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DYNAMICAL CODIMENSION-2 BRANE IN A WARPED SIX DIMENSIONAL SPACE-TIME

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Abstract. *In this paper we present the Einstein equation extended in six-dimensions (6D) from the formalism of codimension-2 brane, which is created by a 4-brane and 4-anti brane moving in the warped 6D bulk space-time. The system of equations of motion for the dynamical codimension-2 brane has been derived to describe the cosmological evolution on the probe branes. Some cosmological consequences are investigated.*

Keywords: six-dimensions (6D), codimension-2 brane, Schwarzschild AdS ($SAdS_6$).

Classification numbers: 98.80.-k; 11.30.Rd.

I. INTRODUCTION

Besides the extended models of Standard Model (SM) and Standard Cosmology, the extra dimensions have been considered as the ways to investigate both old and open problems of Particle physics and Cosmology, in the new physics beyond the SM to unify gravity with gauge interactions [1–3]. Particularly, in the brane-world theories, the D-brane inflation in warped compactification has attracted much attention recently [4, 6]. In that case, the inflation is identified as the separation between a brane and an antibrane, which is described by codimension-2 brane coupled to bulk metric [7, 8].

In this paper we consider the time and external spatial-dependent dimensions in a warped six-dimensional bulk. Based on the form of the brane-world energy momentum tensor proposed by Shiromizu *et al.* in five dimensions (5D) [9], the theory to the codimension-2 brane was extended to six dimensions (6D). Here 4-brane and 4-anti brane have been assumed to be an Anti de Sitter (AdS) space. The Einstein equations in 6D show that the velocities, the curvature and the geometry of extra space affect the evolution of Universe.

To describe the motion of two 4-branes moving in 6D Schwarzschild-AdS space, we consider the dynamical codimension-2 brane in a static bulk. The constrain relating of scalar factors in (1, 3+2) space is directly derived from these equations of motion.

A question which has also attracted recently much attention [10–12] is whether inflation is triggered by the energy density in the bulk or in the branes. We try discussing dynamics of inflation in the case where the matter content of the codimension-2 brane dominates that of the bulk.

This paper is organized as follows. In Sec. II we present the Einstein equation extended in six dimensions from the formalism of codimension-2 brane. The dynamical codimension-2 brane in a static bulk have been considered in Sec. III, where the system of equations of motion was derived. Sec. IV is devoted to discussing some cosmological consequences when the brane and antibrane energy density dominate that of the bulk. The conclusion is given in Sec. V.

II. FORMALISM

We start from the dynamical bulk metric in six dimensional space - time

$$ds^2 = -n^2dt^2 + R^2(t,z)d\Sigma_k^2 + \chi^2(t,z)dzd\bar{z} \tag{1}$$

where $z = y_1 + iy_2$, with y_1, y_2 are two extra dimensions, $d\Sigma_k^2$ represents the 3 dimensional spatial section metric.

We assume that the geometry of our theory is the 6D Riemann manifold. At $y_1 = 0$ or $y_2 = 0$ this manifold contains two 5D space - time, the 6D metric was compactified and considered by the 4D effective theory.

We impose $y_1 = \eta\sin\phi$, $y_2 = \eta\cos\phi$, here η and ϕ are the polar coordinates of two external dimensions, which could be compactified on a dimensional sphere (S^2/Z^2) [11].

Two probe brane and antibrane may be add in the bulk and can be centred around the axis of symmetry, like as around one of the background codimension - 2 brane.

We focus the cosmological dynamics on the brane and antibrane (located at $y_1 = y_2 = 0$) of the bulk metric, where the scalar factor R, χ of $(1,3+2)$ dimensions depend on both the time and two extra dimensions, i.e.

$$ds^2 = -n^2dt^2 + R^2(t,\eta,\phi)\left(\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right) + \chi^2(t,\eta,\phi)\left(d\eta^2 + \eta^2 d\phi^2\right) \tag{2}$$

where $k = -1,\ 0,\ 1$ is the curvature corresponding to the hyperbolic, flat and elliptic spaces, respectively.

