \frac{d}{d\rho} + \frac{L(L+N-2)}{2\rho^2} + \frac{1}{2} \omega^2 \rho^2 \right\} \mathscr{R}(\rho) = \mathscr{E} \mathscr{R}(\rho), \tag{45}$$

where the solutions are written in the associated Laguerre polynomial [35, 36] as

$$\mathscr{R}_{m,L}(\rho) \sim \rho^\ell e^{-\omega\rho^2/2} \mathscr{L}_m^{L+N/2-1}\left(\omega\rho^2\right), \quad \mathscr{E}_{m,L} = (2m+L+N/2)\,\omega. \tag{46}$$

The Schrödinger equation of the n-dimensional Kepler Coulomb (nD-KC) problem is given in dimensionless units:

$$\left\{ -\frac{1}{2} \triangle_n - \frac{Z}{r} \right\} \Psi = E \Psi, \tag{47}$$

where r is the radius from the origin in $\mathbb{R}^n$ space. Thus, the radius part of nD-KC wavefunction $R(r)$ is governed by the equation [16]:

$$\left\{ -\frac{1}{2}\frac{d^2}{dr^2} - \frac{n-1}{2r}\frac{d}{dr} + \frac{\ell(\ell+n-2)}{2r^2} - \frac{Z}{r} \right\} R(r) = ER(r), \tag{48}$$

where the solution is written in the associated Laguerre polynomial [35, 36] as

$$\begin{cases} R_{m,\ell}(r) \sim r^L \exp\left(\frac{-2Zr}{m+\ell+(n-1)/2}\right) \mathscr{L}_m^{2\ell+n-2}\left(\frac{4Zr}{m+\ell+(n-1)/2}\right), \\ E_{m,\ell} = -\frac{Z^2}{m+\ell+(n-1)/2}. \end{cases} \tag{49}$$

Comparing the wavefunctions of these two problems (46) and (49), one may easily pay attention to the associated Laguerre polynomial and then find out that the ND-HO and nD-KC problems are related to each other. Bergmann and Frishman [16] first observed this duality connection between two problems. They found that if we put

$$r = \rho^2, \quad E = -\frac{\omega}{8}, \quad Z = \frac{\mathscr{E}}{4}, \quad n = \frac{N+2}{2}, \quad \ell = \frac{L}{2}, \tag{50}$$

the radial part of Schrödinger equation (45) of the ND-HO problem is identical to the one (48) of the nD-KC problem. Hence, there should be a dual map between the HO problem in the N-space and the KC problem in the n-space, where $N = 2, 4, 6, \ldots$ and $n = (N+2)/2 = 2, 3, 4, \ldots$. For this duality, the question now is about the angular parts of the wavefunctions?

Looking back at the dual transform (50) connecting ND-HO and nD-KC problems, the variable transformation $r = \rho^2$ is actually the Euler identity (21) mentioned before. Thus, one possible solution for the angular part of wavefunctions should be from the Hurwitz transformation (31). As this transformation connects space $\mathbb{R}^n$ to space $\mathbb{R}^{2n-2}$, the dimensionality of the latter space $N = 2n - 2$ agrees with the dual transform between n and N in (50). Therefore, the Hurwitz transformations should be good candidates to connect ND-HO and nD-KC problems. According to the four generations of the Hurwitz transformations, or precisely the four generations of the normed division algebras, the duality between ND-HO and nD-KC problems should occur in four cases of (N, n): $(2, 2), (4, 3), (8, 5)$ and $(16, 9)$.

Except for the trivial case $(2, 2)$, the other three cases exhibit the inequality of the dimensionality between HO and KC problems. This circumstance suggests that the dual mapping from the ND-HO problem to the nD-KC problem must contain $(N - n)$ extra variables. To understand the influences of the extra variables on the duality, we should look for the inverse map of the Hurwitz transformation (31). In the language of the Hopf map $S^{n-2} \hookrightarrow S^{2n-3} \to S^{n-1}$, the extra variables belong to bundle S^{n-2}. The action of extra variables on the nD-KC problems can be interpreted as the monopole arising from bundle S^{n-2} and, consequently, the presence of the magnetic monopole in the KC problem. It means that besides the scalar potential $-Z/r$, there is a gauge field from the monopole corresponding to the Lie group of the bundle S^{n-2}, e.g., $U(1)$, $SU(2)$, or $SO(8)$ gauge fields for $n = 3, 5, 9$. We will revisit this interpretation in the next Section on the MICZ-Kepler problems.

