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Control power in controlled hybrid teleportation between a discrete-variable state and a continuous-variable state under decoherence effects

Cao Thi Bich[†]

Institute of Physics, Vietnam Academy of Science and Technology,
10 Dao Tan, Giang Vo, Hanoi, Vietnam

E-mail: [†]ctbich@iop.vast.vn

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Abstract. *One of our works, [C. T. Bich and N. B. An, Pramana – Journal of Physics 96 (2022) 33], proposed a linear optics scheme for controlled teleportation of two different types of qubits. This was achieved by using a four-partite hybrid entangled state enabling the control to be taken by two controllers operating in two distinct types of Hilbert spaces: a finite-dimensional space and an infinite-dimensional space. In this work, the power of the two controllers is assessed through the analysis of the average fidelity of the teleportation protocol in their absence. It is worth noting that the power of the controller holding a discrete-variable state is equal to or greater than that of the controller holding a continuous-variable state.*

Keywords: hybrid teleportation; hybrid entanglement; DV state; CV state; control power.
Classification numbers: 03.67.Bg; 03.67.Mn; 03.65.Ud.

1. Introduction

Quantum entanglement [1] lies at the heart of quantum physics and lays the groundwork for future quantum technologies. It serves as a crucial element in potential advancements in quantum information theory, such as quantum dense coding [2], quantum cryptography [3], quantum information splitting [4], and other related protocols. Among these genuine applications, quantum teleportation (QT) [5], which was initially proposed by Bennett *et al.*, stands out as one of the most groundbreaking achievements. In its original version, QT enables the secure and precise transmission of a quantum state from a sender, called Alice, to a spatially distant receiver, named Bob, without the direct physical transfer of the state. This transmission can be achieved through local operations and classical communication, based on a previously shared entangled state. After its initial emergence, QT not only has garnered substantial theoretical attention [6–9] but has also yielded significant experimental results [10–12] by using various types of quantum resources, such as Einstein-Podolsky-Rosen (EPR) pair states [13], Greenberger-Horne-Zeilinger (GHZ) trio states [14], W states [15], cluster states [16] and so on. In 1998, as a new extension of the QT protocol, the first scheme for controlled teleportation was proposed [17]. In this protocol, a supervisor oversees the teleportation process, ensuring that an arbitrary qubit state can be transferred between two remote locations only with the supervisor's active participation, utilizing a GHZ trio state. Following the development of this GHZ-based approach, various controlled teleportation protocols have been proposed [18–22].

All of the entangled states mentioned above are encoded in the discrete-variable (DV) state [23, 24] with a single DV degree of freedom. In 1998, Furusawa et al. realized QT with continuous-variable (CV) state [25]. From that, many of the QT schemes for CV state have been investigated [26–28]. For more than a decade, a lot of scientists have turned to a different type of entanglement which is called hybrid entangled state [29]. It combines the advantages as well as overcomes the intrinsic disadvantages of a DV state and a CV state. Hybrid entangled states have proven to be crucial for various important applications, such as enabling heterogeneous system implementations [30–33], generating non-Gaussian states [34], and facilitating teleportation [35–38]. For example, [38] investigated quantum teleportation involving a single-rail qubit and a coherent-state qubit, utilizing hybrid entanglement between these two types of qubits. In [39], we presented a protocol for controlled teleportation that also involves a DV state and a CV state. This protocol utilizes a four-partite hybrid entangled state and is supervised by two controllers operating in distinct Hilbert spaces. Namely, Alice and Charlie work exclusively with DV qubits, while Bob and David handle CV qumodes. The aim is to enable teleportation between a CV qubit and a DV state in a noise-affected environment, with both David and Charlie jointly monitoring the process.

In this work, we examine the power of two controllers, Charlie and David, in the quantum teleportation processes of [39] by virtue of evaluating how genuine the states that the receiver obtained when the controllers, for some reason, do not cooperate. To verify such genuine property, the conventional average fidelities and their formulas are utilized. After calculations and analyses, the results show that the role of the controller who works in finite-dimensional space is always larger than or equal to that of the other controller who works in infinite-dimensional space for both processes of teleportation from a CV state to a DV state and vice versa.

This paper is organized as follows. Section 2 introduces controlled hybrid teleportation using four-party hybrid entangled states, following the framework in [39]. Section 3 investigates the control power of Charlie and David in hybrid teleportation from a DV state to a CV state, while Section 4 addresses their control power in the reverse process, from a CV state to a DV state. Finally, Section 5 provides some discussions and concludes the paper.

2. Controlled hybrid teleportation between DV and CV states

In our previous work [39], we introduced two quantum teleportation procedures as outlined in the following. In the first process, Alice has a DV state given by

$$|\psi_1\rangle_{A'} = (x|0\rangle + y|1\rangle)_{A'}, \tag{1}$$

where $|0\rangle$ and $|1\rangle$ represent the vacuum and single-photon states, respectively. The coefficients x and y are unknown and must satisfy the normalization condition $|x|^2+|y|^2=1$. The objective is to transfer this quantum information from Alice's DV qubit to Bob, such that Bob obtains a CV state of the form

$$|\psi_2\rangle_{B'} = N\big(x|\alpha\rangle + y|-\alpha\rangle\big)_{B'}, \tag{2}$$

where $|\pm\alpha\rangle$ represent coherent states with complex amplitudes $\pm\alpha$. The normalization factor is $N=\left[1+2Re(x^*y)e^{-2|\alpha|^2}\right]^{-1/2}$, ensuring that $|\psi_2\rangle$ is properly normalized. The second process, which is the inverse of the first, involves Bob who has an unknown coherent-state qubit in the state given by (2) and needs to transmit the coefficients x and y to Alice, who then should receive them as part of a single-rail qubit state (1). Both teleportation procedures are performed simultaneously under supervision by two controllers, David and Charlie, who operate within the CV and DV spaces, respectively.

