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RELIABILITY INDEX OF DISTRIBUTED PARAMETER SYSTEM

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ABSTRACT. In this paper, a method to determine the reliability index of distributed parameter systems by using a method of approximation the multi-condition dependent probability by a one-condition dependent probability is proposed. Therefore, the problem of the reliability index of distributed parameter system is transformed into the defined case.

To illustrate for the method, the reliability index of air tube is considered.

1. Introduction

The reliability of a system in general, and that of structures in particular, have been studied for many years. Some works have either directly or indirectly referred to the reliability of distributed parameter system (DPS) [1-9].

DPS herein means a system of which parameters depend on space and time variables.

Until now, studies on DPS have limited themselves to the level of formulation of problem or consideration of simple cases.

In general, mechanical systems are systems with distributed parameters, whose solutions have to satisfy a system of differential equations. It is not easy however, to modelize DPS by electrical networks, in order to facilitate the reliability computation. On the other hand, DPS is composed of elements linked together by relations of equational and inequational forms.

Consequently, to compute the reliability of DPS, we have to consider the problem of the cross, a level in multidimensional space of a stochastic process [9]. Many difficulties are therefore faced.

The many problems of mechanics, such as; optimal design problem; limit analysis problem; shakdown problem; etc., the number of conditions expressed by equation forms is less than that of the unknowns; therefore, the schema for computation of reliability given in [2, 5] is not available.

As we know [1], in case the probability is dependent upon one condition (an inequation), there exists a one to one relation between the probability of failure P_f and the reliability index β

$$P_f = \Phi(-\beta) \Longleftrightarrow \beta = -\Phi^{-1}(P_f),$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\Big(- \frac{1}{2}t^2\Big) dt$$

is standard normal distribution function.

Otherwise, when the probability has to satisfy a system of equations and inequations, the relation between P_f and β have yet been established, so far.

In this paper, a method to determine the reliability index of DPS by using a method of approximation presenting the multi-condition dependent probability into a one-condition dependent one, is proposed. Therefore, the problem is transformed into the case β of a defined.

To illustrate the method, the reliability index of an air tube is considered.

2. Reliability of DPS

Reliability is the most significant indicator of a system. The quality of a system is shown by several characteristic quantities. If quality indicators are safeguarded to some conditions, then the system is considered as guaranteed. Thus, to ensure quality, quality variables need to satisfy a system of equations and inequations.

Eventually, the reliability of a system is the joint probability of an equation and inequation of state, quality and time variables.

According to [5], reliability of a system is the probability:

$$P(t) = \text{Prob.} \left\{ \begin{array}{c} L\vec{u} = \vec{q} \\ M\vec{u} = \vec{v} \\ \vec{v} \in \Omega_0 \\ \forall \vec{x} \in V \\ \forall \vec{\tau} \in [0, t] \end{array} \right\} \tag{2.1}$$

where:

$L\vec{u} = \vec{q}$ is the state equation of the system,

$M\vec{u} = \vec{v}$ - transformation of state variable $\vec{u}$ into quality $\vec{v}$,

Ω_0 - quality domain,

V - the space area occupied by the system,

t - time variable,

$\vec{x} = [x_1, x_2, x_3]^T$ - space variable.

In case, the number of unknowns is greater than the number of condition expressed by equation forms, principly there is no unique solution. Consequently, we have to solve problems by methods of the choice of plans (optimization problem).

In particular case, we can find $\vec{u}$ from state equation $L\vec{u} = \vec{q}$, to find v from $M\vec{u} = \vec{v}$, then (2.1) becomes

$$P(t) = \text{Prob.} \left\{ \begin{array}{c} f(\vec{v}) \le C \\ \forall \vec{x} \in V \\ \forall \vec{\tau} \in [0, t] \end{array} \right\} \tag{2.2}$$

$L\vec{u} = \vec{q}$ generally, is a system of stochastic different, it is not simple to find $\vec{u}$ in an analytic expression, this could only be realized for a narrow class of problems.

Generally, we replace $L\vec{u} = \vec{q}$ by $\left\{ \begin{array}{ll} L\vec{u} & \le \vec{q}, \\ L\vec{u} & \ge \vec{q}. \end{array} \right.$

Therefore the reliability probability of the system (2.1) takes the form

$$P(t) = \text{Prob.} \left\{ \begin{array}{c} L\vec{u} \ge \vec{q} \\ L\vec{u} \le \vec{q} \\ M\vec{u} \ge \vec{v} \\ M\vec{u} \le \vec{v} \\ f(\vec{v}) \le C \\ \forall \vec{x} \in V \\ \forall \vec{\tau} \in [0, t] \end{array} \right\} \tag{2.3}$$

In the discrete forms, obtained by the finite-difference method or the finite-element method [11], we have

$$P(t) \approx \text{Prob.} \left\{ \begin{array}{c} G_j(\vec{u}, \vec{v}) \le 0 \\ j = \overline{1, m} \end{array} \right\} \tag{2.4}$$

3. Reliability index of DPS

As we know, if the probability (2.3) depends on one condition, the reliability index β can be represented as follows [1]

$$\beta = \frac{\mu_G}{\sigma_G}, \tag{3.1}$$

where G is called the safety margin, μ_G and σ_G the mean value and standard deviation of G.