We assume that the energy - momentum tensor, which satisfies the Einstein equation

$$G_{MN} = \kappa_6^2 T_{MN} \tag{3}$$

can be decomposed in two parts corresponding to the bulk and the branes as

$$T_N^M = T_N^M|_{Bulk} + T_N^M|_{branes} \tag{4}$$

where the Latin letters denote the 6D indices $M,N = 0,1,2,3$ (4D) and 5, 6 (for two extra dimensions)

$$T_N^M|_{Bulk} = \mathrm{diag}\left(-\rho_B, -\rho_B, -\rho_B, -\rho_B, -\rho_B, -\rho_B\right) \tag{5}$$

Here the energy density ρ_B is a constant, κ_6 is related with the 6D gravitational constant G_6 and the fundamental scale $M_{(6)}$ by

$$\kappa_6^2 = 8\pi G_{(6)} = M_{(6)}^{-4} \tag{6}$$

$T_N^M|_{branes}$ just is the energy - momentum tensor on the codimension - 2 brane

$$T_N^M|_{brane} = \frac{\delta(\rho)}{\chi}\mathrm{diag}\,(-\rho_b, p, p, p, p, \hat{p}, 0) \tag{7}$$

and

$$T_N^M|_{antibrane} = \frac{\delta(\phi-\pi)}{\rho\chi}\mathrm{diag}\,(-\rho_b, p, p, p, 0, \hat{p}) \tag{8}$$

where p is the pressure in 3D space, ρ_b is the energy density and $\hat{p}$ - pressure in the extra brane and antibrane directions.

The Einstein equations extended in 6D take the form

$$\begin{aligned} 3\left[\left(\frac{\dot{R}}{R}\right)^2 + \left(\frac{R'\eta}{R}\right) + \left(\frac{R'\phi}{R}\right)^2 + k\frac{n^2}{R^2}\right] + \left[\left(\frac{\dot{\chi}}{\chi}\right)^2 + \left(\frac{\chi'_\eta}{\chi}\right)^2 + \left(\frac{\chi'_\phi}{\chi}\right)^2\right] \\ -3\left(\frac{R''_\rho}{R} + \frac{R''_\phi}{R}\right)\left(\frac{\chi''_\eta}{\chi} + \frac{\chi''_\phi}{\chi}\right) - 3\frac{\dot{n}}{n}\frac{\dot{R}}{R} - 2\frac{\dot{n}}{n}\frac{\dot{\chi}}{\chi} + 6\frac{\dot{R}}{R}\frac{\dot{\chi}}{\chi} = \kappa_6^2\rho_B \end{aligned} \tag{9}$$

$$\begin{aligned} 2\frac{\ddot{R}}{R} + 2\frac{\ddot{\chi}}{\chi} - \left(\frac{R''_\eta}{R} + \frac{R''_\phi}{R} + \frac{\chi''_\eta}{\chi} + \frac{\chi''_\phi}{\chi}\right) \\ + \left[\left(\frac{\dot{R}}{R}\right)^2 + \left(\frac{\dot{\chi}}{\chi}\right)^2 + 3\left(\frac{R'_\eta}{R}\right)^2 + \left(\frac{\chi'_\eta}{\chi}\right)^2 + \left(\frac{\chi'_\phi}{\chi}\right)^2 + \frac{k}{R^2}\right] \\ -3\frac{\dot{n}}{n}\frac{\dot{R}}{R} - 2\frac{\dot{n}}{n}\frac{\dot{\chi}}{\chi} + 4\frac{\dot{R}}{R}\frac{\dot{\chi}}{\chi} = \kappa_6^2 p. \end{aligned} \tag{10}$$