To facilitate these tasks, it is necessary for the sender, receiver and controllers to share a hybrid four-partite quantum resource in advance. This resource is expressed as

$$|Q\rangle_{BDCA} = \frac{1}{\sqrt{2}}(|\alpha,\alpha,0,0\rangle + |-\alpha,-\alpha,1,1\rangle)_{BDCA}, \tag{3}$$

where qumodes B and D are held by Bob and David, while qubits C and A by Charlie and Alice, respectively. As these modes are distributed, interactions with the environment can lead to photon dissipation, potentially affecting purity of the quantum channel which becomes a mixed state

$$\begin{aligned}\rho_{BDCA}(\tau) \;=\; & \frac{1}{2}\Big[|\alpha\tau\rangle_B\langle\alpha\tau|\otimes|\alpha\tau\rangle_D\langle\alpha\tau|\otimes|0\rangle_C\langle 0|\otimes|0\rangle_A\langle 0| \\ & +\lambda\tau^2|\alpha\tau\rangle_B\langle-\alpha\tau|\otimes|\alpha\tau\rangle_D\langle-\alpha\tau|\otimes|0\rangle_C\langle 1|\otimes|0\rangle_A\langle 1| \\ & +\lambda\tau^2|-\alpha\tau\rangle_B\langle\alpha\tau|\otimes|-\alpha\tau\rangle_D\langle\alpha\tau|\otimes|1\rangle_C\langle 0|\otimes|1\rangle_A\langle 0| \\ & +|-\alpha\tau\rangle_B\langle-\alpha\tau|\otimes|-\alpha\tau\rangle_D\langle-\alpha\tau| \\ & \otimes((1-\tau^2)\,|0\rangle_C\,\langle 0|+\tau^2\,|1\rangle_C\,\langle 1|) \\ & \otimes((1-\tau^2)\,|0\rangle_A\,\langle 0|)+\tau^2\,|1\rangle_A\,\langle 1|\Big],\end{aligned} \tag{4}$$

where $\lambda=e^{4\alpha^2(\tau^2-1)}$ and $\tau=e^{-\gamma t/2}$. Here, γ is the decay constant determined by the strength of the interaction between the quantum channel and its environment, τ denotes the interaction time with the optical environment, and $\rho_{BDCA}(t)$ represents the density matrix of the quantum channel at time t. Since the initially pure quantum channel state $\rho_{BDCA}(0)=|Q\rangle_{BDCA}\langle Q|$ loses its coherence over time and becomes the mixed state $\rho_{BDCA}(\tau)$ as described in Eq. (4), the amplitudes $\pm\alpha$ of the initial coherent states in Eq. (2) are diminished to $\pm\alpha\tau$ when the process begins. Consequently, we will develop the protocol for the two tasks considering that the state in Eq. (2) evolves to

$$|\psi_2(\tau)\rangle_{B'} = N'(x\,|\alpha\tau\rangle + y\,|-\alpha\tau\rangle)_{B'}, \tag{5}$$

where $N'=\left[1+2Re(x^*y)e^{-2|\alpha|^2\tau^2}\right]^{-1/2}$. Note that, photon loss in single-rail qubits transforms the state $|1\rangle\langle 1|$ into $|0\rangle\langle 0|$ while keeping it within the qubit space $\{|0\rangle,|1\rangle\}$. Such an error is equivalent to a bit-flip and can be corrected using a quantum error-correction code.

