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QUATERNIONIC REFORMULATION OF GENERALIZED SUPERLUMINAL ELECTROMAGNETIC FIELDS OF DYONS

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Abstract. *Superluminal electromagnetic fields of dyons are described in T^4- space and Quaternion formulation of various quantum equations is derived. It is shown that on passing from subluminal to superluminal realm via quaternion the theory of dyons becomes the Tachyonic dyons. Corresponding field Equations of Tachyonic dyons are derived in consistent, compact and simpler form.*

I. INTRODUCTION

The question of existence of monopoles (dyons) [1, 2, 3] has become a challenging new frontier and the object of more interest in connection with the current grand unified theories [4, 5], quark confinement problem of quantum chromo dynamics [6], the possible magnetic condensation of vacuum [7], their role as catalyst in proton decay [8] and supersymmetry [9, 10]. The eight decades of twentieth century witnessed a rapid development of the group theory and gauge field theory to establish the theoretical existence of monopoles and to explain their group properties and symmetries. Keeping in mind t' Hooft-Polyakov and Julia-Zee solutions [11, 12] and the fact that despite the potential importance of monopoles, the formalism necessary to describe them has been clumsy and not manifestly covariant, a self consistent quantum field theory of generalized electromagnetic fields associated with dyons (particle carrying electric and magnetic charges) of various spins has been developed [13, 14] by introducing two four potentials [15] and avoiding string variables [16] with the assumption of the generalized charge, generalized four-potential, generalized field tensor, generalized vector field and generalized four-current density associated with dyons as complex quantities with their real and imaginary parts as electric and magnetic constituents. On the other hand, there has been continuing interest [17, 18, 19, 20] in higher dimensional kinematical models for proper and unified theory of subluminal (bradyon) and superluminal (tachyon) objects [21, 22]. The problem of representation and localizations of superluminal particles has been solved only by the use of higher dimensional space [23, 24, 25] and it has been claimed that the localization space for tachyons is T^4- space with one space and three times while that for bradyon is R^4-space in view of localizability and of these particles.

Introducing the concepts of superluminal Lorentz transformations, need of higher dimensional space-time, localizability of bradyons and tachyons, in the present paper, we have under taken the study of generalized fields of dyons under superluminal Lorentz transformations ($SLTs$). It has been shown that the generalized electromagnetic fields behave in frame K' (i.e. the superluminal frame) as subluminal fields do in frame K (subluminal frame). As such, the generalized fields, when viewed upon by an observer in bradyonic frame, appear as superluminal fields and thus, satisfy the field equations different from Maxwell's equations used for electric charge (or magnetic monopole) and generalized Dirac-Maxwell's (GDM) equations of dyons. Hence, it is concluded that the superluminal electromagnetic fields are not same as the familiar electric and magnetic fields associated with electric charge (or magnetic monopole) and dyons of our every day experience. It is shown that the superluminal electromagnetic fields and field equations are no more invariant under SLTs with the chronological mapping of space-time on passing from subluminal to superluminal realm. It is has been emphasized that in order to retain the Lorentz invariance of field equations, we are forced to include extra negative sign to the components of four-current densities for electric charge (or magnetic monopole) and dyons respectively. It is also concluded that though the roles of electric and magnetic charges are not changed while passing from subluminal to superluminal realm under $SLTs$, a dyon interacting with superluminal electromagnetic fields behaves neither as electric charge, nor as pure magnetic monopole but having the mixed behaviour of electric and magnetic charges, rather, namely a tachyonic dyon. Describing the need of higher dimensional and localizability spaces for bradyons and tachyons respectively as $R^4-$and $T^{4}-$ spaces, we have obtained superluminal electromagnetic fields in T^4-spaces and derived the consistent and manifestly covariant field equations and equation of motion where the velocity is described as reversed velocity. Starting with the quaternionic form of generalized four-potential of dyons, we have developed the simple and compact quaternionic form of Maxwell's equations and it has been shown that while passing from usual four space to quaternionic formulation the signature of four-vector is changed from $(+,-,-,-)$ to $(-,-,-,+)$. Hence, the quaternionic formulation and superluminal behaviour have striking similarities. The corresponding quaternionic field equations of bradyonic and tachyonic dyons are derived consistently in R^4- and T^4- spaces respectively in consistent, simple and compact formulations. These quaternionic formulations reproduce the theories of electric (magnetic) charge in the absence of magnetic (electric) charge or vice versa on dyons in $R^4-$and T^4- spaces.

II. FIELD ASSOCIATED WITH DYONS

Let us define the generalized charge on dyons as [13, 14]

$$q = e - ig \; (i = \sqrt{-1}) \tag{1}$$

where e and g are respectively electric and magnetic charges. Generalized four - potential $\{V_\mu\} = \{\phi, \overrightarrow{V}\}$associated with dyons is defined as,

$$V_\mu = A_\mu - iB_\mu \tag{2}$$

where $\{A_\mu\} = \{\phi_e, \overrightarrow{A}\}$ and $\{B_\mu\} = \{\phi_g, \overrightarrow{B}\}$ are respectively electric and magnetic four - potentials. We have used throughout the natural units $c = \hbar = 1$ and Minikowski space id described with the signature $(-, +, +, +)$. Generalized electric and magnetic fields of dyons are defined in terms of components of electric and magnetic potentials as,

$$\begin{aligned}
\overrightarrow{E} &= -\frac{\partial \overrightarrow{A}}{\partial t} - \overrightarrow{\nabla}\phi_e - \overrightarrow{\nabla} \times \overrightarrow{A}, \\
\overrightarrow{H} &= -\frac{\partial \overrightarrow{B}}{\partial t} - \overrightarrow{\nabla}\phi_g + \overrightarrow{\nabla} \times \overrightarrow{B}.
\end{aligned} \tag{3}$$