Before ending this section, we note another approach for the HO-KC duality, called two-time physics; see [20] and references therein for more details. The basic idea of two-time physics is investigating the hidden dynamical symmetry of the problem and utilizing the corresponding two-time spacetime of this dynamical symmetry. Particularly, the hidden dynamical symmetries

for the *ND*-HO and *nD*-KC problems are $Sp(2N, \mathbb{R})$ and $SO(n+1, 2)$ [20]. Apart from four cases of the Hurwitz transformation (the normed division algebras), there are some other cases of (N, n) such as $(4, 1)$ or $(6, 4)$. However, the gauge fields arising from the extra variables in these cases are not fundamental. For example, when $(N, n) = (6, 4)$, the gauge field is not pure fundamental $U(1) \times U(1)$ that why the case of $(6, 4)$ can split into two maps $(3, 2)$ with one constraint [20]. Henceforth, we only focus on four cases where the gauge fields are fundamental and correspond to the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

V. MICZ-KEPLER PROBLEMS

In the nonrelativistic quantum mechanics regime, Zwanziger [17] and McIntosh and Cisneros [18] had independently investigated how a Dirac magnetic monopole [1] influencing the Kepler problem, consequently, first established the so-called McIntosh-Cisneros-Zwanziger Kepler (Coulomb) problem, named after their works in 1969-1970. The McIntosh-Cisneros-Zwanziger Kepler problem is often called the MICZ-Kepler problem (MICZ KP) for short.

Following the Kepler problem generalization to multidimensional spacetime [37–40], the MICZ-Kepler problem has also been generalized from the $(3 + 1)$-spacetime to the $(n + 1)$-spacetime by Meng [41–44]. Among these *nD* MICZ-Kepler problems, only three-, five-, and nine-dimensional MICZ-Kepler problems are of our interest because of their duality with four-, eight-, and sixteen-dimensional isotropic harmonic oscillators via the Kustaanheimo-Stiefel [22, 45], Hurwitz [23, 30, 31], and generalized Hurwitz transformations [11, 12, 24]. In the latter two MICZ-Kepler problems, *5D* and *9D*, the electron interacts with a dyon via the gauge fields belonging to the $SU(2)$ Yang monopole [6] and $SO(8)$ monopole [8, 46] instead of the $U(1)$ Dirac monopole [1] in the original three-dimensional MICZ-Kepler problem. The general Schrödinger equation for the *n*-dimensional MICZ-Kepler problem is as follows:

$$\left\{ \frac{1}{2} \left(-\iota \frac{\partial}{\partial x_j} + A_j^a(\mathbf{r}) \hat{I}_a(\phi_s) \right) \left(-\iota \frac{\partial}{\partial x_j} + A_j^a(\mathbf{r}) \hat{I}_a(\phi_s) \right) - \frac{1}{2} \frac{\partial^2}{\partial x_n^2} \right.$$
$$\left. + \frac{\hat{I}_a(\phi_s) \hat{I}_a(\phi_s)}{2r^2} - \frac{Z}{r} \right\} \Psi(\mathbf{r}, \phi_s) = E\Psi(\mathbf{r}, \phi_s), \tag{51}$$

where the Einstein summation is used for convenience. From here now, the Latin indices $i, j, k, \ldots$ run from 1 to $n - 1$ while the Greek indices run from 1 to n. $(n - 2)$ angles ϕ_s ($s = 1, 2, \ldots (n - 2)$) describe the internal space belonging to the monopole actions while $(n - 2)$ differential operators $I_a(\phi_s)$ are the monopole generators.