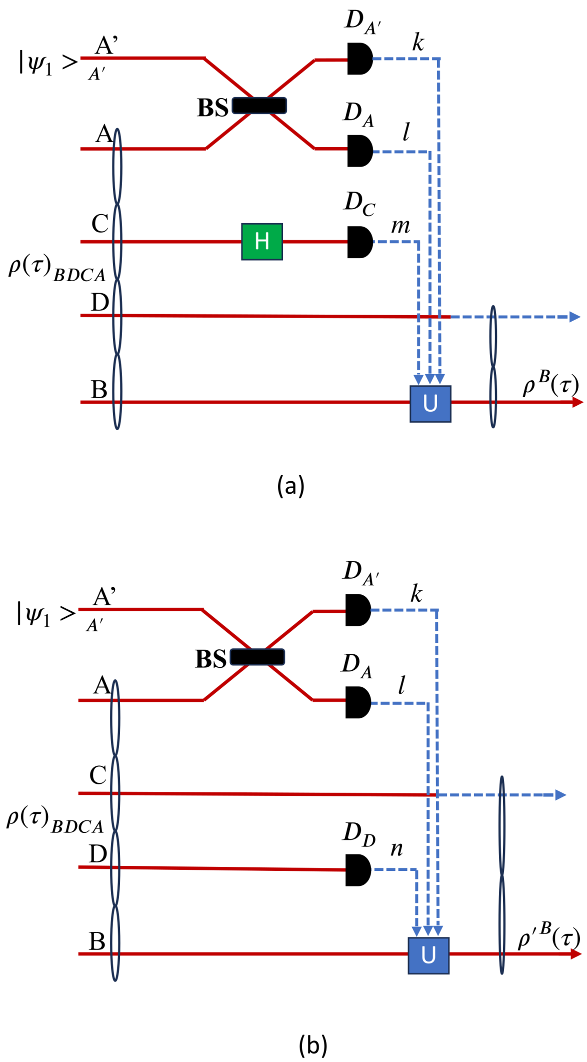

Fig. 1. Diagram of the quantum teleportation process from a DV state to a CV state. (a) Consider the role of the CV controller, David, when he does not participate in the quantum teleportation process. (b) Consider the role of Charlie when she does not participate in the quantum teleportation process. In the diagram, **BS** is a linear optical device consisting of a balanced beam-splitter sandwiched between two $-\pi/2$-phase-shifters, acting on two modes as $\mathbf{BS}_{ij} = P_j(-\pi/2)BS_{ij}(\pi/4)P_j(-\pi/2)$. H is the Hadamard operator and U is an operator that can be X, Z or XZ. $D_{A'}$, D_A, D_C and D_D are photodetectors.

3. Control power in hybrid teleportation from a DV state to a CV state

In this section, we evaluate the control power of David and Charlie in the teleportation process where the quantum information initially encoded in Alice's DV qubit is transferred to

Bob, yielding the CV state specified in Eq. (5). In order to do that, we will calculate how much information the receiver Bob can obtain from the sender Alice without the collaboration of the controllers. First, we will examine the power of David, who holds the qumode D in the CV space. To achieve that, we will perform the tasks sequentially as shown in Fig. 1a. The details are as follows. In the first step, Alice employs the optical device $\mathbf{BS}_{AA'}$ to mix modes A' and A, then measures the photon numbers k and l of these modes using two photodetectors $D_{A'}$ and D_A. Since each mode, A' and A, can hold at most one photon and the beam-splitter is balanced, there are only five possible combinations for (k,l)= $\{(0,0), (0,1), (1,0), (0,2), (2,0)\}$. In the second step, as the DV controller, Charlie first applies a Hadamard gate to her mode C. She then measures the photon number m in mode C using a photodetector D_C and publicly discloses this measurement to Bob for further use. Notably, m can only be 0 or 1, as mode C is a DV state defined by the number states $|0\rangle$ and $|1\rangle$. Here, $\mathbf{BS}$ denotes an optical device that consists of a balanced beam-splitter sandwiched between two $-\pi/2$-phase-shifters, acting on two modes as $\mathbf{BS}_{ij} = P_j(-\pi/2)BS_{ij}(\pi/4)P_j(-\pi/2)$. The state of two modes B and D of Bob and David, respectively, after the actions of Alice and Charlie will be

$$\rho_{BD}(\tau) = \frac{{}_C\langle m|H\rho_{BDC}(\tau)H^+|m\rangle_C}{Tr\{|m\rangle_C\langle m|[H\rho_{BDC}(\tau)H^+]\}}, \tag{6}$$

where

$$\rho_{BDC}(\tau) = \frac{{}_{A'A}\langle k,l|\mathbf{BS}_{A'A}[\rho_{A'}\rho_{BDCA}(\tau)]\mathbf{BS}^+_{A'A}|k,l\rangle_{A'A}}{P_{k,l}}, \tag{7}$$

with

$$P_{k,l} = Tr\left\{|k,l\rangle_{A'A}\langle k,l|[\mathbf{BS}_{A'A}[\rho_{A'}\rho_{BDCA}(\tau)]\mathbf{BS}^+_{A'A}]\right\}. \tag{8}$$

Here, we analyze the situation in which Alice and Bob attempt to perform teleportation process without David's authorization. In this scenario, David declines from measuring his qubit. Consequently, the density matrix for mode B, resulting from tracing out mode D, will be expressed as $\rho^B_{klm}(\tau)$, depending on the values of k, l and m. The four possible combinations for k, l and m are given as follows:

If $klm = 010, 101$ we have

$$\begin{aligned}\rho^B_{010}(\tau) = \rho^B_{101}(\tau) &= M_1(\tau)\{|y|^2|\alpha\tau\rangle_B\langle\alpha\tau| \\ &+\lambda\tau^2 e^{-2\alpha^2\tau^2}(x^*y|\alpha\tau\rangle_1\langle-\alpha\tau| + xy^*|-\alpha\tau\rangle_B\langle\alpha\tau|) \\ &+[|y|^2(1-\tau^2)+|x|^2\tau^2]|-\alpha\tau\rangle_1\langle-\alpha\tau|\} = \rho^B_1(\tau),\end{aligned} \tag{9}$$

where

$$M_1(\tau) = [|y|^2(2-\tau^2)+|x|^2\tau^2+\lambda\tau^2(x^*y+y^*x)e^{-4\alpha^2\tau^2}]^{-1}. \tag{10}$$

