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QUASI THREE-PARAMETRIC *R*-MATRIX AND QUANTUM SUPERGROUPS $GL_{p,q}(1/1)$ AND $U_{p,q}[gl(1/1)]$

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Abstract. *An over-parametrized (three-parametric) R-matrix satisfying a graded Yang-Baxter equation is introduced. It turns out that such an over-parametrization is very helpful. Indeed, this R-matrix with one of the parameters being auxiliary, thus, reducible to a two-parametric R-matrix, allows the construction of quantum supergroups* $GL_{p,q}(1/1)$ *and* $U_{p,q}[gl(1/1)]$ *which, respectively, are two-parametric deformations of the supergroup* $GL(1/1)$ *and the universal enveloping algebra* $U[gl(1/1)]$. *These two-parametric quantum deformations* $GL_{pq}(1/1)$ *and* $U_{pq}[gl(1/1)]$, *to our knowledge, are constructed for the first time via the present approach. The quantum deformation* $U_{p,q}[gl(1/1)]$ *obtained here is a true two-parametric deformation of Drinfel'd-Jimbo's type, unlike some other one obtained previously elsewhere.*

Keywords: quantum supergroup, R-matrix; Drinfel'd-Jimbo deformation; multi- parametric quantum deformation.

Classification numbers: 02.20.-a; 03.65.Fd.

I. INTRODUCTION

The discovery of the Higgs boson by the LHC collaborations ATLAS and CMS [1,2] shows once again the might of the symmetry principle in physics (see, for instance, [3] and references therein for a review on the Higgs boson's search and discovery). In particular, the standard model (SM) based on the gauge symmetry $SU(3)\otimes SU(2)\otimes U(1)$ (see, for example, [4,5]), has been verified by the experiment, specially, after the discovery of the Higgs model, as an excellent model of elementary particles and their interactions [6]. There are, however, a number of problems, which cannot be explained or described by the existing symmetry, for example, within the SM the problems like CP-violation (matter-antimatter asymmetry), neutrino masses and mixing, dark matter, dark energy, etc. cannot be solved. Such problems may require an extension by size, or even, a generalization by concept, of an underlying symmetry[1] adopted to a physics system. One of such generalizations is the concept of quantum deformed symmetry. Mathematically, if ordinary (or classical) symmetry is described by classical groups such as the above-mentioned group $SU(3)\otimes SU(2)\otimes U(1)$, the quantum deformed symmetry is described by the so-called quantum groups [7–12] (see, for example, [13, 14] for some physics applications of quantum groups).

Using the R-matrix formalism [7] is one of the approaches to quantum groups which can be interpreted as a kind of (quantum) deformations of ordinary (classical) groups or algebras. It has proved to be a powerful method in investigating quantum groups and related topics such as noncommutative geometry [8, 11, 12, 15], integrable systems [7, 13, 14], etc. A physical meaning of this approach is the so-called (universal) R-matrix associated to a quantum group satisfies the famous Yang-Baxter equation (YBE) representing an integrability condition of a physical system [7, 13, 14]. A mathematical advantage of this approach is both the algebraic and co-algebraic structure of the corresponding quantum group can be expressed in a few compact (matrix) relations. Quantum groups as symmetry groups of quantum spaces [7, 8, 15] or as deformations of universal enveloping algebras, called also Drinfel'd-Jimbo (DJ) deformation [9, 10], can be also derived in an elegant way in the framework of the R-matrix formalism. The DJ deformation, which is originally one-parametric, has an advantage that it has a clear algebraic structure (as a deformation from the classical algebraic structure) and it is convenient for a representation construction and a multi-parametric generalization. Combined with the supersymmetry idea [16–18] (see, for example, [19, 20], among a vast literature, for a more detailed introduction), the quantum deformations lead to the concept of quantum supergroups [21–24]. In this case, an R-matrix becomes graded and satisfies a graded YBE [24].