These generators $I_a(\phi_s)$ have a similar form to the angular momentum operators and act on the extra variables ϕ_s only. $A_j^a(\mathbf{r})$ are components of the vector potential belonging to the monopole. The explicit expressions of $A_j^a(\mathbf{r})$ and $I_a(\phi_s)$ for $n = 3, 5$ and 9 can be found in Refs. [11,17,19]. Notably, in the last MICZ-KP, the nine-dimensional one, seven generators $I_a(\phi_s)$ found for the first time in Ref. [11] do not form a closed algebra. Thus, actions $A_j^a(\mathbf{r}) I_a(\phi_s)$ have been rewritten as $A_j^{ab}(\mathbf{r}) Q_{ab}(\phi_s)$ in Ref. [12], where 56 generators $Q_{ab}(\phi_s)$ form an $SO(8)$ algebra. As a consequence, the term $\hat{I}_a(\phi_s) \hat{I}_a(\phi_s)/(2r^2)$ in the 9*D*-MICZ KP becomes $\hat{Q}_{ab}(\phi_s) \hat{Q}_{ab}(\phi_s)/(8r^2)$ [12]. As mentioned in the previous Section, we will revisit the duality between the *ND*-HO and *nD*-MICZ KP problems under the Hurwitz transformation. This duality allows us to deeply understand the influence of the magnetic monopole on the *nD*-KC problem.

V.1. Inverse mapping from the isotropic harmonic oscillator to the Kepler Coulomb problems

Interestingly, investigating the duality between the ND-HO and nD-KC problems and establishing the nD-MICZ KP problems relate to the history of finding the Hurwitz transformations. In 1904, Levi-Civita built a transformation for regularizing the classical two-dimensional Kepler motion to a two-dimensional classical isotropic harmonic oscillator [21]. Later, in 1959, Cisneros and McIntosh showed the relation between the two-dimensional Kepler and two-dimensional isotropic harmonic oscillator problems by the Levi-Civita transformation [47]. The Levi-Civita transformation has recently been used to construct a family of quantum mechanics problems that exhibits quasi-exact solutions [48–50].

After being built in 1965, the $(3-4)$ Hurwitz transformation had been used by Kustaanheimo and Stiefel to examine the three-dimensional classical Kepler problem, then by J. Moser to regularize the Kepler motion in celestial mechanics in 1970 [22, 51]. Two years later, Boiteaux applied Kustaanheimo – Stiefel transformation to connect the three-dimensional quantum Kepler problem and the four-dimensional quantum isotropic harmonic oscillator [52, 53]. This connection requires a constraint which has later been interpreted as the $U(1)$ gauge field of Dirac monopole [54, 55]. Thus, the four-dimensional quantum isotropic harmonic oscillator actually connects to the three-dimensional MICZ-Kepler established by McIntosh, Cisneros, and Zwanziger in 1969 and 1970 [17, 18].

From 1986 to 1991, the $(5-8)$ Hurwitz transformation was built, represented and used to connect the five-dimensional quantum Kepler problem and the eight-dimensional quantum isotropic harmonic oscillator problem in several papers by Kibler *et al.*, Davtyan et al. and Van-Hoang Le *et al.* [23, 30–32]. These studies led Mardoyan *et al.* to determine the relationship between the Schrödinger equation for the five-dimensional MICZ-Kepler problem, which contains Yang $SU(2)$ monopole, and that for the eight-dimensional harmonic oscillator problem [19].

In 1993, the $(9-16)$ Hurwitz transformation was constructed in a simple graphic way by Le and Komarov [24]. The generalized Hurwitz transformation has recently been used to build a relationship between the nine-dimensional Kepler problem and 16-dimensional isotropic harmonic oscillator problem in quantum mechanics. This achievement led Le, Nguyen, and Phan to rediscover a non-Abelian monopole in $9D$-space, the $SO(8)$ monopole, which was first found by Grossman *et al.* and re-constructed by Bernevig *et al.* [8, 10–12].

Using the mapping (50) and the inverse Hurwitz transformations, the Schrödinger equation (51) of the nD-MICZ-KP was shown to be completely equivalent to the one of the $(2n-2)D$-HO problem (44). For $n=3, 5$, and 9, the details of calculations were given in Refs. [11, 19, 55]. What we emphasize here is the influence of the magnetic monopole in the nD-MICZ-KP. The normed division algebras with the Hopf maps, the Hurwitz transformation, the $2T$ physics, and the monopole physics provide some ingredients:

- The normed division algebras, Hopf maps, and Hurwitz transformations show that the extra variables arising from the inverse Hurwitz transformations between $\mathbb{R}^{2n-2}$ and $\mathbb{R}^n$ belong to the bundle S^{n-2};
- The Hopf map $S^{n-2} \hookrightarrow S^{2n-3} \longrightarrow S^{n-1}$ states that the sphere S^{2n-3} is locally $S^{n-1} \times S^{n-2}$ [33, 34];

- Monopole physics shows that the connection between the bundle S^{n-2} and the symmetry group G_n of the monopole in the nD-space must be $G_n \cong S^{n-2}$ [33, 34];
- The $2T$ physics suggests that the dynamical symmetry of the $(2n+2)D$-HO and the nD-MICZ-KP should relate to each other as follows [20]:

$$Sp(2n-2,\mathbb{R}) \supset SO(n+1,2) \otimes G_n, \tag{52}$$

  where G_n is a gauge symmetry group induced from this duality;
- When applying the inverse Hurwitz transformation to the $(2n-2)D$-HO to obtain the nD MICZ-KP, appear some differential operators that are generators of the gauge symmetry group G_n. These operators only act on the extra variables or the angular part of wavefunctions and come with the gauge vector potential of the monopole. Notably, these generators have a similar form to the angular momentum in S^{n-2} sphere [12, 19, 55].

Meanwhile, at every point in $\mathbb{R}^n$ space, the electron in the Kepler Coulomb problem "feels" a locally abstract space S^{n-2}. The magnetic monopole can act on the phase or the angular part (ϕ_s) of the electron's wavefunction via the angular-momentum-like generators $\hat{I}^a(\phi_s)$ of this abstract space. Combining with the generators $\hat{I}^a(\phi_s)$ of the symmetry group $G_n \cong S^{n-2}$, the presence of the monopole also exhibits by the gauge vector potential $A_j^a(\mathbf{r})$ in the term $A_j^a(\mathbf{r})\hat{I}^a(\phi_s)$. It interprets how the G_n magnetic monopole influences the Kepler Coulomb problem. To summarize, Table 3 below shows the duality between the ND-HO and the nD MICZ-KP in the context of the normed division algebras, Hopf maps, and $2T$ physics.

Table 3. The duality between the ND-HO and the nD MICZ-KP in the context of the normed division algebras, Hopf maps, and $2T$ physics.

Algebra	HO	MICZ-KP	Hopf bundle	G_n	Monopole	Dynamical symmetry
Real numbers $\mathbb{R}$	$2D$	$2D$	S^0	$\mathbb{Z}_2$	–	$Sp(2,\mathbb{R}) \supset SO(2,2) \otimes \mathbb{Z}_2$
Complex numbers $\mathbb{C}$	$4D$	$3D$	S^1	$U(1)$	Dirac monopole	$Sp(4,\mathbb{R}) \supset SO(4,2) \otimes U(1)$
Quaternions $\mathbb{H}$	$8D$	$5D$	S^3	$SU(2)$	Yang monopole	$Sp(8,\mathbb{R}) \supset SO(6,2) \otimes SU(2)$
Octonions $\mathbb{O}$	$16D$	$9D$	S^7	$SO(8)$	$SO(8)$ monopole	$Sp(16,\mathbb{R}) \supset SO(10,2) \otimes SO(8)$

V.2. Three-dimensional MICZ-Kepler problem

The three-dimensional MICZ-Kepler problem was independently established by McIntosh, Cisneros, and Zwanziger in 1968 and 1970 when adding Dirac monopole on the conventional Kepler-Coulomb problem. The Schrödinger of this $3D$-MICZ-KP [17, 18]

$$\left\{ \frac{1}{2} \left(\hat{\mathbf{p}} + \mathbf{A}(\mathbf{r})\hat{S} \right)^2 + \frac{\hat{S}^2}{2r} - \frac{Z}{r} \right\} \Psi(\mathbf{r},\phi) = E\Psi(\mathbf{r},\phi), \tag{53}$$

was shown to be transformed into that of the four-dimensional isotropic harmonic oscillator [54]

$$\left\{ -\frac{1}{8} \left(\frac{\partial^2}{\partial u_s^2} + \frac{\partial^2}{\partial v_s^2} \right) + \frac{\omega^2}{2} \left(u_s^2 + v_s^2 \right) \right\} \Psi(u_s,v_s) = E\Psi(u_s,v_s), \tag{54}$$

through the Kustaanheimo-Stiefel transformation [22]

$$
\begin{aligned}
x_1 &= 2(u_1v_1 + u_2v_2), \\
x_2 &= 2(u_1v_2 - u_2v_1), \\
x_3 &= u_1^2 + u_2^2 - v_1^2 - v_2^2.
\end{aligned}
\tag{55}
$$

