ANALYSIS OF EXCEEDANCE PROBABILITY OF DISPLACEMENT RESPONSE OF RANDOMLY NONLINEAR STRUCTURES

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SUMMARY. The paper presents the estimation of the exact exceedance probability (EEP) of stationary responses of some white noise-randomly excited nonlinear systems whose exact probability density function can be known. Consequently, the approximate exceedance probabilities (AEPs) are evaluated based on the analysis of equivalent linearized systems using the traditional Caughey method and the extension technique of LOMSEC. Comparisons of the AEPs versus the EEP are demonstrated. The obtained results indicate important characters of the exceedance probability (EP) as well as the influence of nonlinearity over EP. The evaluation of the applied possibility of the proposed linearization techniques for estimating AEPs are made.

1. Introduction

One of the most concerned problems in the design process of types of structures, is the estimation of the extreme demands on the structure during a specified period of time. This is the same meaning with the estimation of exceedance probability of the extreme responses during the period of time. In general, this is a very difficult problem and usually, only indicative answers can be obtained in practice. However, in the context of civil engineering, structures subjected to environmental loads such as wind and ocean waves, a remarkable developments over the last two decades in modelling both the structure, the loading process and the interaction between them has been made.

The framework usually adopted for the estimation of extreme responses of civil engineering structures for the purpose of design, is that of modelling the loading processes on the structure as stochastic processes. In cases where the dynamic behaviour of the structure can be modelled by linear equations of motion, the response statistics can be analysed in a rather satisfactory manner. However, this is usually an exception, especially for the estimation of extreme responses. Since stochastic response analysis of nonlinear structures is very difficult, methods of stochastic linearization have been developed.

When analysing nonlinear random systems using the equivalent linearization
techniques, the analysis of the second-order moments were very much investigated; whereas the researches on the exceedance probability of the extreme responses were rarely made. Naess [2-4] has presented results from initial efforts to develop a stochastic linearization procedure specifically designed for making predictions of large responses.

This paper presents the estimation of the EEP of the stationary responses of some white noise-randomly excited nonlinear systems, whose exact probabil­ity density function can be found. Through the obtained EEP, some important characters of the EP, especially the influence of nonlinearity over EP are intro­duced. Consequently, the AEPs are evaluated based on the analysis of equivalent linearized systems using the traditional Caughey method [1] and the extension technique of LOMSEC [9-11]. Comparisons of the AEPs versus the EEP are given in order to evaluate the applied possibility of the proposed linearization techniques for estimating AEPs. The systems considered in this paper, and relative data for estimation of the exceedance probability are originated from the previous publications of the Author himself [10-12]. The numerical calculations are added by a special software of Mathematica 3.0 [13].

2. Estimation of exceedance probability of displacement response

The extreme value distribution of a stationary process $X(t)$ is assumed, for simplicity, to be given as [7]:

$$F(x) = \text{Prob}\{M(T) \leq x\} = \exp\{-\nu(x)T\}, \quad (2.1)$$

where $M(T) = \max\{X(t); 0 \leq t \leq T\}$ is the largest value of $X(t)$ during a time interval of length $T$; $\nu(x)$ denotes the mean up-crossing rate of $X(t)$. At level $x_p$, the exceedance probability $p^+$ during time $T$ is defined by:

$$F(x_p) = 1 - p^+. \quad (2.2)$$

From (2.1), (2.2) we get:

$$p^+ = 1 - F(x) = 1 - \exp\{-\nu(x)T\}$$

$$\Rightarrow 100(1 - F(x)) = 100[1 - \exp\{-\nu(x)T\}]\%.$$  \hspace{1cm} (2.3)

A typical range of exceedance probabilities for design purposes is from 1% to 20%. The time interval chosen here is $T = 3h$.

The mean up-crossing rate $\nu(x)$ is defined by [8]:

$$\nu(x) = \int_0^\infty \dot{x}p(x, \dot{x})d\dot{x}, \quad (2.4)$$
\[ p(x, \dot{x}) \] is joint probability density function (PDF) of responses \( X(t) \) and \( \dot{X}(t) \), \( \dot{x} = dx/dt \). However, it is difficult to obtain the exact PDF of randomly excited nonlinear systems in practice. Even if the response can be modeled as a Markov process, the possibility of an exact PDF solution is still limited. Therefore, some approximate methods were developed and investigated for estimating the mean up-crossing rate [5, 6, 8, 12].

**Exact exceedance probability (EEP)**

The EEP of the stationary responses of the randomly nonlinear structures is defined by using formulas (2.3), (2.4); with the assumption that the exact joint probability density function \( p_e(x, \dot{x}) \) of the responses \( x(t), \dot{x}(t) \) can be known by solving Fokker-Planck equation [10-12]:

\[
p_e^+ = 100 [1 - \exp\{-\nu_e(x)T\}]\%.
\]  

(2.5)

where \( \nu_e(x) \) is the exact mean up-crossing rate (EMCR):

\[
\nu_e(x) = \int_0^\infty \dot{x}p_e(x, \dot{x})d\dot{x}.
\]  

(2.6)

**Approximate exceedance probability (AEP)**

The AEPs are also defined by using formulas (2.3), (2.4) but for the equivalent linearized systems (obtained by Caughey or LOMSEC, respectively). In this case, the joint probability density functions are approximate and considered as the norm:

\[
p_A(x, \dot{x}) = \frac{1}{2\pi \sigma_x \sigma_{\dot{x}}} \exp \left\{ -\frac{x^2}{2\sigma_x^2} + \frac{\dot{x}^2}{2\sigma_{\dot{x}}^2} \right\},
\]  