By original construction, a quantum (super) group depends on a, complex in general, parameter, but the concept of one-parametric quantum (super) groups can be generalized to that of multi-parametric quantum (super) groups. For about three decades quantum groups have been investigated in great details in many aspects. These investigations were carried out first and mainly on the one-parametric case and they were extended later to on the multi-parametric deformations [8, 25]. Having in principle richer structures, multi-parametric quantum groups are also a subject of interest of a number of authors (see Refs. [26–36] and references therein) and have been applied to considering some physics models (see in this context, for example, Refs. [36–39]) but in

[1] *Here we do not discuss the global- and the local symmetry separately.*

comparison with the one-parametric quantum groups, they are considerably less understood (even, in some cases they can be proved to be equivalent to one-parametric deformations). Moreover, most of the multi-parametric deformations considered so far are two-parametric ones including those of supergroups [27–35] (it is clear that two-parametric deformations of supergroups cannot be always reduced to one-parametric ones [27–29, 31]). Here, we continue to deal with the case of two-parametric deformations, in particular, a two-parametric deformation of the supergroup $GL(1/1)$, which was also considered in [27]. The two-parametric deformation obtained there, however, does not lead to a "standard" DJ form of a two-parametric deformation of $U[gl(1/1)]$ obtaining which is the purpose of the present work. It will be shown that such a two-parametric deformation of DJ type can be found via a quasi three-parametric deformation of $Gl(1/1)$.

II. DRINFEL'D-JIMBO QUANTUM SUPERGROUPS AND THEIR TWO-PARAMETRIC GENERALIZATION

A quantum group as a DJ deformation [9,10] of an universal enveloping algebra of a (semi-) simple superalgebra g of rank r can be defined via a set of $3r$ Cartan-Chevalley generators h_i, e_i, f_i, $i = 1,2,...,r$, subject to the following defining relations (see, for example, [40, 41] and references therein):

a) *the quantum Cartan-Kac supercommutation relations*,

$$\begin{array}{l}[h_i,h_j]=0,\ [h_i,e_j]=a_{ij}e_j,\\ [h_i,f_j]=-a_{ij}f_j,\ [e_i,f_j\}=\delta_{ij}[h_i]_{q_i},\end{array} \tag{1}$$

b) *the quantum Serre relations*,

$$(ad_qE_i)^{1-\tilde{a}_{ij}}\mathscr{E}_j=0,\ (ad_q\mathscr{F}_i)^{1-\tilde{a}_{ij}}\mathscr{F}_j=0, \tag{2}$$

with $\mathscr{E}_i = e_iq_i^{-h_i}$, $\mathscr{F}_i = f_iq_i^{-h_i}$, and

c) *the quantum extra-Serre relations* which for g being $sl(m/n)$ or $osp(m/n)$ have the form,

$$\begin{array}{l}\{[e_{m-1},e_m]_q,[e_m,e_{m+1}]_q\}=0,\\ \{[f_{m-1},f_m]_q,[f_m,f_{m+1}]_q\}=0,\end{array} \tag{3}$$

where

$$[X]_q=\frac{q^X-q^{-X}}{q-q^{-1}}, \tag{4}$$

denotes a (one-parametric) quantum deformation of a number or operator X, and $(\tilde{a}_{ij})$ is a matrix obtained from the non-symmetric Cartan matrix (a_{ij}) of g by replacing the strictly positive elements in the rows with 0 on the diagonal entry by -1, while ad_q is the q-deformed adjoint operator given by the formula (2.8) in [40]. Here $q_i = q^{d_i}$ where d_i are rational numbers symmetrizing the Cartan matrix, $d_ia_{ij} = d_ja_{ji}$, $1 \leq i, j \leq r$. They take, for example, in case $g = sl(m/n)$, the values

$$d_i=\begin{cases}1 & \text{for } 1\leq i\leq m,\\ -1 & \text{for } m+1\leq i\leq r=m+n-1.\end{cases} \tag{5}$$