The 3D-MICZ-KP has the symmetry of $SO(4)$ and the dynamical symmetry of $SO(4,2)$. This problem is also maximally superintegrable. Its integrals of motion are the Poincare vector, the Laplace-Runge-Lenz vector, and the Hamiltonian as its cousin 3D-KC problem [45, 56–58]. The exact analytical solution of this problem was found in 1970, and the 3D-MICZ-KP is shown to be separable in spherical, parabolic, and prolate spheroidal coordinates [18, 59]. The Green function of the 3D-MICZ-KP was also constructed as applying the Kustaanheimo-Stiefel transformation on Feynmann path integrals of 4D-HO problem [24]. Furthermore, some algebraic approaches also provided the solution for the 3D-MICZ-KP, for example, separating its symmetry group $SO(4) \sim SO(3) \oplus SO(3)$ or using the Casimir invariance of its symmetry group $SO(4)$ [17, 56].

V.3. Five-dimensional MICZ-Kepler problem

The 5D-MICZ-Kepler problem was first introduced and examined by Mardoyan *et al.* in 1997 when generalizing the 5D-Kepler Coulomb problem by adding the Yang monopole [19]. The Schrödinger equation of the 5D-MICZ-KP

$$
\left\{ \frac{1}{2} \left(\hat{\mathbf{p}} + \mathbf{A}^\lambda(\mathbf{r})\hat{T}_\lambda \right)^2 + \frac{\hat{T}^2}{2r} - \frac{Z}{r} \right\} \Psi(\mathbf{r}, \phi_1, \phi_2, \phi_3) = E\Psi(\mathbf{r}, \phi_1, \phi_2, \phi_3),
\tag{56}
$$

is connected to the 8D isotropic harmonic oscillator [19, 60]

$$
\left\{ -\frac{1}{8} \left(\frac{\partial^2}{\partial u_s^2} + \frac{\partial^2}{\partial v_s^2} \right) + \frac{\omega^2}{2} \left(u_s^2 + v_s^2 \right) \right\} \Psi(u_s, v_s) = E\Psi(u_s, v_s),
\tag{57}
$$

via the Hurwitz transformation obtained by Kibler *et al.* [23, 31, 61]

$$
\begin{aligned}
x_1 &= 2(u_1v_4 - u_2v_3 + u_3v_2 - u_4v_1), \\
x_2 &= 2(-u_1v_3 - u_2v_4 + u_3v_1 + u_4v_2), \\
x_3 &= 2(u_1v_2 - u_2v_1 - u_3v_4 + u_4v_3), \\
x_4 &= 2(u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4), \\
x_5 &= u_1^2 + u_2^2 + u_3^2 + u_4^2 - v_1^2 - v_2^2 - v_3^2 - v_4^2.
\end{aligned}
\tag{58}
$$

The 5D-MICZ-KP symmetry and dynamical symmetry are $SO(6)$ and $SO(6,2)$ respectively. It is also a maximally superintegrable system with the constants of motion, the Poincare vector, the Laplace-Runge-Lenz vector, and the Hamiltonian [62–65]. From 1997 to 2000, the exact analytical solutions of the 5D MICZ-KP were found in spherical, parabolic, and prolate spheroidal coordinates [19, 66]. Its Green function was also constructed using the quaternion algebra [67]. The Casimir invariance of its $SO(6)$ symmetry was also used to obtain its energy spectrum [62]. Table 4 summarizes notable results in the investigation of the 3D and 5D MICZ-Kepler problems.

Table 4. Notable results in the investigation of the 3*D* and 5*D* MICZ-Kepler problems.