These electric and magnetic fields of dyons are invariant under generalized duality transformation i.e.

$$\begin{aligned}
A_\mu &\longrightarrow A_\mu \cos\theta + B_\mu \sin\theta \\
B_\mu &\longrightarrow A_\mu \sin\theta + B_\mu \cos\theta.
\end{aligned} \tag{4}$$

The expression of generalized electric and magnetic field given by equation (3) are symmetrical and both the electric and magnetic field of dyons may be written in terms of longitudinal and transverse components. The generalized field vector $\overrightarrow{\psi}$ associated with dyons is defined as

$$\overrightarrow{\psi} = \overrightarrow{E} - i\overrightarrow{H} \tag{5}$$

and accordingly, we get the following differential form of generalized Maxwell's equations for dyons i.e.

$$\begin{aligned}
\overrightarrow{\nabla}.\overrightarrow{\psi} &= J_0, \\
\overrightarrow{\nabla} \times \overrightarrow{\psi} &= -i\overrightarrow{J} - i\frac{\partial \overrightarrow{\psi}}{\partial t}
\end{aligned} \tag{6}$$

where J_0 and $\overrightarrow{J}$, are the generalized charge and current source densities of dyons, given by

$$J_\mu = j_\mu - ik_\mu \equiv \{J_0, \overrightarrow{J}\}. \tag{7}$$

Substituting relation (3) into equation (5) and using equation (2), we obtain the following relation between generalized field vector and generalized potential of dyons i.e.

$$\overrightarrow{\psi} = -\overrightarrow{\nabla}\phi - \frac{\partial \overrightarrow{V}}{\partial t} - i\overrightarrow{\nabla} \times \overrightarrow{V}. \tag{8}$$

In equation (8) , $\{j_\mu\} = (\rho_e, \overrightarrow{j})$ and $\{k_\mu\} = (\rho_g, \overrightarrow{k})$ are electric and magnetic four current densities. Thus we write the following tensor forms of generalized Maxwell's - Dirac equations of dyons [13, 14] i.e.

$$\begin{aligned}
F_{\mu\nu,\nu} &= j_\mu \\
\widetilde{F_{\mu\nu,\nu}} &= k_\mu
\end{aligned} \tag{9}$$

where

$$\begin{aligned} F_{\mu\nu} &= E_{\mu\nu} - \widetilde{H_{\mu\nu}}, \\ \widetilde{F_{\mu\nu}} &= H_{\mu\nu} + \widetilde{E_{\mu\nu}} \end{aligned} \tag{10}$$

with

$$\begin{aligned} E_{\mu\nu} &= \partial_\nu A_\mu - \partial_\mu A_\nu, \\ H_{\mu\nu} &= \partial_\nu B_\mu - \partial_\mu B_\nu, \\ \widetilde{E_{\mu\nu}} &= \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda} E^{\rho\lambda}, \\ \widetilde{H_{\mu\nu}} &= \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda} H^{\rho\lambda}. \end{aligned} \tag{11}$$

The tidle ($\sim$) denotes the dual part while $\varepsilon_{\mu\nu\rho\lambda}$ are four index Levi - Civita symbol. Generalized fields of dyons given by equations (3) may directly be obtained from field tensors $F_{\mu\nu}$ and $F^d_{\mu\nu}$ as,

$$\begin{aligned} F_{0i} &= E^i, \\ F_{ij} &= \varepsilon_{ijk} H^k, \\ H^d_{0i} &= -H^i, \\ H^d_{ij} &= -\varepsilon_{ijk} E^k. \end{aligned} \tag{12}$$

Taking the curl of second equation of (6) and using first equation of (6), we obtain a new vector parameter $\overrightarrow{S}$ (say) i.e.

$$\overrightarrow{S} = \Box \overrightarrow{\psi} = -\frac{\partial \overrightarrow{\psi}}{\partial t} - i \overrightarrow{\nabla} \times \overrightarrow{J} - \overrightarrow{\nabla} \rho \tag{13}$$

where $\Box$ represents the D'Alembertian operator i.e.

$$\Box = \nabla^2 - \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}. \tag{14}$$

Defining generalized field tensor as

$$G_{\mu\nu} = F_{\mu\nu} - i \widetilde{F_{\mu\nu}} \tag{15}$$

one can directly obtain the following generalized field equation of dyons i.e.

$$\begin{aligned} G_{\mu\nu,\nu} &= J_\mu, \\ \widetilde{G_{\mu\nu,\nu}} &= 0 \end{aligned} \tag{16}$$

where $G_{\mu\nu} = V_{\mu,\nu} - V_{\nu,\mu}$ is called the generalized electromagnetic field tensor of dyons. Equation (16) may also be written as follows like second order Klein-Gordon equation for dyonic fields

$$\Box V_\mu = J_\mu \qquad (in\, Lorentz\, gauge) \tag{17}$$

Equations (9) and (16) are also invariant under duality transformations;

$$(F, F^d) \longrightarrow (F\cos\theta + F^d \sin\theta, F\sin\theta - F^d\cos\theta), \tag{18}$$

$$(j_\mu, k_\mu) \longrightarrow (j_\mu\cos\theta + k_\mu\sin\theta, j_\mu\sin\theta - k_\mu\cos\theta) \tag{19}$$

where

$$\frac{g}{e} = \frac{B_\mu}{A_\mu} = \frac{k_\mu}{j_\mu} = -\tan\theta = constant. \tag{20}$$