In the case, where probability (2.3) depends on multi-conditions ($m \geq 2$), then the reliability index is yet to be defined. Now, we would like to put forward an approximate expression of reliability index β for the general case.

3.1. The approximation of safety domain

In view of geometry, the system of inequations

$$G_j(\vec{u},\vec{v}) \leq 0 \qquad i = \overline{1,m}$$

determines a G-domain in $\{\vec{u},\vec{v}\}$ space.

Generally, G is a convex domain, filled with the origin of coordinates.

In the case where G is not convex, we shall approximate G by some convex sub-domains.

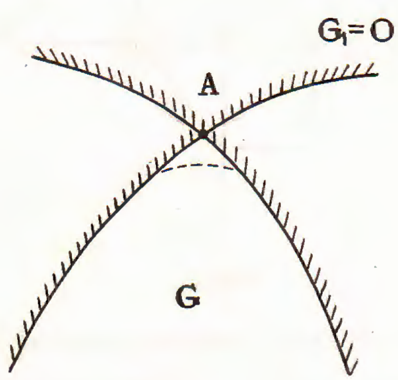

Fig. 1

Without the loss of generality, we can take an illustration for argument in a two-dimensional case (Fig. 1).

G domain is determined by two inequations

$$\begin{cases} G_1 & \leq 0, \\ G_2 & \leq 0. \end{cases}$$

Therefore, the G domain has an angular point at A.

Now, we replace a small piece of boundary of G in the neighbourhood of an angular A by a curve (dotted line), so that the G domain is smooth at A (i.e. boundary of G is differentiable at A). Thus, we approximated two conditions by one condition.

Similar results hold for probability over any finite numbers of m.

At last, we approximate a super-surface, which is defined by m equations, by other defined by one equation. Now, approximate safety domain is differentiable at every boundary point. In this case, the reliability index is defined by (3.1).

The remining in (3.1), the equation of super-surface is $G = 0$. In order to determine β, we have to know the analytical form of G, its mean value and normal deviation. Our function G however, is only a conventional function.

Next, we shall prove that, with an approximate calculation of β, it is unnecessary to know G function in a clear form, that is very convenient for the computation.

3.2. Determination approximate value of β

For the simplicity of exposition, the safety condition is chosen by the strength condition. For example, the strength condition is Tresca's yield condition [12] (Fig. 2), and the safety domain is a polygon with six angular points. Now, we approximate Tresca's polygon by Mises's ellipse (dotted line), thus,

$$P = \text{Prob.}\left\{\begin{array}{l} \sigma_1 \leq a \\ \sigma_1 \geq -a \\ \sigma_2 \geq -b \\ \sigma_2 \leq b \\ \dfrac{\sigma_2}{b} - \dfrac{\sigma_1}{a} \leq 1 \\ \dfrac{\sigma_1}{a} - \dfrac{\sigma_2}{b} \geq 1 \end{array}\right\}$$

is replaced by

$$P \approx \text{Prob.}\left\{\frac{\sigma_1^2}{a^2} + \frac{\sigma_2^2}{b^2} \leq 1\right\}$$

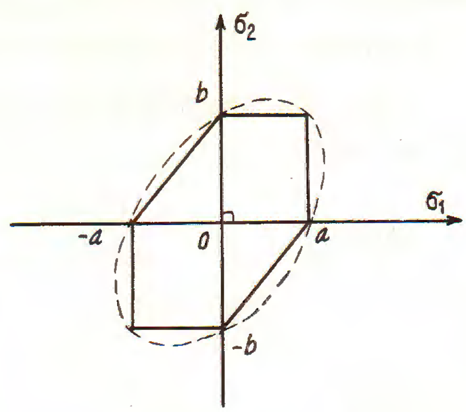

Fig. 2

We approximated a polygon by an ellipse. Conversely, we can say that the ellipse is an approximation of a polygon. In general, G_i $(i = \overline{1,m})$ are nonlinear functions, the system of inequations $G_i \leq 0$ $(i = \overline{1,m})$ determines a domain in n-dimensional space. By an argument analogous to that used for the proof above, we can approximate safety domain by a super-surface.

Suppose that, random variables are independent, uncorrelated and normal distributed. If they are not normal and correlated, we transform them into normal variables by Rachwiz-Fiessler's transformation [1].

As we know [1, 2], in the space of normalized random variables, the shortest distance from the origin to the failure surface is equal to reliability index β. Because the normalized transformation is linear, the above geometrical interpretation is hold.

From now on we study the space of normalized random variables.

Denote β_i $(i = \overline{1,m})$ are the shortest distances from the origin to G_i-surfaces ($G_i = 0$ surface). The origin safety domain is also an approximate form of smooth approximate surface.

We now combine the above results to obtain the following expression

$$\beta \approx \min_{\{i\}} \beta_i \tag{3.2}$$

The expression (3.2) proves that, to determine reliability index β of DPS, we only should find index β_i $(i = \overline{1,m})$.