MICZ-KP	3*D*	5*D*
Monopole	Dirac $U(1)$	Yang $SU(2)$
Symmetry	$SO(4)$	$SO(6)$
Dynamical symmetry	$SO(4,2)$	$SO(6,2)$
Superintegrability	maximal	
Algebraic solution	Casimir invariance	
Analytical solution	spherical, parabolic, spheroidal	
Green function obtained?	yes	

VI. NINE-DIMENSIONAL MICZ-KEPLER PROBLEMS

The nine-dimensional MICZ-Kepler problem (9*D* MICZ-KP) was first introduced a decade ago by Van-Hoang Le *et al.* when using their $(9-16)$ Hurwitz transformation to convert the 16*D* HO problem [11, 12]. The time-independent wavefunction $\Psi(\boldsymbol{r}, \phi)$ describing the bound state ($E < 0$) of an electric charge in the 9*D* Euclidean space under the interaction of the self-dual $SO(8)$ monopole field is governed by the following dimensionless Schrödinger equation [1]:

$$\left\{ \frac{1}{2}\hat{\boldsymbol{\pi}} \cdot \hat{\boldsymbol{\pi}} + \frac{\hat{Q}^2}{8r^2} - \frac{Z}{r} \right\} \Psi(\boldsymbol{r}, \phi) = E\Psi(\boldsymbol{r}, \phi). \tag{59}$$

Here, Z is the electric charge, and $\hat{Q}_{ij}$ are operators describing the $SO(8)$ monopole charges. $\hat{Q}^2 = \hat{Q}_{ij}\hat{Q}_{ij}$ ($1 \le i < j \le 8$) is the Casimir operator of the $SO(8)$ monopole generators. The generalized momentum operators are defined as

$$\hat{\boldsymbol{\pi}} = \left(-\imath\partial_j + A_k(\boldsymbol{r})\hat{Q}_{kj}, -\imath\partial_9 \right), \quad j,k = 1,2,\ldots,8, \tag{60}$$

where

$$A_k(\boldsymbol{r}) = \frac{x_k}{r(r+x_9)} \tag{61}$$

is the $SO(8)$ monopole vector potential.

It is noted that $\hat{Q}_{ij}$ are differential operators in terms of variables ϕ_s ($s = 0, 1, \ldots, 6$). Hence, the wavefunction $\Psi(\boldsymbol{r}, \phi)$ depends not only on coordinates of the real space ($\mathbf{r} \in \mathbb{R}^9$) but also on seven additional angles $(\phi_6, \phi_5, \ldots, \phi_0) \in [0, \pi]^6 \times [0, 2\pi]$ of a unit sphere S^7 arisen from the generalized $(9, 16)$ Hurwitz transformation $\mathbb{R}^{16} \to \mathbb{R}^9 \times S^7$, connecting the 16*D* HO with the 9*D* KC problem.

Since the operator $\hat{Q}^2$ commutes with the Hamiltonian in equation (59), the wavefunction Ψ also obeys the equation

$$\hat{Q}^2\Psi = Q(Q+6)\Psi. \tag{62}$$

---

[1]Notice that the octonion $\mathbb{O}$ representation of the 9*D* MICZ-KP has also been proposed in our talk [68].

Here, the quantum number Q is an integer because it has to close the abstract monopole space S^7.

Following our works [69–72], we expand the Hamiltonian of the 9D MICZ-KP into a more specific form

$$\hat{H} = -\frac{1}{2}\Delta_{\mathbb{R}^9} + \frac{1}{2r(r+x_9)}\hat{L}_{jk}\hat{Q}_{jk} + \frac{1}{4r(r+x_9)}\hat{Q}^2 - \frac{Z}{r}, \tag{63}$$

where $\hat{L}_{jk} = -\iota(x_j\partial_k - x_k\partial_j)$ are projectors of the angular momentum on the sphere S^7, $x_1^2 + x_2^2 + \cdots + x_8^2 = 1$, of the real space $\mathbb{R}^9$. This expansion helps us to observe the influences of the $SO(8)$ monopole on the 9D KC system. As can be seen from Equation (63), the coupling term $\hat{L}_{jk}\hat{Q}_{jk}$ is similar to the spin-orbital coupling; thus, we claim that the $SO(8)$ monopole interacts with the 9D KC system as an "isospin" interaction.