Consequently the generalized charge of dyons may be written as

$$q = |q| e^{-i\theta}. \tag{21}$$

The suitable Lagrangian density, which yields the field equation (16) under the variation of field parameters i.e. potential only without changing the trajectory of particle, may be written as follows;

$$\mathcal{L} = -m_0 - \frac{1}{4} G_{\mu\nu} G^{\star}_{\mu\nu} + V^{\star}_{\mu} J_\mu \tag{22}$$

where m_0 is the rest mass of the particle and $(\star)$ denotes the complex conjugate. Lagrangian density given by equation (22) directly follows the following form of Lorentz force equation of motion for dyons i.e.

$$f_\mu = m_0 \ddot{x}_\mu = Re\,(q^{\star} G_{\mu\nu}) u^{\nu} \tag{23}$$

where Re denotes real part, $\{\ddot{x}_\mu\}$ is the four - acceleration and $\{u^\nu\}$ is the four - velocity of the particle.

III. DYONIC FIELD EQUATIONS UNDER SUPERLUMINAL LORENTZ TRANSFORMATION

Special theory of relativity has been extended in a straight forward manner to superluminal inertial frames and it has been shown that the existence of tachyons (particles moves faster than light) does not violate the theory of relativity while their detection may require a modification in certain established motion of causality. In deriving Superluminal Lorentz transformation with relative velocity between two frames greater than velocity of light, two main approaches are adopted by different authors. In the first one adopted by Recami et al [26], the components of a four vector field in the direction perpendicular to relative motion become imaginary on passing from subluminal to superluminal realm while in the second approach followed by Antippa-Everett [27] the real superluminal Lorentz transformation are used. In the light of Gorini's theorem [28] and the conclusion and Pahor and Strnad [29], that with the real transformations either the speed of light is not invariant or relative velocity between the frames does not have a meaning, the superluminal Lorentz transformation of Racami et al [26] are closer to the spirit of relativity in comparison to the real ones . In order to examine the invariance of generalized Maxwell's equations of dyons under imaginary superluminal Lorentz transformations [26], let us introduce two inertial frames K and K' whose axes are parallel and whose origins coincide at $t = t' = 0$. Let K' moves with respect to K with a superluminal velocity $v > 1$ directed along Z - axis, the transformation equations between the coordinates of an event as seen in K' and those of the same event in K, may be written as follows [26],

$$\begin{aligned} x'_j &= \pm i x_j, \; (j = 1, 2) \\ x'_3 &= \pm \gamma (x_3 - vt), \; (v > 1) \\ t' &= \pm \gamma (t - x v_3) \end{aligned} \tag{24}$$

where

$$\gamma = (v^2 - 1)^{-\frac{1}{2}}. \tag{25}$$

From these transformations we have

$$-(t')^2 + (x')_1^2 + (x')_2^2 + (x')_3^3 = t^2 - x_1^2 - x_2^2 - x_3^2 \tag{26}$$

which shows that the reference metric $(-1, +1, +1, +1)$ in frame K is transformed to the metric $(1, -1, -1, -1)$in frame K' with the transformations (24) and the roles of space and time get interchanged for superluminal transformations. In other words, the superluminal transformations lead to chronological mapping [23, 24, 26, 30, 17],

$$(3, 1) \longleftrightarrow (1, 3) \tag{27}$$

for the components of space and time on passing from subluminal to superluminal realm or vice versa and thus describes

$$(x, y, z, t) \rightarrow (t', ix', iy', iz') \tag{28}$$

from which we get

$$\Box = -\Box' \tag{29}$$

and the mapping

$$(\overrightarrow{\nabla}, i\frac{\partial}{\partial t}) \rightarrow (\frac{\partial}{\partial t'}, i\overrightarrow{\nabla}). \tag{30}$$

Similar superluminal transformations may be derived for the components of four potentials (electric, monopole, dyon) and we may obtain that

$$\begin{aligned} |A'_\mu|^2 &= -|A_\mu|^2, \\ |B'_\mu|^2 &= -|B_\mu|^2, \\ |V'_\mu|^2 &= -|V_\mu|^2 \end{aligned} \tag{31}$$

with the correspondence $(3, 1) \longleftrightarrow (1, 3)$ mapping we get

$$\begin{aligned} (A_1, A_2, A_3, i\phi_e) &\rightarrow (\phi'_e, A'_1, A'_2, A'_3), \\ (B_1, B_2, B_3, i\phi_g) &\rightarrow (\phi'_g, B'_1, B'_2, B'_3), \\ (V_1, V_2, V_3, i\phi) &\rightarrow (\phi', V'_1, V'_2, V'_3). \end{aligned} \tag{32}$$