By using the conditions (7) and (8), we derive the equations on the brane and antibrane, respectively

$$\begin{aligned} 3\frac{\ddot{R}}{R} + \frac{\ddot{\chi}}{\chi} - 3\frac{R''_\phi}{R} + 3\left[\left(\frac{\dot{R}}{R}\right)^2 + \left(\frac{R'_\eta}{R}\right)^2 + \left(\frac{R'_\phi}{R}\right)^2 + \frac{k}{R^2}\right] \\ -3\frac{\dot{n}}{n}\frac{\dot{R}}{R} - \frac{\dot{n}}{n}\frac{\dot{\chi}}{\chi} + 3\left(\frac{\dot{R}}{R}\frac{\dot{\chi}}{\chi} + \frac{R'_\eta}{R}\frac{\chi'_\eta}{\chi}\right) = \kappa_6^2\hat{p}. \end{aligned} \tag{11}$$

$$\begin{aligned} 3\frac{\ddot{R}}{R} + \frac{\ddot{\chi}}{\chi} - 3\frac{R''_\rho}{R} + 3\left[\left(\frac{\dot{R}}{R}\right)^2 + \left(\frac{R'_\eta}{R}\right)^2 + \left(\frac{R'_\phi}{R}\right)^2 + \frac{k}{R^2}\right] \\ -3\frac{\dot{n}}{n}\frac{\dot{R}}{R} - \frac{\dot{n}}{n}\frac{\dot{\chi}}{\chi} + 3\left(\frac{\dot{R}}{R}\frac{\dot{\chi}}{\chi} + \frac{R'_\phi}{R}\frac{\chi'_\phi}{\chi}\right) = \kappa_6^2\hat{p}. \end{aligned} \tag{12}$$

It shows that the evolution of Universe depends on the velocities $a(t) = \frac{\dot{R}}{R}$, $b(t) = \frac{\dot{\chi}}{\chi}$, $c(t) = \frac{\dot{n}}{n}$; on the curvature and the geometry of extra space. However, these equations can not be integrated analytically.

III. DYNAMICAL CODIMENSION - 2 BRANE IN A STATIC BULK

We consider the motion of codimension 2 brane as the separation between a brane and antibrane moving in 6D Schwarzschild - AdS ($SAdS_6$) space, whose metric is given by

$$ds^2 = -h(z)dt^2 + \frac{z^2}{L^2}d\Sigma_\kappa^2 + h^{-1}(z)dzd\bar{z}. \tag{13}$$

We focus on configuration, where the symmetry in the brane and antibrane directions are maximum, and $dzd\bar{z} = d\eta^2 + \eta^2 d\phi^2$, $z = |\eta|$. So we have

$$h(\eta) = k + \frac{\eta^2}{L^2}d\Sigma_k^2 - \frac{M_{(6)}}{\eta^4}. \tag{14}$$

In the Schwarzschild coordinates, the metric (13) reads

$$ds^2 = -h(\eta)dt^2 + \frac{\eta^2}{L^2}d\Sigma_\kappa^2 + h^{-1}(\eta)d\eta^2. \tag{15}$$

where

$$d\Sigma_\kappa^2 = \frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin\theta^2 d\varphi^2\right) + (1-kr^2)\eta^2 d\phi^2 \tag{16}$$

Comparing (16) with the metric (1) yields

$$n = \sqrt{h(\eta)}; \quad R = \frac{\eta^2}{L^2}; \quad \chi = \sqrt{h^-1(\eta)}. \tag{17}$$

Applying Israel junction conditions [12] for two moving branes

$$\left[K_N^M\right] = -\kappa_6^2\left(T_N^M - \frac{1}{4}Th_N^M\right) \tag{18}$$

where $T \equiv T_{MN}g^{MN}$; h_M^N is the induced on each 4-branes. Similarly to the case of five dimensions (5D) for $z = \eta(t)$, one have got [13]

$$-\frac{h' - 2\ddot{\eta}}{\sqrt{h^2 + \dot{\eta}^2}} = -\,\kappa_6^2\left(\frac{3}{4}\rho_b - p\right), \tag{19}$$

$$-2\frac{\sqrt{h + \dot{\eta}^2}}{\eta} = \frac{\kappa_6^2}{4}\rho_b. \tag{20}$$

The Friedmann - type equation in six dimensions takes the form

$$2\frac{\ddot{\eta}}{\eta} + 3\left(\frac{\dot{\eta}}{\eta}\right)^2 = -3\frac{\kappa_6^4}{64}\rho_b^2 + \frac{\kappa_6^4}{8}p\rho_b - 3\frac{k}{\eta^2} - \frac{5}{L^2} \tag{21}$$

The last term in (21) corresponds to energy - momentum tensor in the codimension - 2 brane.