If $klm = 100, 011$ we have

$$\begin{aligned}\rho^B_{100}(\tau) = \rho^B_{011}(\tau) &= M_2(\tau)\{|y|^2|\alpha\tau\rangle_B\langle\alpha\tau| \\ &-\lambda\tau^2 e^{-2\alpha^2\tau^2}(x^*y|\alpha\tau\rangle_1\langle-\alpha\tau| + xy^*|-\alpha\tau\rangle_B\langle\alpha\tau|) \\ &+[|y|^2(1-\tau^2)+|x|^2\tau^2]|-\alpha\tau\rangle_1\langle-\alpha\tau|\} = \rho^B_2(\tau),\end{aligned} \tag{11}$$

where

$$M_2(\tau) = [|y|^2(2-\tau^2)+|x|^2\tau^2-\lambda\tau^2(x^*y+y^*x)e^{-4\alpha^2\tau^2}]^{-1}. \tag{12}$$

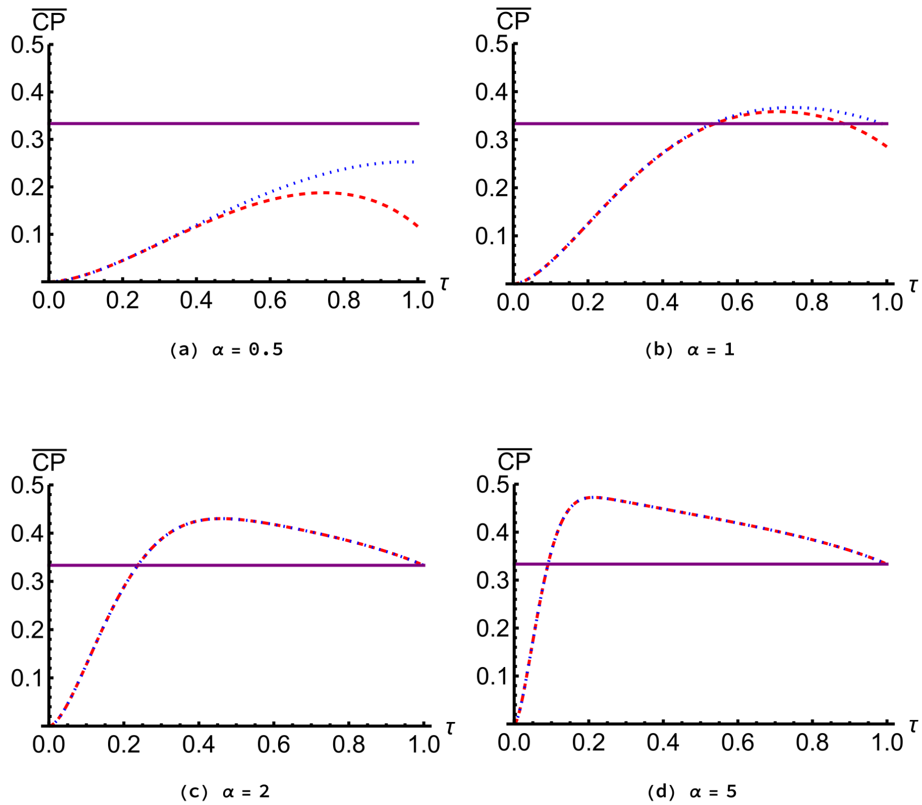

Fig. 2. (Color online) The average control power in the teleportation process from a single-rail qubit to a coherent-state qubit is analyzed as a function of the scaled dimensionless τ for different values of α: (a) $\alpha = 0.5$, (b) $\alpha = 1$, (c) $\alpha = 2$ and (d) $\alpha = 5$. The blue dotted curve depicts Charlie's control power $\overline{CP}_C$, who operates within the DV space, while the red dashed curve shows David's control power $\overline{CP}_D$, who works in the CV space. The purple horizontal line at $1/3$ represents the highest classical value that can be achieved.

If $klm = 000, 001$ we have

$$\rho^B_{000}(\tau) = \rho^B_{001}(\tau) = \frac{|\alpha\tau\rangle_B\langle\alpha\tau| + (1-\tau^2)|-\alpha\tau\rangle_B\langle-\alpha\tau|}{2-\tau^2} \tag{13}$$

If $klm = 020, 021, 200, 201$ we have

$$\rho^B_{020}(\tau) = \rho^B_{021}(\tau) = \rho^B_{200}(\tau) = \rho^B_{201}(\tau) = |-\alpha\tau\rangle_B\langle-\alpha\tau|. \tag{14}$$

Looking at Eqs. (9), (11), (13) and (14), we see that only when $klm = 010, 101, 100, 011$, Bob have a chance to receive the desired state (see the details in [39]) by applying the operators X or XZ to Eqs. (9) and (11), respectively. Note that, X maps $|\pm\tau\alpha\rangle$ to $|\mp\tau\alpha\rangle$ and XZ maps $|\pm\tau\alpha\rangle$

to $\pm|\pm\tau\alpha\rangle$. So the fidelity between the output state and the target state in the absence of David's collaboration determined by

$$\begin{aligned} F_D &= {}_{B'}\langle\psi_2(\tau)|\rho_1^B(\tau)|\psi_2(\tau)\rangle_{B'} \\ &= (N'^2 M_1)\{|y(y+xe^{-2\tau^2\alpha^2})|^2 + ((1-\tau^2)|y|^2+\tau^2|x|^2)|(ye^{-2\tau^2\alpha^2}+x)|^2 \\ &+ 2\lambda\tau^2 e^{-2\alpha^2\tau^2} Re[xy^*(xe^{-2\tau^2\alpha^2}+y)(y^*+y^*e^{-2\tau^2\alpha^2})]\}. \end{aligned} \quad (15)$$

To calculate the average fidelity across all input states, which are assumed to occur with equal frequency, we redefine the parameters in polar coordinates as $x=\cos\theta$ and $y=e^{i\varphi}\sin\theta$. The average fidelity can then be computed by [40]

$$\overline{F}(\theta,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\pi/2} F(\theta,\varphi)\sin(2\theta)d\theta. \quad (16)$$

The power of controller David is determined by $CP_D = 1-F_D$, so the average control power is $\overline{CP}_D = 1-\overline{F}_D$. We have performed numerical calculations for several values of α, and plot the result, which is red curve in Fig. 2.