Using relation (26-32) and the similar mapping for the component four current densities (electric, monopole, dyon) we may transform the Maxwell's equation given by equation (17) in frame K to the following equation in frame K' i.e.;

$$\begin{aligned} \Box' A'_\mu &= -j'_\mu, \\ \Box' B'_\mu &= -k'_\mu, \\ \Box' V'_\mu &= -J'_\mu, \end{aligned} \tag{33}$$

which are the equations according to which the superluminal electromagnetic fields associated respectively with electric charge , magnetic monopole and dyon are coupled to their tachyonic counterparts which may be considered as bradyons in superluminal frame K' in view of tachyon-bradyon reciprocity and, therefore, these electromagnetic fields behave in frame K' as the subluminal fields do in frame K. These fields, when viewed upon by an observer in frame K (Bradyonic frame), appear as superluminal electromagnetic fields and satisfy the field equations (33), which differs from the usual filed equations respectively for electric charge, magnetic monopole and dyon. As such, it may be concluded that the superluminal electromagnetic fields are not the same as the familiar electric and magnetic fields of electric charge, magnetic monopole and dyons of our daily experience and obey Maxwell type equations in subluminal frame of reference. Consequently the field equations are no more invariant under imaginary superluminal transformations and to retain the Lorentz invariance of field equations we are forced to include extra negative sign to the components of four current densities for electric charge, magnetic monopole and dyon respectively with incorporating the following mappings;

$$\begin{aligned}(j_1, j_2, j_3, i\rho_e) &\rightarrow -(\rho'_e, j'_1, j'_2, j'_3),\\ (k_1, k_2, k_3, i\rho_g) &\rightarrow -(\rho'_g, k'_1, k'_2, k'_3),\\ (J_1, J_2, J_3, i\rho) &\rightarrow -(\rho', J'_1, J'_2, J'_3).\end{aligned} \tag{34}$$

Despite the change in sign, the real and imaginary components of four current densities lead to corresponding real and imaginary components of the four potentials. The change in sign of charge and current densities leaves the total charge invariant as the volume element also changes the sign under imaginary superluminal Lorentz transformations. This change of sign in the components of four current densities may lead to the conclusion that the filed equations may be treated as invariant on passing from subluminal to superluminal realm or vice versa [31]. If we use the mappings given by equations (27,28,30 and 32) the generalized electric and magnetic fields of dyons for superluminal case take the following expressions for generalized superluminal electromagnetic fields;

$$\begin{aligned}\overrightarrow{E}' &= -grad'\phi'_e - \frac{\partial \overrightarrow{A}'}{\partial t'} - \frac{\partial}{\partial t'}\phi'_g\hat{n_g},\\ \overrightarrow{H}' &= -grad'\phi'_g - \frac{\partial \overrightarrow{B}'}{\partial t'} - \frac{\partial}{\partial t'}\phi'_e\hat{n}_e\end{aligned} \tag{35}$$

where $\hat{n}_e$ and $\hat{n}_g$ are unit vectors in the direction of electric and magnetic fields associated with electric and magnetic charges. These equations are different from those obtained earlier by Negi-Rajput [23] derived for electric charge only. These are also not exactly same as given by equations (3) for generalized subluminal electric and magnetic fields of dyons but shows the striking symmetry between the electric and magnetic fields of dyons under superluminal transformations and may thus be visualized as the generalized superluminal electromagnetic field of dyons in frame K' when viewed from subluminal frame K. As such, it may be concluded that though the roles of electric and magnetic charges are not changed while passing from subluminal to superluminal realm under the superluminal transformations, a dyon interacting with superluminal electromagnetic fields containing symmetrical electromagnetic fields behaves neither as electric charge nor as pure magnetic monopole [32, 33] but with mixed behaviors of electric and magnetic charges rather namely a tachyonic dyon. As such we agree with Negi-Rajput [23] that even in the case of a dyon interacting with generalized superluminal electromagnetic fields, a tachyonic electric charge can not behave as a bradyonic magnetic monopole or vice versa. Neither it behaves exactly as dyons interacting with superluminal fields as the components of electric and magnetic potential get mixed in different manner for generalized superluminal electromagnetic fields. We do not have any alternative left but to say that it is a kind tachyonic dyon interacting with inconsistent natures of superluminal electromagnetic fields where rotational (*curl*of vector potentials) counter parts of electric and magnetic field do not occur. Transforming the force equation of a dyon in frame K i.e.

$$\overrightarrow{F} = e(\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{H}) + g(\overrightarrow{H} - \overrightarrow{v} \times \overrightarrow{E}), \tag{36}$$

we get the following equation of force in frame K'i.e.,

$$\overrightarrow{F}' = e(\overrightarrow{E}' + \overrightarrow{v}' \times \overrightarrow{H}') + g(\overrightarrow{H}' - \overrightarrow{v}' \times \overrightarrow{E}'), \tag{37}$$

under the mapping of fore said superluminal transformations in the case of electric charge [23] where the velocity becomes the inverse velocity $\overrightarrow{v}' = \overleftarrow{\omega} = \frac{dt}{dz}$. Equations (35) for superluminal electromagnetic fields derived by using transformations (24) and corresponding mappings are not consistent and do not describe the isotropic components of electric and magnetic field vectors in all directions. On the other hand, the components of a position four - vector become imaginary in the direction perpendicular to relative motion between frame K and K'. Similarly the perpendicular components of four - potential, four - force, four - current and electromagnetic fields become imaginary on passing from sub to superluminal realm via these transformations. A lot of literature is also available [17, 34, 35] for the justification of imaginary superluminal transformations. Despite of justifications, it is concluded that when we are prepared to consider the tachyonic objects , we must give up the idea that dynamical quantities or variables in classical mechanics are always real.