Now let us define a two-parametric DJ deformation as a direct generalization of the above-defined one-parametric deformation (1)–(3) by extending (4) to

$$[X]_{p,q}=\frac{q^X-p^{-X}}{q-p^{-1}}, \tag{6}$$

where p and q are, in general, independent complex parameters. Thus $[h]_{q_i}$ in (1) becomes

$$[h_i]_{p_i,q_i} \equiv \frac{q_i^{h_i} - p_i^{-h_i}}{q_i - p_i^{-1}}, \tag{7}$$

with q_i defined above and $p_i = p^{d_i}$. This kind two-parametric generalization of the DJ deformation was considered earlier in, for example, [29,31,32]. Next, according to this definition, we will derive via the R-matrix formalism a two-parametric DJ deformation of $U[gl(1/1)]$ which, to our knowledge, has not yet been constructed. Since $gl(1/1)$ is a rank-1 ($r = 1$) superalgebra, the index i will be omitted.

As discussed in the previous section, one of the two-parametric quantum deformations of $GL(1/1)$ was obtained elsewhere [27], however, the corresponding two-parametric deformation of the universal enveloping algebra $U[gl(1/1)]$ has no DJ form. In fact, the two-parametic deformation of $U[gl(1/1)]$ in [27] can be transformed to an one-parametric DJ deformation by re-scaling its generators appropriately. Indeed, starting from the defining relations of the deformation of $U[gl(1/1)]$ given in [27],

$$[K,H] = 0,\ [K,\chi_\pm] = 0,\ [H,\chi_\pm] = \pm 2\chi_\pm,$$
$$\{\chi_+,\chi_-\}_{q/p} = \left(\frac{q}{p}\right)^{H/2} [K]_{\sqrt{qp}}\ ,$$

where

$$\{\chi_+,\chi_-\}_{q/p} \equiv \left(\frac{q}{p}\right)^{1/2} \chi_+\chi_- + \left(\frac{q}{p}\right)^{-1/2} \chi_-\chi_+,$$
$$[K]_{\sqrt{qp}} = \frac{(qp)^{K/2} - (qp)^{-K/2}}{(qp)^{1/2} - (qp)^{-1/2}}$$

and making re-scaling $\chi_\pm \to \chi'_\pm = \left(\frac{q}{p}\right)^{-H/4} \chi_\pm$, we get

$$[K,H] = 0,\ [K,\chi'_\pm] = 0,$$
$$[H,\chi'_\pm] = \pm 2\chi'_\pm,\ \{\chi'_+,\chi'_-\} = [K]_{\sqrt{qp}}.$$

The latter relations are (conventional) defining relations of an one-parametric DJ deformation of $U[gl(1/1)]$ with parameter $\sqrt{qp}$ (cf. (1)–(4)). To obtain a true two-parametric deformation of both $GL(1/1)$ and $U[gl(1/1)]$ we start from a three-parametric R-matrix satisfying a graded YBE. This R-matrix approach will allow us to construct a (quasi) three-parametric deformation of $GL(1/1)$ which in fact is equivalent upto a rescaling to a true two-parametric deformation of $GL(1/1)$. It also leads to a true two-parametric DJ deformation of $U[gl(1/1)]$.

III. A QUASI THREE-PARAMETRIC DEFORMATION OF $GL(1/1)$

As the maximal number of parameters of a quantum deformation of $GL(1/1)$ is two, the below-obtained deformation of $GL(1/1)$ is in fact quasi-three parametric (so is the corresponding R-matrix). We will see below that such an over-parametrization is very convenient.

Let us start with the operator

$$\begin{aligned} R &= q(e_1^1 \otimes e_1^1) + r(e_1^1 \otimes e_2^2) + s(e_2^2 \otimes e_1^1) \\ &\quad + \lambda(e_2^1 \otimes e_1^2) + p(e_2^2 \otimes e_2^2), \end{aligned} \tag{8}$$

where p, q, r, s and λ are complex deformation parameters, while e_j^i, $i, j = 1, 2$, are Weyl generators of $GL(1|1)$ with a Z_2-grading given as follows:

$$[e_j^i] = [i] + [j] \ (\text{mod } 2), \ [i] = \delta_{i2}. \tag{9}$$

We call the latter operator an R-matrix although it has a (finite) matrix form only in a finite-dimensional representation. In the fundamental representation e_j^i are super-Weyl matrices, $(e_j^i)_k^h = \delta_k^i \delta_j^h$, and R is a 4×4 matrix. Three of the five parameters, say, p, q and r, can be chosen to be independent, while the remaining parameters, s and λ, are subject to the constraints

$$rs = pq, \ \lambda = q - p. \tag{10}$$

By this choice of the parameters, the R-matrix (1) satisfies the graded YBE

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{11}$$

with

$$\begin{aligned} R_{12} &= R \otimes I \equiv R \otimes e_i^i, \ i = 1, 2, \\ R_{13} &= q(e_1^1 \otimes e_i^i \otimes e_1^1) + r(e_1^1 \otimes e_i^i \otimes e_2^2) + s(e_2^2 \otimes e_i^i \otimes e_1^1) \\ &\quad + (-1)^{[i]} \lambda (e_2^1 \otimes e_i^i \otimes e_1^2) + p(e_2^2 \otimes e_i^i \otimes e_2^2), \\ R_{23} &= I \otimes R \equiv e_i^i \otimes R, \end{aligned} \tag{12}$$

where repeated indices are summation indices, I is the identity operator and the Z_2-grading is given in (9).

Now suppose the operator subject

$$T = a\, e_1^1 + \beta\, e_2^1 + \gamma\, e_1^2 + d\, e_2^2 \equiv t_i^j e_j^i \tag{13}$$

obeys the so-called RTT equation

$$RT_1T_2 = T_2T_1R, \tag{14}$$

where

$$\begin{aligned} T_1 &= T \otimes I \equiv (ae_1^1 + \beta e_2^1 + \gamma e_1^2 + de_2^2) \otimes e_j^i, \\ T_2 &= I \otimes T \\ &\equiv e_j^i \otimes [ae_1^1 + (-1)^{[i]} \beta e_2^1 + (-1)^{[i]} \gamma e_1^2 + de_2^2]. \end{aligned} \tag{15}$$

The Eq. (14) leads to the supercommutation relations between the elements of T :

$$\begin{aligned} a\beta &= \frac{r}{p}\beta a, \ a\gamma = \frac{q}{r}\gamma a, \ ad = da + \frac{\lambda}{r}\gamma\beta, \ \beta^2 = 0 = \gamma^2, \\ \beta\gamma &= -\frac{s}{r}\gamma\beta \equiv -\frac{pq}{r^2}\gamma\beta, \ \beta d = \frac{p}{r}d\beta, \ \gamma d = \frac{r}{q}d\gamma. \end{aligned} \tag{16}$$

Let us denote G a set of all operators (13) satisfying (14) and let T and T' be two independent copies of (13) in the sense that all elements t_j^i of T commute with all those of T'. The fact that the multiplication $T.T'$ preserves the relation (14), that is, the relations (16), reflects the group nature of G. Next, since the quantity

$$\begin{aligned} D(T) &\equiv (a-\beta d^{-1}\gamma)d^{-1} = d^{-1}(a-\beta d^{-1}\gamma) \\ &= a(d-\gamma a^{-1}\beta)^{-1} \end{aligned} \tag{17}$$

commutes with T and has the "multiplicative" property $D(T.T') = D(T).D(T')$ it can be identified with a representation of a quantum superdeterminant. Thus we can take G with $D(T) \neq 0, \forall T \in G$, as a quasi three-parametric deformation, denoted by $GL_{p,q,r}(1/1)$, of a representation of $GL(1/1)$. The latter deformation is equivalent upto a rescaling (e.g., $p/r \to p$, $q/r \to q$) to a two-parametric deformation, say $GL_{p,q}(1/1)$, but we keep the quasi three-parametric form until obtaining a true two-parametric deformation of $U[gl(1/1)]$. When we set $D(T) = 1$ we get a quasi three-parametric deformation of $SL(1/1)$. We note that the form of the quantum superdeterminant $D(T)$ is the same as that given in [27], that is, it remains non-deformed and belongs to the center of $GL_{p,q,r}(1/1)$. The Hopf structure is straightforward and given by the following maps:

- *the co-product*:

$$\Delta(T) = T\dot{\otimes}T, \tag{18}$$

- *the antipode*:

$$S(T).T = I, \tag{19}$$

- *the counit*:

$$\varepsilon(T) = I. \tag{20}$$

In components they read

$$\Delta(t^i_j) = t^k_j \otimes t^i_k, \tag{21}$$

$$\begin{aligned} S(t^j_i e^i_j) &= S(t^j_i)e^i_j \\ &= a^{-1}(1+\beta d^{-1}\gamma a^{-1})e^1_1 - (a^{-1}\beta d^{-1})e^1_2 \\ &\quad -(d^{-1}\gamma a^{-1})e^2_1 + d^{-1}(1-\beta a^{-1}\gamma d^{-1})e^2_2, \end{aligned} \tag{22}$$

$$\varepsilon(t^i_j) = \delta^i_j. \tag{23}$$

A quantum superplane with symmetry (authomorphism) group $GL_{p,q,r}(1/1)$ is given by the coordinates

$$\begin{pmatrix} x \\ \theta \end{pmatrix} \text{ or } \begin{pmatrix} \eta \\ y \end{pmatrix} \tag{24}$$

subject to the commutation relations

$$x\theta = \frac{q}{r}\theta x \equiv \frac{s}{p}\theta x, \; \theta^2 = 0 \text{ or } \eta^2 = 0, \; \eta y = \frac{p}{r} y\eta, \tag{25}$$

respectively. Note that these quantum superplanes (which are "two-dimensional") are still two-parametric (of course, we cannot make relations between two coordinates to depend on more than two parameters). Finally, in order to complete our program we must construct a true two-parametric DJ deformation of the universal enveloping algebra $U[gl(1/1)]$. It can be obtained from a quasi-three parametric DJ deformation, denoted as $U_{p,q,r}[gl(1/1)]$, corresponding to the R-matrix (8).

IV. A TWO-PARAMETRIC DRINFEL'D–JIMBO DEFORMATION OF $U[gl(1/1)]$

First, following the technique of [7], we introduce two auxilary operators

$$\begin{aligned} L^+ &= H_1^+ e_1^1 + H_2^+ e_2^2 + \lambda X^+ e_2^1, \\ L^- &= H_1^- e_1^1 + H_2^- e_2^2 + \lambda X^- e_1^2, \end{aligned} \tag{26}$$

with $H_i^\pm$ and $X^\pm$ belonging to $U_{p,q,r}[gl(1/1)]$ to be constructed. Then, demanding

$$\begin{aligned} L_1^\pm &= L^\pm \otimes e_i^i, \\ L_2^+ &= e_i^i \otimes [H_1^+ e_1^1 + H_2^+ e_2^2 + (-1)^{[i]} \lambda X^+ e_2^1], \\ L_2^- &= e_i^i \otimes [H_1^- e_1^1 + H_2^- e_2^2 + (-1)^{[i]} \lambda X^- e_1^2] \end{aligned}$$

to obey the equations

$$RL_1^{\epsilon_1} L_2^{\epsilon_2} = L_2^{\epsilon_2} L_1^{\epsilon_1} R, \tag{27}$$

where $(\epsilon_1, \epsilon_2) = (+,+), (-,-), (+,-)$, we get the following commutation relations between $H_i^\pm$ and $X^\pm$:

$$\begin{aligned} & H_i^{\epsilon_1} H_j^{\epsilon_2} = H_j^{\epsilon_2} H_i^{\epsilon_1}, \\ & pH_i^+ X^+ = rX^+ H_i^+, \qquad qH_i^- X^+ = rX^+ H_i^-, \\ & rH_i^+ X^- = pX^+ H_i^+, \qquad rH_i^- X^- = qX^- H_i^-, \\ & rX^+ X^- + sX^- X^+ = \lambda^{-1} (H_2^- H_1^+ - H_2^+ H_1^-), \end{aligned} \tag{28}$$