Since $\delta(\rho) = 0$, $\delta(\phi - \pi) = 0$ in the bulk, then

$$\kappa_6^2 T_{\eta\eta} = \kappa_6^2 T_{\phi\phi} = \kappa_6^2 \Lambda_6, \tag{22}$$

where Λ_6 is the 6D bulk cosmological constant.

The L size of $SAdS_6$ space is determined from

$$\kappa_6^2\Lambda_6 = -\frac{10}{L^2}. \tag{23}$$

To describle the motion of codimension - 2 brane as two 4-branes living in a 6D space - time, we consider the metric (1), where the position of codimension - 2 brane at any bulk time t is noted by $z = \eta(t)$. The proper time in each brane is defined by

$$n^2(t)\dot{t}^2 - \chi^2(t,\eta(t))\,\dot{\eta}^2 = 1, \tag{24}$$

$$n^2(t)\dot{t}^2 - \eta^2\chi^2(t,\eta(t))\,\dot{\phi}^2 = 1, \tag{25}$$

where $\dot{t} = \frac{dt}{d\tau}$, $\dot{\eta} = \frac{d\eta}{d\tau}$, $\dot{\phi} = \frac{d\phi}{d\tau}$.

The metric on the brane takes the form

$$ds^2 = -d\tau^2 + R^2(t,\eta)d\Sigma_3^2 + \chi^2(t,\eta)d\eta^2 \tag{26}$$

Similarly, on the antibrane it reads

$$ds'^2 = -d\tau^2 + R^2(t,\eta)d\Sigma_3^2 + \eta^2\chi^2(t,\eta)d\phi^2 \tag{27}$$

The extrinsic curvature tensor on codimension 2 brane is given by [14]

$$\kappa_N^M = h_M^L\nabla_L n_N = {h'}_M^L\nabla_L n'_N \tag{28}$$

where n and n' are unitary vector fields normal to the brane and antibrane world - sheet

$$h_{MN} = g_{MN} - n_M n_N; {h'}_{MN} = g_{MN} - {n'}_M {n'}_N. \tag{29}$$

We introduce the unitary velocity vectors corresponding to these branes

$$u^M = \{\dot{t},0,0,0,\dot{\eta},0\}, \tag{30}$$

$$u'^M = \{\dot{t},0,0,0,0,\dot{\eta}\}. \tag{31}$$

Due to the conditions

$$g_{MN}n^M n^N = 1, \quad g_{MN}n^M n^N = 0, \tag{32}$$

$$g_{MN}n'^M n'^N = 1, \quad g_{MN}n'^M n'^N = 0 \tag{33}$$

the componets of unit vector fields are determined

$$n_M = -\{n\chi\dot{\eta},0,0,0,n\chi\dot{t},0\}, \tag{34}$$

$$n_M = -\left\{n'\chi\eta\dot{\eta},0,0,0,0,n\chi\eta\dot{t}\right\}. \tag{35}$$

By using the metric (25), (26) for the brane and the antibrane, we can find the components of the extrinsix curvature (see Appendix A).