To over come the various problems associated with both type of superluminal Lorentz transformations, six - dimensional formalism [36, 37, 38, 39, 40] of space - time is adopted with the symmetric structure of space and time having three space and three time components of a six dimensional space time vector. The resulting space for bradyons and tachyons is identified as the R^6- or $M(3,3)$ space where both space and time and hence energy and momentum are considered as vector quantities. Superluminal Lorentz transformations (SLTs) between two frames K and K' moving with velocity $v > 1$ are defined in R^6- or $M(3,3)$ space as follows;

$$\begin{aligned} x' &= \pm t_x, \\ y' &= \pm t_y, \\ z' &= \pm\gamma(z - vt), \\ {t_x}' &= \pm x, \\ {t_y}' &= \pm y, \\ {t_z}' &= \pm\gamma(t - vz). \end{aligned} \tag{38}$$

These transformations lead to the mixing of space and time coordinates for transcendental tachyonic objects, ($|\overrightarrow{v}| \to \infty$ or $\overrightarrow{\omega} \to 0$) where equation (38) takes the following form;

$$\begin{aligned} +dt_x &\rightarrow dt_{x'} = dx + \\ +dt_y &\rightarrow dt_{y'} = dy + \\ +dt_z &\rightarrow dt_{z'} = dz + \\ -dz &\rightarrow dz' = dt_x - \\ -dy &\rightarrow dy' = dt_y - \\ -dx &\rightarrow dy' = dt_z - . \end{aligned} \tag{39}$$

It shows that we have only two four dimensional slices of R^6- or $M\,(3,3)$ space $(+,+.+,-)$ and $(-,-,-,+)$. When any reference frame describes bradyonic objects it is necessary to describe

$$M(1,3) \;=\; [t,x,y,z]. \qquad (R^4 - space)$$

So that the coordinates t_x and t_y are not observed or couple together giving $t = (t_x^2 + t_y^2 + t_z^2)^{\frac{1}{2}}$. On the other hand when a frame describes bradyonic object in frame K, it will describe a tachyonic object (with velocity $(|\overrightarrow{v}| \rightarrow \infty$ or $\overrightarrow{\omega} \rightarrow 0)$ in K' with $M'(1,3)$ space i.e.

$$M'(1,3) \;=\; [t_{z'}, x', y', z'] = [z, t_x, t_y, t_z]. \qquad (T^4 - space)$$

We define $M'(1,3)$ space as T^4- space or $M(3,1)$ space where x and y are not observed or coupled together giving rise to $r = (x^2 + y^2 + z)^{2\frac{1}{2}}$. As such, the spaces R^4 and T^4 are two observational slices of R^6- or $M(3,3)$ space but unfortunately the space is not consistent with special theory of relativity. Subluminal and superluminal Lorentz transformations loose their meaning in R^6- or $M(3,3)$ space with the sense that these transformations do not represent either the bradyonic or tachyonic objects in this space. It has been shown earlier [23, 24, 25] that the true localizations space for bradyons is R^4 - space while that for tachyons is T^4 - space. So a bradyonic $R^4 = M(1,3)$ space now maps to a tachyonic $T^4 = M'(3,1)$ space or vice versa.

$$R^4 = M(1,3) \overset{SLT}{\rightarrow} M'(3,1) = T^4. \tag{40}$$

In a similar manner the corresponding mapping for the components of electromagnetic potential in six - dimensional space may be written as

$$(\overrightarrow{V}, i\phi) \;\rightarrow\; (\overrightarrow{\phi}, iV) \tag{41}$$

where

$$\begin{aligned} \phi = |\overrightarrow{\phi}| &= (\phi_x^2 + \phi_y^2 + \phi_z^2)^{\frac{1}{2}} \\ V = |\overrightarrow{V}| &= (V_x^2 + V_y^2 + V_z^2)^{\frac{1}{2}}. \end{aligned}$$

The generalized four potential $\{\phi_\mu\} = \{\overrightarrow{\phi}, iV\}$ associated with tachyonic dyon defined as

$$\{\phi_\mu\} \;=\; \{\phi_\mu^e\} - i\{\phi_\mu^m\} \tag{42}$$

where

$$\{\phi_\mu^e\} = \{\overrightarrow{\phi^e}, iA\},$$
$$\{\phi_\mu^m\} = \{\overrightarrow{\phi^m}, iB\} \quad (43)$$

are the four - potentials associated with superluminal electric and magnetic charges respectively with

$$\phi^e = |\overrightarrow{\phi^e}| = (\phi_x^{e2} + \phi_y^{e2} + \phi_z^{e2})^{\frac{1}{2}},$$
$$A = |\overrightarrow{A}| = (A_x^2 + A_y^2 + A_z^2)^{\frac{1}{2}},$$
$$\phi^m = |\overrightarrow{\phi^m}| = (\phi_x^{m2} + \phi_y^{m2} + \phi_z^{m2})^{\frac{1}{2}},$$
$$B = |\overrightarrow{B}| = (B_x^2 + B_y^2 + B_z^2)^{\frac{1}{2}}. \quad (44)$$

Then the superluminal electric and magnetic fields of dyons in this formalism will be described as

$$\overrightarrow{E}_T = -\overrightarrow{\nabla}_t A - \frac{\partial \overrightarrow{\phi_e}}{\partial r} - \overrightarrow{\nabla}_t \times \overrightarrow{\phi_m},$$
$$\overrightarrow{H}_T = -\overrightarrow{\nabla}_t B - \frac{\partial \overrightarrow{\phi_m}}{\partial r} + \overrightarrow{\nabla}_t \times \overrightarrow{\phi_e}. \quad (45)$$