Next we examine the power of Charlie, who is a DV controller. To achieve that, we will perform the tasks sequentially as shown in Fig. 1b. Firstly, Alice uses $\mathbf{BS}_{AA'}$ to mix modes A' and A, then counts the photon numbers of these modes k and l by two photodetectors $D_{A'}$ and D_A). Secondly, David employs a photodetector D_D to measure the photon count in mode D. The result, denoted as n, is also announced publicly for Bob's subsequent use. Since mode D represents a coherent state, n can take on any non-negative integer value and is categorized into even or odd, i.e., n = {even, odd}. After the actions of Alice and David, the state of modes B and C will be

$$\rho_{BC}(\tau) = \frac{{}_D\langle n|\rho_{BDC}(\tau)|n\rangle_D}{Tr\{|n\rangle_D\langle n|\rho_{BDC}(\tau)\}}, \quad (17)$$

where $\rho_{BDC}(\tau)$ is determined in Eq. (7). The density matrix describing mode B, obtained by tracing over mode C, will only depend on k, l as $\rho'^B_{kl}(\tau)$. There are three possible cases for the values of k, l as follows:
If $kl = 01, 10$ we have

$$\begin{aligned} \rho'^B_{01}(\tau) = \rho'^B_{10}(\tau) &= M'_1(\tau)\{|y|^2|\alpha\tau\rangle_B\langle\alpha\tau| \\ &+[|y|^2(1-\tau^2)+|x|^2\tau^2]|-\alpha\tau\rangle_1\langle-\alpha\tau|\} = \rho_1'^B(\tau), \end{aligned} \quad (18)$$

where

$$M'_1(\tau) = [|y|^2(2-\tau^2)+|x|^2\tau^2]^{-1}. \quad (19)$$

If $kl = 00, 11$ we have

$$\rho'^B_{00}(\tau) = \rho'^B_{11}(\tau) = \frac{|\alpha\tau\rangle_B\langle\alpha\tau| + (1-\tau^2)|-\alpha\tau\rangle_B\langle-\alpha\tau|}{2-\tau^2}. \quad (20)$$

If $kl = 02, 20$ we have

$$\rho'^B_{02}(\tau) = \rho'^B_{20}(\tau) = |-\alpha\tau\rangle_B\langle-\alpha\tau|. \quad (21)$$

Looking at Eqs. (18), (20), (21) we see that only when $kl = 01, 10$ Bob have a chance to receive the desired state by applying the operators X or XZ to Eqs. (18) and (20), respectively. The fidelity between the output state and the target state without the collaboration of the second controller Charlie who holds qubit in mode C can be determined.

$$\begin{aligned} F_C &= {}_{B'}\langle\psi_2(\tau)|\rho_1^{\prime B}(\tau)|\psi_2(\tau)\rangle_{B'} = (N'^2M_1')\{|y(y+xe^{-2\tau^2\alpha^2})|^2 \\ &+ ((1-\tau^2)|y|^2+\tau^2|x|^2)|(ye^{-2\tau^2\alpha^2}+x)|^2\}. \end{aligned} \tag{22}$$

The average power of the second controller, Charlie, denoted as $\overline{CP}_C = 1-\overline{F}_C$, was numerically evaluated for various values of α and plotted as a function of the noise parameter τ in Fig. 2, where Charlie's role is represented by the blue curve. A detailed analysis of these results will be presented in the discussion section (Section 5).

4. Control power in hybrid teleportation from a CV state to a DV state

In this subsection, we aim to consider the power of David and Charlie in the teleportation process, where the quantum information initially encoded in Bob's CV qubit is transferred to Alice, yielding the DV state specified in Eq. (1). First, we also will examine the power of David, who holds the qubit D in the CV space. The steps of this process are illustrated in detail in Fig. 3a. Initially, Bob starts the procedure by combining modes B' and B using a $\mathbf{BS}_{B'B}$. He then finds the photon numbers in the resulting modes using photodetectors $D_{B'}$ and D_B, recording the counts as p and q, respectively. Meanwhile, as the DV controller, Charlie first applies a Hadamard gate to her mode C. She then measures the photon number m in mode C with a photodetector labeled D_C. Charlie publicly reveals this information for Alice to use later. After Bob and Charlie have performed their respective actions, the state of modes D and A will be

$$\rho_{DA}(\tau) = \frac{{}_C\langle m|H\rho_{DCA}(\tau)H^+|m\rangle_C}{Tr\{|m\rangle_C\langle m|[H\rho_{DCA}(\tau)H^+]\}}, \tag{23}$$

where

$$\rho_{DCA}(\tau) = \frac{{}_{B'B}\langle p,q|\mathbf{BS}_{B'B}[\rho_{B'}(\tau)\rho_{BDCA}(\tau)]\mathbf{BS}^+_{B'B}|p,q\rangle_{B'B}}{P_{p,q}}, \tag{24}$$

with

$$P_{p,q} = Tr\left\{|p,q\rangle_{B'B}\langle p,q|[\mathbf{BS}_{B'B}[\rho_{B'}(\tau)\rho_{BDCA}(\tau)]\mathbf{BS}^+_{B'B}]\right\}. \tag{25}$$