Applying the Darmoris - Israel condition

$$\kappa_{MN} = -\kappa_6^2 T_{MN}, \tag{36}$$

where the energy - momentum tensor T_{MN} on each brane is defined by (18) and was given in (7), (8), we obtain the system of equation of motion of codimension - 2 brane

$$\frac{\chi^2\dot{\chi}\dot{\eta}^3-\chi\ddot{\eta}}{\sqrt{1+\chi^2\dot{\eta}^2}}-\frac{\sqrt{1+\chi^2\dot{\eta}^2}}{n}\left(\dot{n}\dot{\eta}\chi-\frac{d_\eta n}{\chi}-\dot{\eta}^2\left(\chi d_\eta n-nd_\eta\chi\right)\right)=-\frac{\kappa_6^2}{8}\left(3\left(\rho_b+p\right)+\hat{p}\right), \quad (37)$$

$$\frac{d_\eta R}{\chi R}\sqrt{1+\chi^2\dot{\eta}^2}=-\frac{\kappa_6^2}{8}\left(\rho_b+p-\hat{p}\right) \quad (38)$$

$$\frac{1}{\chi}\left(\frac{d_\eta\eta}{\eta}+\frac{d_\eta\chi}{\chi}\right)\sqrt{1+\chi^2\dot{\eta}^2}=-\frac{\kappa_6^2}{8}\left(\rho_b-3\left(p-\hat{p}\right)\right) \quad (39)$$

In the case of a 4-brane, this approach leads to result equivalent to that obtained for static brane in a dynamical bulk [15].

It is easily to see that the static 4-brane and 4-antibrane $(\dot{n}=\dot{\eta}=\dot{\chi}=0)$ are special cases of codimension 2-brane. Futhermore, it is shown that the contribution of matter on branes $(\hat{p}\neq 0)$ is significantly in the evolution of Universe.

Combine (38) and (39), we derive the relation of scalar factors $R(t,\eta)$ and $\chi(t,\eta)$ in a $(1,3+2)$ dimensional bulk (see Appendix B)

$$R=C\left[\eta\chi\right]^{(\rho_b+p+\hat{p})/(\rho_b-3(p-\hat{p}))} \quad (40)$$

where C is an integration constant.

If $R(\eta)=\chi(\eta)=\dfrac{\eta}{L}$ then the 6D bulk has S^4 symmetry and a 6D SAdS black hole solution [16]

IV. SOME COSMOLOGICAL CONSEQUENCES

Let us discuss the consequences when assume that Friedmann - type equation

$$2\frac{\ddot{\eta}}{\eta}+3\left(\frac{\dot{\eta}}{\eta}\right)^2=-3\frac{\kappa_6^4}{64}\rho_b^2-\frac{\kappa_6^4}{8}p\rho_b-3\frac{k}{\eta^2}+\frac{\kappa_6^2}{3}\Lambda_6 \quad (41)$$

can be applied to our present universe. The energy density ρ_b is rewritten as a fraction of the critical density ρ_c which is usually defined by

$$\rho_c=\frac{3H^2}{8\pi G_{(6)}} \quad (42)$$

such that

$$\rho_b=\Omega\rho_c \quad (43)$$

and

$$H^2=\frac{8\pi G_{(6)}}{3}\rho_b-\frac{k}{\eta^2}+\frac{\Lambda_6}{3} \quad (44)$$

Equation (41) becomes

$$2\dot{H}+5H^2=-\frac{1}{64}\lambda^2H^2\Omega^2\left(3+8\omega\right)-3\frac{k}{\eta^2}+\frac{\kappa_6^2}{3}\Lambda_6 \quad (45)$$

where

$$\lambda = \frac{M_p H}{M_{(6)}^2} \tag{46}$$

$$M_p^2 = \left(8\pi G_{(6)}\right)^{-1} = 2M_{(6)}^4 R_{(6)} \tag{47}$$

here the crossover distance $R_{(6)}$ between two regimes is [17]

$$R_{(6)} = \frac{M_p}{2M_{(6)}^2} \tag{48}$$

The left hand side of Eq. (11) can be expressed in terms of acceleration parameter defined by

$$q = -\left(\frac{2\ddot{a}a}{3\dot{a}^2}\right)_{\text{now}} \tag{49}$$

so that

$$\left(2\dot{H} + 5H^2\right)_{\text{now}} = 3(1-q)H^2 \tag{50}$$

Since the cosmological observations constrain $|q|$ to be of the order 1 (Ω_{now} being of the order 1) then the present matter content has influence to the dynamic of its geometry if λ was at least of order 1, when sixth dimension radius is of the order the present Hubble radius. A possible way out is to assume that the bulk term dominated in Eq (41) drive the dynamics of our universe [18].