The vector wave function $\overrightarrow{\psi_T}$ associated with generalized electromagnetic fields in superluminal transformation is defined as

$$\overrightarrow{\psi_T} = \overrightarrow{E_T} - i\overrightarrow{H_T}. \quad (46)$$

Then we get the following pair of generalized Maxwell's equation for generalized fields of dyons in T^4- space (for tachyonic dyons via superluminal transformation) i.e.

$$\overrightarrow{\nabla_t}.\overrightarrow{\psi_T} = \Im_0$$
$$\overrightarrow{\nabla_t} \times \overrightarrow{\psi_T} = -i\overrightarrow{\Im} - i\frac{\partial \overrightarrow{\psi_T}}{\partial r} \quad (47)$$

where $\Im_0$ and $\overrightarrow{\Im}$ are the components of generalized four - current source densities of dyons in T^4 - space , given by

$$\{\rho_\mu\} = \{\rho_\mu^e\} - i\{\rho_\mu^m\} = \{\Im_0, \overrightarrow{\Im}\}. \quad (48)$$

Substituting relation (45) into equation (46) as we have done earlier and using equation (44), we obtain the following expression for generalized vector field in terms of the components of generalized four potential of dyon in T^4 - space i.e.

$$\overrightarrow{\psi_T} = -\frac{\partial \overrightarrow{\phi}}{\partial r} - \overrightarrow{\nabla}_t V - i\overrightarrow{\nabla_t} \times \overrightarrow{\phi}. \quad (49)$$

As such, we can write the following tensorial forms of generalized Maxwell's - Dirac equations of dyons under the influence of superluminal transformation (in T^4- space) i.e.

$$f_{\mu\nu,\nu} = \rho_\mu^e,$$
$$\widetilde{f_{\mu\nu,\nu}} = \rho_\mu^m \quad (50)$$

where $f_{\mu\nu}$ and $\widetilde{f_{\mu\nu}}$ are described (like equations (10)) as superluminal electric and magnetic fields tensors in T^4 - space giving rise to Maxwell's equations coupled to electric and magnetic four currents in T^4- space.With the help of equation (47) we may obtain a new vector parameter (current) $\overrightarrow{S_T}$, in T^4- space i.e.

$$\overrightarrow{S_T} = \square_t \overrightarrow{\psi_T} = \frac{\partial \overrightarrow{\Im}}{\partial r} - \overrightarrow{\nabla}_t \Im_0 - i \overrightarrow{\nabla}_t \times \overrightarrow{\Im} \tag{51}$$

where

$$\square_t = \partial_r^2 - \nabla_t^2 = \frac{\partial}{\partial r^2} - \frac{\partial}{\partial t_x^2} - \frac{\partial}{\partial t_y^2} - \frac{\partial}{\partial t_z^2}. \tag{52}$$

We may also define the generalized field tensor for tachyonic dyons associated with generalized superluminal electromagnetic fields in T^4- space as

$$g_{\mu\nu} = f_{\mu\nu} - i\,\widetilde{f_{\mu\nu}}. \tag{53}$$

It gives directly the following form of field equation (parallel to Maxwell's equations associated with generalized superluminal electromagnetic fields of dyons) in T^4- space i.e.

$$\begin{aligned} g_{\mu\nu,\nu} &= \rho_\mu, \\ \widetilde{g_{\mu\nu,\nu}} &= 0 \end{aligned} \tag{54}$$

where $g_{\mu\nu} = \phi_{\mu,\nu} - \phi_{\nu,\mu}$ is denoted as the generalized electromagnetic field tensor and $\rho_\mu = \rho_\mu^e - i\rho_\mu^m$ is described as the generalized four current associated with superluminal electromagnetic fields of dyons in T^4- space. Equation (54) may also be written as follows (like second order Klein-Gordon equation), for dyonic fields in T^4- space, i.e.

$$\square_t \phi_\mu = \rho_\mu \qquad (in\, Lorentz\, gauge). \tag{55}$$

Lorentz gauge condition and continuity equation in T^4- space are written as

$$\frac{\partial V}{\partial r} + \overrightarrow{\nabla}_t . \overrightarrow{\phi} = 0 \qquad (Lorentz\, gauge\, condition) \tag{56}$$

$$\frac{\partial J}{\partial r} + \overrightarrow{\nabla}_t . \overrightarrow{\rho} = 0 \qquad (Continuity\, equation). \tag{57}$$

The suitable manifestly covariant Lagrangian density, which yields the field equation (40) under the variation of field parameters i.e. potential only without changing the trajectory of particle, may be written as follows in T^4- space;

$$L_T = -m_0 - \frac{1}{4} g_{\mu\nu} g_{\mu\nu}^{\star} + \phi_\mu^{\star} \rho_\mu \tag{58}$$

where m_0 is the rest mass of the particle, $(\star)$ denotes the complex conjugate and

$$\widetilde{g_{\mu\nu}} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} g^{\rho\sigma}. \tag{59}$$

Lagrangian density given by equation (58) directly yields the following Lorentz force equation of dyons under superluminal transformations i.e.