Let's now consider the scenario where Alice and Bob aim to carry out teleportation without David's permission. In such a case, David absents from measuring his qubit. In that case, the density matrix describing mode A depending on p,q and m obtained by tracing over mode D, will be:

If ($m=0, q=0$, p= even $\neq 0$) or ($m=1, q=0$, p= odd), then

$$\begin{aligned} \rho_1^A(\tau) &= L_1(\tau)\{(|x|^2+|y|^2(1-\tau^2))|0\rangle_A\langle 0| \\ &\quad +\lambda\tau^2e^{-2\alpha^2\tau^2}(xy^*|0\rangle_A\langle 1|+yx^*|1\rangle_A\langle 0|) \\ &\quad +|y|^2\tau^2|1\rangle_A\langle 1|\}, \end{aligned} \tag{26}$$

where

$$L_1(\tau) = [1+(x^*y+y^*x)\lambda\tau^2e^{-2\alpha^2\tau^2}]^{-1}. \tag{27}$$

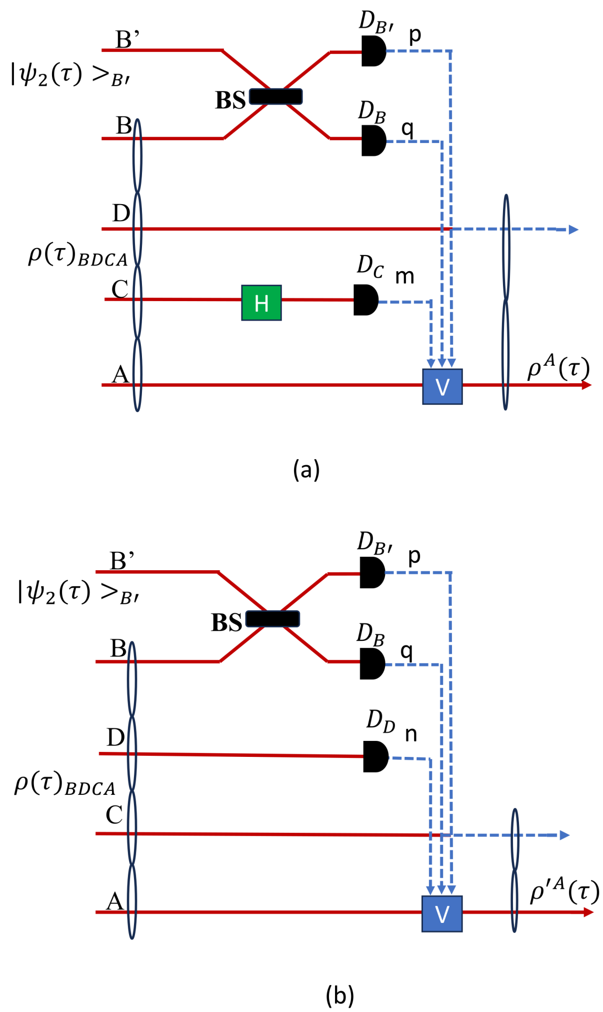

Fig. 3. (Color online) Diagram of the quantum teleportation process from a CV state to a DV state. (a) Consider the role of the controller, David, when he does not participate in the quantum teleportation process. (b) Consider the role of Charlie when she does not participate in the quantum teleportation process. In the diagram, V is an operator that can be X, Z or XZ.

If ($m=0, q=0$, p= odd) or ($m=1, q=0$, p= even $\neq 0$), then

$$\begin{aligned}\rho_2^A(\tau) \quad = \quad & L_2(\tau)\{(|x|^2+|x|^2(1-\tau^2))|0\rangle_A\langle 0| \\ & -\lambda\tau^2 e^{-2\alpha^2\tau^2}(xy^*|0\rangle_A\langle 1|+yx^*|1\rangle_A\langle 0|) \\ & +|y|^2\tau^2|1\rangle_A\langle 1|\}, \end{aligned} \tag{28}$$