The sphere having a radius β and centre at origin, is inside - tangent of the safety domain. From that, β is a low boundary of reliability index.

3.3. Example

Consider an air tube subjected to internal pressure P_a (Fig. 3) [13].

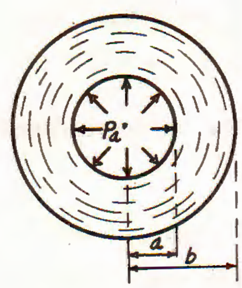

Fig. 3

The state equation

$$\frac{du^2}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \tag{a}$$

The transformation from state variables into quality variables

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \\ \sigma_\theta &= \frac{R}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right) \end{aligned} \tag{b}$$

Boundary conditions:

$$\begin{aligned} \sigma_r(r=a) &= -P_a, \\ \sigma_r(r=b) &= 0. \end{aligned} \tag{c}$$

Quality condition is Tresca's yield condition

$$\begin{aligned} -\sigma_0 \le \sigma_r &\le \sigma_0 \\ -\sigma_0 \le \sigma_\theta &\le \sigma_0 \\ -\sigma_0 \le \sigma_\theta - \sigma_r &\le \sigma_0 \\ a \le r &\le b \end{aligned} \tag{d}$$

The reliability of air tube is

$$P = \text{Prob.} \left\{ \begin{array}{l} \dfrac{du^2}{dr^2} + \dfrac{1}{r}\dfrac{du}{dr} - \dfrac{u}{r^2} = 0 \\ \sigma_r = \dfrac{E}{1-\nu^2}\Big(\dfrac{du}{dr} + \nu\dfrac{u}{r}\Big) \\ \sigma_\theta = \dfrac{E}{1-\nu^2}\Big(\dfrac{u}{r} + \nu\dfrac{du}{dr}\Big) \\ \sigma_r(r=a) = -P_a \\ \sigma_r(r=b) = 0 \\ |\sigma_r| \le \sigma_0 \\ |\sigma_\theta| \le \sigma_0 \\ |\sigma_\theta - \sigma_r| \le \sigma_0 \\ a \le r \le b \end{array} \right\}$$

Suppose that, from experience we have mean values and standard variances of basic variables

$$\begin{aligned} \mu P_a &= 1.200\, N/cm^2, & \sigma P_a &= 100\, N/cm^2, \\ \mu\sigma_0 &= 14.000\, N/cm^2, & \sigma_{00} &= 0, \\ \mu_a &= 150\, cm, & \sigma_a &= 2\, cm \\ \mu_b &= 250\, cm, & \sigma_b &= 2\, cm. \end{aligned}$$

We find u from state equation (a) and use transformation (b), boundary condition (c), thus P becomes

$$P = \text{Prob.} \left\{ \begin{array}{l} -\sigma_0 \le \sigma_r \le \sigma_0 \\ -\sigma_0 \le \sigma_\theta \le \sigma_0 \\ -\sigma_0 \le \sigma_\theta - \sigma_r \le \sigma_0 \\ \quad a \le r \le b \end{array} \right\}$$

where

$$\sigma_r = \frac{P_a a^2}{b^2 - a^2}\left[1 - \frac{b^2}{r^2}\right] \equiv \varphi_1(P_a, a, b, r)$$

$$\sigma_\theta = \frac{P_a a^2}{b^2 - a^2}\left[1 + \frac{b^2}{r^2}\right] \equiv \varphi_2(P_a, a, b, r)$$

The safety margins are:

$$\begin{aligned} &G_1 \equiv \sigma_r - \sigma_0, && G_2 \equiv -\sigma_0 - \sigma_r \\ &G_3 \equiv -\sigma_0 - \sigma_\theta, && G_4 \equiv \sigma_\theta - \sigma_0 \\ &G_5 \equiv \sigma_\theta - \sigma_r - \sigma_0, && G_6 \equiv -\sigma_0 - \sigma_\theta - \sigma_r. \end{aligned}$$

Linearization φ_1 and φ_2 at mean values of basic variables, to find μ_{Gi} and σ_{Gi}, from that we find out β_i. At last, we have

$$\beta \approx 3.25 \Longleftrightarrow P \approx 0.999$$

4. Conclusions

The main idea of the method presented in this paper is to give an approximation of reliability index of DPS. The proposed technique based only on the mathematical form of safety condition, there is no relation with the modelization of system into a series or parallel systems.

In the [1], Palle Thoft - Christensen and others proposed a similar comment for single elements without provement.

The method could be applied to technical, economical systems and to ecological systems as well.

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CHỈ SỐ ĐỘ TIN CẬY CỦA HỆ PHÂN BỐ THAM SỐ

Bài này trình bày một phương pháp tìm giá trị gần đúng của chỉ số độ tin cậy β của hệ phân bố tham số, không cần biết hàm mật độ phân bố đồng thời của các biến cơ bản. Phương pháp nêu ra chỉ dựa trên các dạng toán học của điều kiện an toàn, không liên quan đến việc mô hình hóa hệ. Vì vậy rất thuận lợi cho việc tính toán.

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