which are taken to be the defining relations of $U_{p,q,r}[gl(1/1)]$. Its Hopf structure is given by

$$\Delta(L^\pm) = L^\pm \dot{\otimes} L^\pm, \tag{29}$$

$$S(L^\pm) = (L^\pm)^{-1}, \tag{30}$$

$$\varepsilon(L^\pm) = I, \tag{31}$$

or equivalently (no summation on $i = 1, 2$),

$$\begin{aligned} \Delta(H_i^\pm) &= H_i^\pm \otimes H_i^\pm, \\ \Delta(X^+) &= H_1^+ \otimes X^+ + X^+ \otimes H_2^+, \\ \Delta(X^-) &= H_2^- \otimes X^- + X^- \otimes H_1^-, \end{aligned} \tag{32}$$

$$\begin{aligned} S(H_i^\pm) &= (H_i^\pm)^{-1}, \\ S(X^+) &= -(H_1^+)^{-1} X^+ (H_2^+)^{-1}, \\ S(X^-) &= -(H_2^-)^{-1} X^- (H_1^-)^{-1}, \end{aligned} \tag{33}$$

$$\varepsilon(H_i^\pm) = 1, \ \varepsilon(X^\pm) = 0. \tag{34}$$

At first sight $U_{p,q,r}[gl(1/1)]$ given in (28) is a three-parametric quantum supergroup depending on three parameters p, q and r (or s). However, making the substitution

$$\begin{aligned} &H_1^+ = \left(\frac{r}{p}\right)^{E_{11}},\ H_2^+ = \left(\frac{p}{r}\right)^{E_{22}}, \\ &H_1^- = \left(\frac{r}{q}\right)^{E_{11}},\ H_2^- = \left(\frac{q}{r}\right)^{E_{22}}, \\ &E_{12} = X^+ r^{E_{22}},\ E_{21} = X^- s^{E_{11}}, \end{aligned} \tag{35}$$

in (28) and replacing p by p^{-1} (without loss of generality), we obtain a two-parametric deformation of $U[gl(1/1)]$ generated by E_{ij}, which are two-parametric analogs of the Weyl generators, via the following relations

$$\begin{aligned} [E_{ii}, E_{jj}] &= 0, \\ [E_{ii}, E_{j,j\pm1}] &= (\delta_{ij} - \delta_{i,j\pm1}) E_{j,j\pm1}, \\ \{E_{12}, E_{21}\} &= [K]_{p,q}\ , \end{aligned} \tag{36}$$

where $1 \leq i, j, j \pm 1 \leq 2$ and

$$[K]_{p,q} = \frac{q^K - p^{-K}}{q - p^{-1}},\ K = E_{11} + E_{22}. \tag{37}$$

The latter deformation denoted as $U_{p,q}[gl(1/1)]$ is a true two-parametric DJ deformation of $U[gl(1/1)]$, which we are looking for, as it cannot be made to become one-parametric by a further rescaling of its generators. Of course, (35) is not the only realization of the generators of $U_{p,q,r}[gl(1/1)]$ in terms of the deformed Weyl generators E_{ij}.

V. CONCLUSION

We have suggested in the present paper an R-matrix satisfying a (quasi) three-parametric graded YBE. Using this overparametrized R-matrix we can obtain two-parametric deformations $GL_{p,q}(1/1)$ and $U_{p,q}[gl(1/1)]$, respectively, of the supergroup $GL(1/1)$ and the corresponding universal enveloping algebra $U[gl(1/1)]$, respectively. It is worth noting that the quantum superalgebra $U_{p,q}[gl(1/1)]$ is a true two-parametric deformation of $U[gl(1/1)]$ generalizing the Drinfel'd–Jimbo deformation $U_q[gl(1/1)]$ which is one-parametric. That is $U_{p,q}[gl(1/1)]$ cannot be reduced to any one-parametric deformation by any re-scaling or re-definition of generators. Physics interpretations and applications of these two-parametric deformations are a subject of our current interest.

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