$$\zeta_\mu = m_0 x_\mu = Re(\ddot{q} * g_{\mu\nu}) u^\nu \tag{60}$$

where q is the generalized charge of dyon, Re denotes real part, $\ddot{x_\mu}$ is the four acceleration and u^ν is the inverse velocity of the particle in T^4- space and is given by,

$$u^{\nu} = \frac{dt^{\nu}}{dz}$$

or

$$\overleftarrow{u} = (\frac{dt_x}{dz}, \frac{dt_y}{dz}, \frac{dt_z}{dz}). \tag{61}$$

Equation (60) describes the following form of Lorentz force for dyons interacting with superluminal electromagnetic fields in T^4- space,

$$\overrightarrow{\zeta} = e(\overrightarrow{E_T} + \overleftarrow{u} \times \overrightarrow{H_T}) + g(\overrightarrow{H_T} - \overleftarrow{u} \times \overrightarrow{E_T}). \tag{62}$$

IV. QUATERNIONIC FORMULATION OF DYONS IN T^4- SPACE

A quaternion q is defined as

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \tag{63}$$

where $q_{\alpha}(\alpha = 0, 1, 2, 3)$ are real or complex numbers with three imaginary units e_1, $e_2$and e_3 satisfy the relations,

$$e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k. \tag{64}$$

All laws of algebra with the exception of commutative law of multiplication are satisfied by quaternions, which form a division ring. The set of quaternions is called a 4 - dimensional real space essentially because 4- real numbers are required to specify a quaternion. The complex conjugate of q i.e. $q^{\star}$ is defined by

$$q^{\star} = q_0 + q_1^{\star} e_1 + q_2^{\star} e_2 + q_3^{\star} e_3 \tag{65}$$

and the quaternion conjugate of q by

$$\overline{q} = q_0 - q_1 e_1 - q_2 e_2 - q_3 e_3 \tag{66}$$

which gives

$$\begin{aligned} (qp)^{\star} &= q^{\star} p^{\star} \\ (\overline{qp}) &= \overline{p}\,\overline{q} \end{aligned} \tag{67}$$

where p and q , are also the quaternions showing that the quaternion conjugate of product of two quaternions is the product of conjugates in reverse order.

The scalar product of a two quaternions is defined as

$$(p, q) = \frac{1}{2}(p\overline{q} + q\overline{p})$$

and the norm of a quaternion is given as

$$|p| = (p.p) = p_0^2 + p_1^2 + p_2^2 + p_3^2. \tag{68}$$

The inverse of a quaternion $q$is also a quaternion and given by

$$q^{-1} = \frac{q}{|q|^2}. \tag{69}$$

As such, it is easy to write a four vector in T^4- space as a quaternion. Adopting the same procedure to write the quantum equation in quaternion formalism, (as we have done earlier [41, 42]), the generalized four-potential and generalized four - current associated with dyons under superluminal Lorentz transformation (T^4 -space) may be written as quaternions i.e.

$$\phi = -iV + e_1\phi_1 + e_2\phi_2 + e_3\phi_3 \quad (70)$$

$$\rho = -iJ + e_1\rho_1 + e_2\rho_2 + e_3\rho_3. \quad (71)$$

In this case quaternionic differential operator is written as

$$\boxdot_t = -i\partial_r + e_1\partial_1 + e_2\partial_2 + e_3\partial_3 \quad (72)$$

where

$$\partial_r = \frac{\partial}{\partial r}, \quad \partial_1 = \frac{\partial}{\partial t_x}, \quad \partial_2 = \frac{\partial}{\partial t_y}, \partial_3 = \frac{\partial}{\partial t_z}. \quad (73)$$

Operating equation (72) on equations (70) and (71) and using equations (49) ,(56) and (57), we get;

$$\begin{aligned}
\boxdot_t\phi &= -(\partial_r V + \partial_1\phi_1 + \partial_2\phi_2 + \partial_3\phi_3) \\
-i \quad & e_1 \quad \{-\partial_r\phi_1 - \partial_1 V - i(\partial_2\phi_3 - \partial_3\phi_2)\} \\
-i \quad & e_2 \quad \{-\partial_r\phi_2 - \partial_2 V - i(\partial_3\phi_1 - \partial_1\phi_3)\} \\
-i \quad & e_3 \quad \{-\partial_r\phi_3 - \partial_3 V - i(\partial_1\phi_2 - \partial_2\phi_1)\} \\
= \quad & -\psi_r \quad -i(e_1\psi_1 + e_2\psi_2 + e_3\psi_3)
\end{aligned} \quad (74)$$

and

$$\begin{aligned}
\boxdot_t\rho &= -(\partial_r J + \partial_1\rho_1 + \partial_2\rho_2 + \partial_3\rho_3) \\
-i \quad & e_1 \quad \{-\partial_r\rho_1 - \partial_1 J - i(\partial_2\rho_3 - \partial_3\rho_2)\} \\
-i \quad & e_2 \quad \{-\partial_r\rho_2 - \partial_2 J - i(\partial_3\rho_1 - \partial_1\rho_3)\} \\
-i \quad & e_3 \quad \{-\partial_r\rho_3 - \partial_3 J - i(\partial_1\rho_2 - \partial_2\rho_1)\} \\
= \quad & -S_r \quad -i(e_1 S_1 + e_2 S_2 + e_3 S_3)
\end{aligned} \quad (75)$$

where

$$\psi_r = \partial_r V + \partial_1\phi_1 + \partial_2\phi_2 + \partial_3\phi_3 = 0 \,(Lorentz\ Gauge\ condition) \quad (76)$$

$$S_r = \partial_r J + \partial_1\rho_1 + \partial_2\rho_2 + \partial_3\rho_3 = 0. \quad (Continuity\, equation) \quad (77)$$