During the last decade, the 9D MICZ-KP has also been investigated in various aspects. Its symmetry and dynamical symmetry are $SO(10)$ and $SO(10,2)$, respectively [73, 74]. The $SO(10,2)$ dynamical symmetry was used to obtain the algebraic solutions [73]. The analytical solutions with the wavefunctions were obtained by the variable separation method in spherical [69], parabolic, and prolate spheroidal coordinates [71]. This problem is also maximally superintegrable as the 3D and 5D MICZ-KP [70]. Within the analytical approach in Ref. [71], the interbasis transformation between the parabolic and spherical bases has been constructed. However, a similar transformation between the prolate spheroidal and spherical bases was not built in an analytical approach since it required a complex integration between confluent Heun [75], associated Laguerre, and generalized Jacobi polynomials [36]. These integrations have not been analytically calculated yet, to the best of our knowledge. This problem has been recently solved by examining the algebraic structure of each basis of the 9D MICZ-KP in our most recent work [72].

VI.1. Hidden symmetry on the nine-dimensional MICZ-Kepler problem

The 9D MICZ-KP has been shown $SO(10)$ symmetric in Ref. [74]. In this work, 54 elements belonging to the following antisymmetric matrix operator $\hat{D}$

$$\hat{D} = \begin{pmatrix} \hat{\Lambda}_{\mu\nu} & -\hat{M}_\mu \\ \hat{M}_\mu & 0 \end{pmatrix}, \quad \mu, \nu = 1, 2, \ldots, 9, \tag{64}$$

were shown to commute with Hamiltonian (59). Here, $\hat{\Lambda}_{\mu\nu}$ and $\hat{M}_\mu$ are generalized angular momentum and Laplace-Runge-Lenz vector operators [72, 74]

$$\hat{\Lambda}_{\mu\nu} = x_\mu\hat{\pi}_\nu - x_\nu\hat{\pi}_\mu + \iota r^2 \left[\hat{\pi}_\mu, \hat{\pi}_\nu\right]_-, \tag{65}$$

$$\hat{M}_\nu = \frac{1}{\sqrt{-2\hat{H}}}\left\{\frac{1}{2}\left[\hat{\pi}_\mu, \hat{\Lambda}_{\mu\nu}\right]_+ + \frac{Zx_\nu}{r}\right\}, \tag{66}$$

where the notation $[\hat{A}, \hat{B}]_\pm = \hat{A}\hat{B} \pm \hat{B}\hat{A}$ is used for commutator/anticommutator between two operators $\hat{A}$ and $\hat{B}$.

Since $\hat{D}$-matrix obeys the $SO(10)$ algebra [70, 72, 74], the 9D MICZ-KP is $SO(10)$ symmetric. This symmetry is higher than the trivial geometric symmetry $SO(9)$; thus, the quantization of the 9D MICZ-KP is degenerated so high that its energy solely depends on the principal quantum number n only. This highest degeneracy has been observed in the energy spectrum of the 9D MICZ-KP

$$E_{n,Q} = -\frac{Z^2}{2(n+4+Q/2)^2}, \quad n = 0, 1, 2, \ldots, \tag{67}$$

obtained by different approaches given in Refs. [69, 71, 73, 74].

VI.2. Superintegrability and multiple separation of the nine-dimensional MICZ-Kepler problem

To examine the superintegrability of the $9D$ MICZ-KP, in the study [70], we aimed to construct at most $2 \times 9 - 1 = 17$ algebraically independent constants of motion. If we reach the number 17, the $9D$ MICZ-KP is called maximally superintegrable.

Indeed, the first constant of motion is the Hamiltonian $\hat{H} \equiv \hat{D}_1^2$. The next seven members were constructed from subgroup $SO(m) \subset SO(10)$ of $\hat{D}$ as follows. We took the $m \times m$ block matrices from the top left of $\hat{D}$ [70]

$$\hat{X}(m) = \begin{pmatrix} \ddots & \vdots & \cdot^{\cdot^{\cdot}} \\ \dots & \hat{\Lambda}_{jk} & \dots \\ \cdot_{\cdot_{\cdot}} & \vdots & \ddots \end{pmatrix}_{j,k=1,2,\dots,m} \quad , \quad m = 2, 3, \dots, 8, \tag{68}$$

and then built seven second-order Casimir operators

$$\hat{D}_m^2 = -\frac{1}{2} X_{jk}(m) X_{kj}(m) = \sum_{1 \leq j < k \leq m} \hat{\Lambda}_{jk}^2, \quad m = 2, 3, \dots, 8. \tag{69}$$