Extending directly the investigation in 5D [15], we focus on the case where the matter content of codimension - 2 brane dominated that of the 6D bulk. The initial conditions in early Universe suggest [14] that the branes inflation might require almost empty bulk following the condition

$$\rho_{\text{bulk}} M_{(6)}^{-2} < \rho_{\text{branes}}. \tag{51}$$

The universe dynamics will be dominated by the brane and antibrane energy density rather than the bulk pressure $p = \omega\rho_b$, which satisfies the condition

$$\rho_{\text{bulk}} \ll \frac{\rho_b^2}{M_{(6)}^2}. \tag{52}$$

That means the domination of codimension - 2 brane over bulk energy density might be imposed on a typical distance scale of order $M_{(6)}^2$

$$\rho_{\text{bulk}} R_{(6)} \ll \rho_{\text{branes}} \tag{53}$$

If the nature bound $\rho_b \leq M_{(6)}^3$ then

$$H \sim M_{(6)}^{-2} \rho_b \leq M_{(6)}. \tag{54}$$

The inflation mass, which must be less than H during inflation in order to fulfil slow - roll condition, will be less than the fundamental scale $M_{(6)}$.

V. CONCLUSION

In the above sections, we have presented a formalism for the probe codimension-2 brane in a warped six-dimensional space-time.

The Einstein equations have been extended in 6D, which shows that the cosmological evolution depends on the velocities, the curvature and the geometry of extra space. The system of equations of motion of 4-brane and 4-antibrane was derived when we consider dynamical codimension-2 brane in a static bulk. In the Schwarzschild coordinates, the 6D bulk has S_4 symmetry and a $SAdS_6$ black hole solution. Some cosmological consequences was discussed in the case where codimension-2 brane energy density dominated that of the bulk. This mechanism can drive the dynamics of Universe. In the case of 6D, the Hubble radius might be less than the fundamental scale $M_{(6)}$.

It is expected that this formalism could pave the way for better understanding of various physical phenomena of high energy physics.

REFERENCES

[1] Kaustubh Agashe, Alex Pomarol, *New J. Phys.* **12** (2010) 070510.
[2] Randall and Sundrum. *Phys. Rev. Lett.* **83** (1999) 4690 [arXiv: help-th/9906064].
[3] E.Witten, *Adv. Theor. Math. Phys.* **2** (1998) 253 [arXiv: help-th/9802150].
[4] H. M. Lee and A. Papazoglou, *Phys. Rev. D* **80** (2009) 043506 [arXiv: 09014692].
[5] C. Csaki, TAS1, Lectures on Extra Dimensions and Branes arXiv:hep-ph/04 04096, (2004).
[6] J. M. Cline, J. Descheneau, M. Jovannini and J. Vinat, *JHEP* **0306** (2003) 048 [arXiv: help-th/0304147].
[7] A. Bayntun, C. P. Burgess, and L. van Nierop, *New J. Phys.* **12** (2010) 075015.
[8] G. Dvali and S. H. Henry, *Phys. Lett. B* **450** (1999) 72.
[9] J. Shiromizu, K. Macda and M. Sasaki, *Phys. Rev. D* **62** (2002) 024012.
[10] C. Csaki, M. Graesser, J.Terning, *Phys. Rev. Lett.* **92** (2004) [arXiv: hep-ph/0308038].
[11] P. H. Lien, T. H. Phat and H.V. Chung, *J. Phys : Conf. Ser.* **627** (2015) 012007.
[12] W. Israel, *Nuovo Cimento B* **44** (1966) 1.
[13] C. Csaki, J. Erlich and C. Grojean, *Nucl. Phys. B* **604** (2001) 312 [arXiv: hep-th/0012143].
[14] E. W. Kolb and M. S. Turner, *The Early Universe* (1990) (Addison - Wesley, New York) pp. 270-275.
[15] Cuadros-Melga.B, Papantonobulous.E, *Brazilian J. Phys.* **35** (2005) 1117.
[16] Makoto Natsuume, "AdS/CFTEquilibrium." AdS/CFT Duality User Guide. Springer, Tokyo, pp. 101-115 (2015).
[17] Arkani-Hamed, S. Dimopoulos, N. Kaloper and J. March-Russel *Nuclear Physics B* **567** (2000) 189; hep-ph/9903224.
[18] P. Binétruy, C. Deffayet, D. Langlois (1999) [arXiv: help-th/9905012].