We may then write equations (74) and (75) in the following quaternionic forms;

$$\boxdot_t\phi = \psi_T \quad (78)$$

and

$$\boxdot_t\rho = S_T. \quad (79)$$

These are the quaternionic differential equations for generalized potential and generalized current in T^4- space under superluminal Lorentz transformations. The conjugate representation of quaternion field equations (78) and (79) in T^4 - space under superluminal Lorentz transformations may then be expressed as,

$$\overline{\boxdot_t}\,\overline{\phi} = \overline{\psi_T} \quad (80)$$
$$\overline{\boxdot_t}\,\overline{\rho} = \overline{S_T} \quad (81)$$

where $\overline{\boxdot_t}$, $\overline{\phi}$, $\overline{\rho}$, $\overline{\psi_T}$ and $\overline{S_T}$ are the quaternion conjugate and defined as,

$$\overline{\boxdot_t} = -i\partial_r - (e_1\partial_1 + e_2\partial_2 + e_3\partial_3) \quad (82)$$
$$\overline{\phi} = -iV - (e_1\phi_1 + e_2\phi_2 + e_3\phi_3) \quad (83)$$
$$\overline{\rho} = -iJ - (e_1\rho_1 + e_2\rho_2 + e_3\rho_3) \quad (84)$$
$$\overline{\psi_T} = -\psi_r + i(e_1\psi_1 + e_2\psi_2 + e_3\psi_3) \quad (85)$$
$$\overline{S_T} = -S_r - i(e_1S_1 + e_2S_2 + e_3S_3). \quad (86)$$

Similarly, we may derive the following quaternionic forms of other fields of dyons in T^4 - space under superluminal Lorentz transformation given by equations (51), (55) (56) and (57) as

$$\overline{\boxdot_t}\,\boxdot_t\,\psi_T = -S_T \quad (87)$$
$$\overline{\boxdot_t}\,\boxdot_t\,\phi = -\rho \quad (88)$$
$$[\boxdot_t, g] = \rho \quad (89)$$
$$q[v, g] = f \quad (90)$$

where

$$v = -iv_0e_0 + e_1v_1 + e_2v_2 + e_3v_3,$$
$$g = -ig_0e_0 + e_1g_1 + e_2g_2 + e_3g_3,$$
$$f = -if_0e_0 + e_1f_1 + e_2f_2 + e_3f_3. \quad (91)$$

In equation (90) v, g and f are respectively the quaternionic forms of inverse velocity, generalized field tensor and Lorentz force associated with dyons in T^4- space under superluminal Lorentz transformations.

As such, the tensorial forms of generalized field equations (89) and (90) are analogous to the following quaternionic forms;

$$[\boxdot_t, g_\mu] = \rho_\mu\,(\mu = 0, 1, 2, 3) \quad (92)$$

and

$$q[v, g_\mu] = f_\mu \quad (93)$$

where ρ_μ and f_μ are the four - current and four - force associated with generalized fields of dyons in T^4- space superluminal Lorentz transformation. The norms of quaternions $\boxdot_t$, ϕ and ρ may then be obtained as

$$\boxdot_t^2 = \frac{1}{2}[\boxdot_t, \boxdot_t] = \overline{\boxdot_t}\boxdot_t = -\partial_r^2 + |\nabla_t|^2 = -\Box_t = -\left|\boxdot\right|^2; \quad (94)$$
$$\phi^2 = = \frac{1}{2}[\phi, \phi] = \overline{\phi}\phi = -V^2 + |\phi|^2 = -|\phi|^2; \quad (95)$$
$$\rho^2 = = \frac{1}{2}[\rho, \rho] = \overline{\rho}\rho = -J^2 + |\rho|^2 = -|\rho|^2 \quad (96)$$

where $\overline{\boxdot_t}$, $\overline{\phi}$and $\overline{\rho}$ are Hamiltonian conjugate of $\boxdot_t$, ϕ and ρ respectively. We may thus see that under the influence of quaternions the norm of four vectors is changed like the norms of four vectors do under imaginary superluminal transformations. The same conclusion may be drawn for the generalized form of Maxwell's equations for dyon on passing from subluminal and superluminal electromagnetic fields. It may therefore be concluded that the quaternionic forms of field equation may be regarded as the Maxwell's equations under the influence of imaginary superluminal transformations, which lead to the mapping $(3,1) \longleftrightarrow (1,3)$ of space-time.

As such, from the fore going analysis we may again draw the same conclusion, also for the theory of generalized fields of dyons , as it has already be drawn [41] that the complex quantum mechanics for time-like particles (bradyons) in subluminal frame of reference reduces to quaternion quantum mechanics for space-like particles (tachyons) in superluminal frame of reference or vice versa. The advantage in expressing the field equations in quaternionic form is that one may extend the theory of bradyons (the Cauchy data at $t = 0$) to the theory of tachyons (the Cauchy data at $r = 0$) directly in this formalism [43] and accordingly the space-time duality and space-time reciprocity be tackled between complex and quaternion quantum mechanics [44]. Finally, it may be pointed out that relativistic equations in quaternionic form will describe the theory of both bradyons tachyons only when making the use of complex (bi) quaternions. The quaternionic formalism described here is thus compact, simpler, unique and consistent. It is also manifestly covariant under quaternion Lorentz transformations.

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