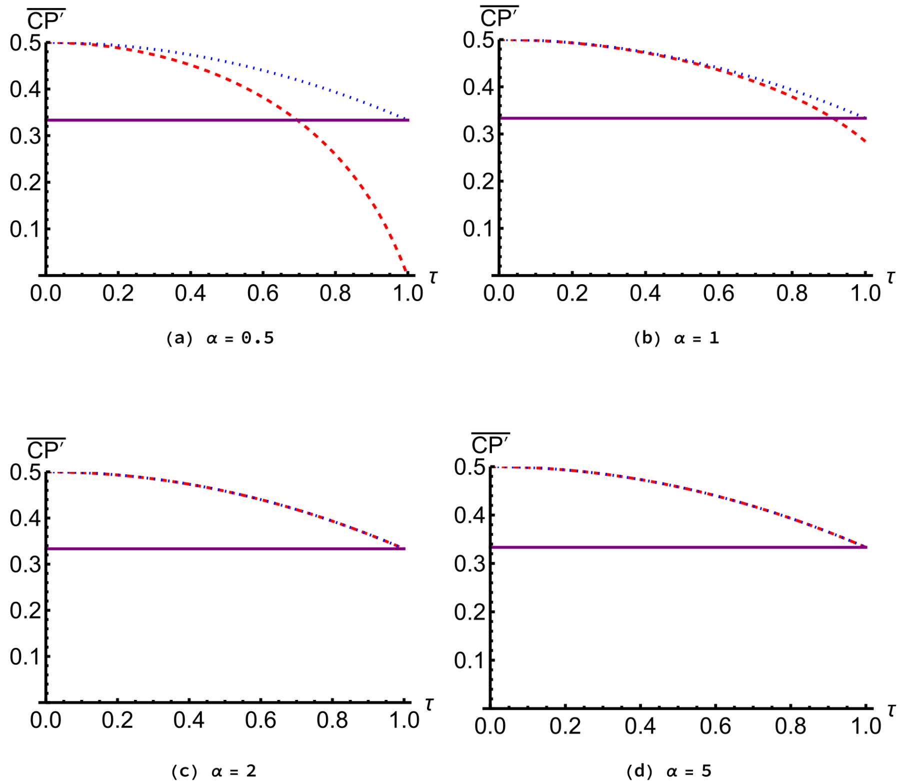

Fig. 4. (Color online) The average control power of the teleportation process from a coherent-state qubit to a single-rail qubit is plotted as a function of the scaled dimensionless parameter τ, with different values of α: (a) $\alpha = 0.5$, (b) $\alpha = 1$, (c) $\alpha = 2$, and (d) $\alpha = 5$. The blue dotted curve indicates Charlie's control power $\overline{CP'}_C$, who operates within the DV space, while the red dashed curve shows David's control power $\overline{CP'}_D$, who functions in the CV space. The purple horizontal line at $1/3$ represents the maximum classical value that can be achieved.

where

$$L_2(\tau) = [1-(x^*y+y^*x)\lambda\tau^2 e^{-2\alpha^2\tau^2}]^{-1}. \tag{29}$$

If ($m=0, p=0, q=$ even $\neq 0$) or ($m=1, p=0, q=$ odd $\neq 0$), then

$$\begin{aligned}\rho_3^A(\tau) \;=\; & L_1(\tau)\{(|y|^2+|x|^2(1-\tau^2))|0\rangle_A\langle 0| \\ & +\lambda\tau^2 e^{-2\alpha^2\tau^2}(yx^*|0\rangle_A\langle 1| + xy^*|1\rangle_4\langle 0|) \\ & +|x|^2\tau^2|1\rangle_A\langle 1|\}. \end{aligned} \tag{30}$$

If ($m=0, p=0$, q= odd) or ($m=1, p=0$, q= even $\neq 0$), then

$$\begin{aligned}\rho_4^A(\tau) &= L_2(\tau)\{(|y|^2+|x|^2(1-\tau^2))|0\rangle_A\langle 0| \\ &-\lambda\tau^2 e^{-2\alpha^2\tau^2}(yx^*|0\rangle_A\langle 1|+xy^*|1\rangle_A\langle 0|) \\ &+|x|^2\tau^2|1\rangle_A\langle 1|\}. \end{aligned} \tag{31}$$

If ($m=0, p=0$, q= 0) or ($m=1, p=0$, q= 0), then

$$\begin{aligned}\rho_5^A(\tau) &= L_2(\tau)[(2-\tau^2))|0\rangle_A\langle 0|+\lambda\tau^2 e^{-2\alpha^2\tau^2}(x^*y+y^*x)(|0\rangle_A\langle 1| \\ &+|1\rangle_A\langle 0|)+\tau^2|1\rangle_A\langle 1|], \end{aligned} \tag{32}$$

where

$$L_3(\tau) = [2+2(x^*y+y^*x)\lambda\tau^2 e^{-2\alpha^2\tau^2}]^{-1}. \tag{33}$$

Examining Eqs. (26), (28), (30), (31) and (32), we observe that the state $\rho_5^A(\tau)$ from Eq. (32) does not approach the desired DV state as indicated in Eq. (1). In contrast, the states $\rho_2^A(\tau)$, $\rho_3^A(\tau)$ and $\rho_4^A(\tau)$ are capable of transitioning to $\rho_1^A(\tau)$, which closely resembles the target state in Eq. (1) by applying the operators Z, X and XZ on qubit A correspondingly. Here, for single-rail qubits, the X operator maps $|j\rangle$ to $|j\oplus 1\rangle$, while the Z operator multiplies $|j\rangle$ by a phase factor $(-1)^j$, where $j\in\{0,1\}$ and $\oplus$ denotes addition modulo 2. Therefore, we will calculate the fidelity between the output state represented by $\rho_2^A(\tau)$ and the target state given in Eq. (1) in the absence of David's collaboration by

$$\begin{aligned}F'_D &= {}_{A'}\langle\psi_1|\rho_1^A(\tau)|\psi_1\rangle_{A'} \\ &= L_1(\tau)(|x|^4+\tau^2|y|^4+(1-\tau^2+2\lambda e^{-2\alpha^2\tau^2}\tau^2)|x|^2|y|^2). \end{aligned} \tag{34}$$

The expression for F'_D in Eq. (34) is relatively simple, and the averaging integration process is straightforward, resulting in

$$\overline{F}_D = \frac{1}{2}+\frac{1}{6}\tau^2(1+2\lambda e^{-2\alpha^2\tau^2}). \tag{35}$$