APPENDIX

Appendix A. The components of the extrinsic curvature

$$K_{\tau\tau} = -n\chi\dot{t}\ddot{\eta} + n\chi\dot{\eta}\ddot{t} - \frac{\dot{t}d_\eta\eta}{\chi} + \dot{\eta}^2\dot{t}\left(\chi d_\eta n - nd_\eta\chi\right) \tag{A.1}$$

$$K_{rr} = K_{\theta\theta} = K_{\varphi\varphi} = \frac{d_\eta R}{\chi R}\sqrt{1+\chi^2\dot{\eta}^2} \tag{A.2}$$

$$K_{\eta\eta} = \frac{d_\eta\eta}{\chi\eta}\sqrt{1+\chi^2\dot{\eta}^2} \tag{A.3}$$

$$K_{\phi\phi} = \frac{d_\eta\chi}{\chi^2}\sqrt{1+\chi^2\dot{\eta}^2} \tag{A.4}$$

Substituting $\dot{t}$, $\ddot{t}$ from Eq. (23) in to Eq. (35), $K_{\tau\tau}$ is written as

$$K_{\tau\tau} = \frac{\chi^2\dot{\chi}\dot{\eta}^3 - \chi\ddot{\eta}}{\sqrt{1+\chi^2\dot{\eta}^2}} - \frac{\sqrt{1+\chi^2\dot{\eta}^2}}{n}\left(\dot{n}\dot{\eta}\chi - \frac{\partial_\eta n}{\chi} - \dot{\eta}^2\left(\chi d_\eta n - nd_\eta\chi\right)\right). \tag{A.5}$$

Appendix B. The equation of motion of codimension - 2 brane

$$K_{\tau\tau} = -\frac{\kappa_6^2}{8}\left(3(\rho_b+p)+\hat{p}\right) \tag{B.1}$$

$$\frac{d_\eta R}{\chi R}\sqrt{1+\chi^2\dot{\eta}^2} = -\frac{\kappa_6^2}{8}\left(\rho_b+p-\hat{p}\right) \tag{B.2}$$

$$\frac{1}{\eta}\left(\frac{d_\eta\eta}{\eta}+\frac{d_\eta\chi}{\chi}\right)\sqrt{1+\chi^2\dot{\eta}^2} = -\frac{\kappa_6^2}{8}\left(\rho_b-3(p-\hat{p})\right) \tag{B.3}$$

Division (B.2) by (B.3) yields

$$\frac{d_\eta R}{R} = \frac{\rho_b+p-\hat{p}}{\rho_b-3(p-\hat{p})}\left(\frac{d_\eta\eta}{\eta}+\frac{d_\eta\chi}{\chi}\right) \tag{B.4}$$

$$\ln R = \frac{\rho_b+p-\hat{p}}{\rho_b-3(p-\hat{p})}\ln\eta\chi + \ln C \tag{B.5}$$

$$R = C(\eta\chi)^{(\rho_b+p-\hat{p})/(\rho_b-3(p-\hat{p}))} \tag{B.6}$$

where C is an integration constant.