Note that in Ref. [72], the above operators are shown to be

$$\hat{D}_m^2 = \sum_{1 \leq j < k \leq m} \left(\hat{L}_{jk} + \hat{Q}_{jk} \right)^2. \tag{70}$$

As in Ref. [70], we chose the ninth constant of motion as the total generalized angular momentum $\hat{D}_9^2 = \hat{\Lambda}^2$. We took other seven block-matrices from the bottom right of $\hat{D}$ and built their Casimir operators:

$$\hat{D}_m^2 = -\frac{1}{2} Y_{jk}(m) Y_{kj}(m) = \sum_{18-m \leq j < k \leq 9} \hat{\Lambda}_{jk}^2, \quad m = 10, 11, \dots, 17. \tag{71}$$

Collecting seventeen constants of motion $\hat{D}_1^2, \hat{D}_2^2, \dots, \hat{D}_8^2, \hat{D}_9^2, \hat{D}_{10}^2, \dots, \hat{D}_{17}^2$, we have proved that these are all second-order operators in momentum and all algebraically independent. Also, the first nine constants of motion $\hat{D}_1^2, \hat{D}_2^2, \dots, \hat{D}_8^2, \hat{D}_9^2$ commute with each other. Hence, the $9D$ MICZ-KP should be maximally superintegrable [70].

In fact, the $9D$ MICZ-KP is a vector degenerated system. Therefore, the above seventeen integrals of motion did not ensure the maximal superintegrability of our system. A remarkable property of a maximally superintegrable system is the multiple separation via different coordinate systems. Fortunately, this is true for the $9D$ MICZ-KP, and hence, the $9D$ MICZ-KP is a good example that the definition of a maximally superintegrable system works for a vector degenerated system [70]. In the series of our studies, we have variable separate the $9D$ MICZ-KP in spherical [69], parabolic [70], and prolate spheroidal [71] coordinates, respectively.

VII. CONCLUSIONS AND OUTLOOK

The normed division algebras such as the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$ and their cousins Hopf maps in the differential topology and the Hurwitz transformations in the quadratic algebraic form have been applied to various aspects of physics.

Many phenomena or problems in physics require the normed division algebras to perform simple, compact, and beautiful explanations or solutions. Under the perspective of the normed division algebras, monopoles in the three-, five-, and nine-dimensional spaces are a direct consequence of the gauge invariance of complex, quaternion, and octonion Hopf maps. Notwithstanding some difficulties in both mathematics and physics sides, we believe that the normed division algebras, e.g., the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$, could be the algebra of nature.

On the other side, in mathematical physics, the duality between the two most fundamental physics problems - harmonic oscillator and Kepler Coulomb problem - has taken the attention for several decades. Amazingly, this duality relates to fascinating topics like the Hurwitz transformations and two-time physics. From the relationship between the Hurwitz transformations and the normed division algebras, the duality leads to the gauge fields and generators of magnetic monopoles in the Kepler Coulomb problem and the establishment of the nD MICZ-Kepler problem.

Three nD MICZ-Kepler problems with dimensionality 3, 5, and 9 are the direct consequence of the existence of the complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, and octonions $\mathbb{O}$. The last case, the nine-dimensional MICZ-Kepler problem, has been established and examined for over a decade, and some results are also aggregated in this review paper.

In this review, we do not go to the low energy regime of condensed matter physics, where the complex numbers $\mathbb{C}$ can also explain the well-known $2D$ quantum Hall effect. According to quaternions $\mathbb{H}$ and octonions $\mathbb{O}$, physicists have proposed $4D$ and $8D$ quantum Hall effects in topological insulators. The explanation of the quantum Hall effect by the normed division algebras and the Hopf maps opened a new era of applying mathematical objects to condensed matter physics. This topic may be included in our other studies.

In the last two years, some studies have used the last normed division algebra - octonions $\mathbb{O}$ - to explore the Standard Model structure and its extension. Some preliminary results from these works show the great potential of explaining the Standard Model by the normed division algebras. However, we will discuss this topic in our next work.

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