The power of controller David is determined by $CP'_D = 1-F'_D$, so the average control power is $\overline{CP}'_D = 1-\overline{F'}_D$. We have the average power for the first controller David

$$\overline{CP'}_D = \frac{1}{2}-\frac{(2\lambda e^{-2\alpha^2\tau^2}+1)\tau^2}{6}. \tag{36}$$

Now, lets examine the power of Charlie, who is a CV controller, which is illustrated in detail in Fig. 3b. Similar to the above calculations, the density matrix describing mode A, while tracing over mode C will be

$$\rho'^A(\tau) = \delta_{0p}\rho_1'^A(\tau)+\delta_{0q}\rho_2'^A(\tau)+\delta_{0p}\delta_{0q}\rho_3'^A(\tau), \tag{37}$$

where

$$\begin{aligned}\rho_1'^A(\tau) &= \{(|x|^2+|y|^2(1-\tau^2))|0\rangle_A\langle 0| \\ &+|y|^2\tau^2|1\rangle_A\langle 1|\},\ l\neq 0 \end{aligned} \tag{38}$$

$$\begin{aligned}\rho_2'^A(\tau) &= \{(|x|^2+|y|^2(1-\tau^2))|0\rangle_A\langle 0| \\ &+|x|^2\tau^2|1\rangle_A\langle 1|\},\ k\neq 0 \end{aligned} \tag{39}$$

and

$$\rho_3^{\prime A}(\tau) = \frac{1}{2}[(2-\tau^2))|0\rangle_A\langle 0| + \tau^2|1\rangle_A\langle 1|]. \tag{40}$$

Analyzing Eqs. (38), (39) and (40), it is evident that the state $\rho_3^{\prime A}(\tau)$ in Eq. (40) does not approximate the desired DV state as specified in Eq. (1). In contrast, the states $\rho_2^{\prime A}(\tau)$ can evolve into $\rho_1^{\prime A}(\tau)$, which is close to the target state in Eq. (1). Therefore, we will assess the fidelity between the state with density matrix $\rho_1^{\prime A}(\tau)$ and the desired state in Eq. (1) in the scenario where Charlie's participation is absent by

$$\begin{aligned} F_C' &= {}_{A'}\langle\psi_1|\rho_1^{\prime A}(\tau)|\psi_1\rangle_{A'} \\ &= |x|^4 + \tau^2|y|^4 + (1-\tau^2)|x|^2|y|^2. \end{aligned} \tag{41}$$

The equation for F_C' given in Eq. (41) is simple, and the averaging integration process is straightforward, yielding

$$\overline{F'}_C = \frac{1}{2} + \frac{1}{6}\tau^2. \tag{42}$$

We have the average power for the second controller Charlie will be determined as

$$\overline{CP'}_C = \frac{1}{2} - \frac{\tau^2}{6}. \tag{43}$$

The roles of David, $\overline{CP'}_D = 1 - \overline{F'}_D$, and Charlie, $\overline{CP'}_C = 1 - \overline{F'}_C$, in the CV-to-DV teleportation process are depicted in Fig. 4, where David's role is represented by the red curve and Charlie's role by the blue curve.

5. Discussion and Conclusion

In this section, we analyze the roles of David and Charlie in two types of hybrid teleportation processes: from a DV state to a CV state and from a CV state to a DV state. For the DV-to-CV process (Fig. 2), the results indicate that Charlie's control power (blue curve) is consistently higher than David's (red curve) when α is small. As α increases, their powers converge and eventually coincide for large values of α. Moreover, the overall control power grows with increasing α. In the small-α regime, both Charlie's and David's powers remain below the classical threshold of $1/3$, which is required to ensure that the teleportation fidelity without the controller's authorization does not surpass that of a purely classical channel. The highest control power is achieved in the regime of large α combined with small values of the noise parameter τ. For the CV-to-DV process (Fig. 4), the behavior of Charlie's control power is distinctly different. His power remains essentially unaffected by variations in α, demonstrating the robustness of his role with respect to this parameter. By contrast, David's control power strongly depends on α: when α is small, his power is significantly weaker than Charlie's. As α increases, David's control power gradually improves, and eventually, for sufficiently large α, the powers of both controllers converge. This contrasting behavior underscores Charlie's superior and more stable influence, particularly in the regime of small α.

In conclusion, we have investigated the control powers of two controllers in quantum controlled teleportation protocols involving hybrid systems, specifically between a single-rail qubit and a coherent-state qubit. By evaluating the average teleportation fidelities in the absence of each controller's participation, we quantified their respective roles in both teleportation directions. Our analysis reveals that, in the single-rail to coherent-state direction, both controllers' powers depend on the coherent-state amplitude parameter α. In contrast, in the coherent-state to single-rail direction, a clear asymmetry emerges: David's control power varies with α, whereas Charlie's power remains independent of it. A general trend can thus be identified: the controller holding the DV quantum state consistently achieves a control power equal to or greater than that of the controller holding the CV quantum state. This asymmetry can be traced to the additional operations required during the decoding process. Specifically, Charlie, who holds the DV state, must perform a Hadamard gate, while David, who holds the CV state, is not required to do so. This operational difference suggests that the complexity or number of quantum operations assigned to a controller directly contributes to their influence over the teleportation process. In other words, the controller tasked with performing more demanding quantum operations tends to exert stronger control.

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Conflict of interest

The author have no conflict of interest to declare.

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