(2.7)

where \( \sigma_x^2 = \langle x^2 \rangle \), \( \sigma_{\dot{x}}^2 = \langle \dot{x}^2 \rangle \) are second moments of the equivalent linearized system. The approximate mean up-crossing rates (AMCRs) as follows:

\[
\nu_A(x) = \int_0^\infty \dot{x}p_A(x, \dot{x})d\dot{x}.
\]  

(2.8)

Using the linearization method of Caughey or LOMSEC, \( \langle x^2 \rangle_G \), \( \langle \dot{x}^2 \rangle_G \) and \( \langle x^2 \rangle_{LG} \), \( \langle \dot{x}^2 \rangle_{LG} \) can be calculated [10-11]. Consequently the AMCRs of \( \nu_G(x) \), \( \nu_{LG}(x) \) are evaluated by using formulas (2.7), (2.8) [12]. Then from (2.3), (2.4), (2.7), (2.8), the AEPs are defined as follows:

\[
p_A^+ = 100 [1 - \exp\{-\nu_A(x)T\}]\%.
\]  

(2.9)
3. Illustrative examples

Example 1. Consider the Duffing oscillator with Gaussian white noise excitation:

\[ \ddot{x} + 2h\dot{x} + \omega_0^2 x + \varepsilon x^3 = \sigma w(t). \] (3.1)

A series of calculation procedures to obtain the mean up-crossing rates (EMCR and AMCR) has been done in [12]. The results are:

The EMCR as follows:

\[ \nu_e(x) = \sqrt{\frac{\sigma}{2\pi}} \int_0^\infty \exp \left\{ - \frac{4h}{\sigma^2} \left( \frac{\omega_0^2}{2} x^2 + \frac{\varepsilon}{4} x^4 \right) \right\} \] (3.2)

The AMCRs according to the linearization criterion of Caughey and LOMSEC are:

\[ \nu_G(x) = \frac{\left( \int_0^\infty \exp \left\{ - \frac{\dot{x}^2}{2\langle x^2 \rangle_G (\omega_0^2 + \lambda_G)} \right\} d\dot{x} \right) \exp \left\{ - \frac{x^2}{2\langle x^2 \rangle_G} \right\}}{2\pi \langle x^2 \rangle_G \sqrt{\omega_0^2 + \lambda_G}}, \] (3.3)

where \( \lambda_G = 3\varepsilon \langle x^2 \rangle_G \)

\[ \nu_{LG}(x) = \frac{\left( \int_0^\infty \exp \left\{ - \frac{\dot{x}^2}{2\langle x^2 \rangle_{LG} (\omega_0^2 + \lambda_{LG})} \right\} d\dot{x} \right) \exp \left\{ - \frac{x^2}{2\langle x^2 \rangle_{LG}} \right\}}{2\pi \langle x^2 \rangle_{LG} \sqrt{\omega_0^2 + \lambda_{LG}}}, \] (3.4)

where \( \lambda_{LG} = K_r \varepsilon \langle x^2 \rangle_{LG} \) and \( K_r = \frac{\int_0^t t^4 n(t) dt}{\int_0^t t^2 n(t) dt} \); \( n(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \).

Substitutions (3.2) into (2.5) and (3.3), (3.4) into (2.9) yields formulas for calculating the EEP and AEPs respectively. Table 1 shows the terminal expressions corresponding with specific value of the parameters.
Fig 1 - 4 show the EEP and AEPs. In order to distinguish graphics, we use different lines as follows [ —— $p_x^+$ (the EEP), —— $p_G^+$ (the AEP of Caughey), and ——— $p_{LG}^+$ (the AEP of LOMSEC)]. Numerical values are given in table 2.

Table 1. Expressions of the exceedance probabilities
($\omega_0^2 = 1; h = 0.25, \sigma = 1; \epsilon$ varies)

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$p_x^+$</th>
<th>$p_G^+$</th>
<th>$p_{LG}^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>$100{1 - \exp{-0.17528 \times \exp{-(0.5x^2 + 0.05x^4)T}} \times \exp{-0.18979 \times \exp{-0.71097x^2}T}} \times \exp{-0.67996x^2}T}$</td>
<td>$100{1 - \exp{-0.18979 \times \exp{-0.71097x^2}T}} \times \exp{-0.67996x^2}T}$</td>
<td>$100{1 - \exp{-0.18744 \times \exp{-0.24151 \times \exp{-1.15138x^2}T}} \times \exp{-1.07117x^2}T}$</td>
</tr>
<tr>
<td>1</td>
<td>$100{1 - \exp{-0.20615 \times \exp{-(0.5x^2 + 0.25x^4)T}} \times \exp{-0.24151 \times \exp{-1.15138x^2}T}} \times \exp{-1.07117x^2}T}$</td>
<td>$100{1 - \exp{-0.24151 \times \exp{-1.15138x^2}T}} \times \exp{-1.07117x^2}T}$</td>
<td>$100{1 - \exp{-0.23252 \times \exp{-0.50894x^2}T}} \times \exp{-2.73613x^2}T}$</td>
</tr>
<tr>
<td>10</td>
<td>$100{1 - \exp{-0.30597 \times \exp{-(0.5x^2 + 2.5x^4)T}} \times \exp{-2.99994x^2}T}} \times \exp{-2.73613x^2}T}$</td>
<td>$100{1 - \exp{-0.3985 \times \exp{-2.99994x^2}T}} \times \exp{-2.73613x^2}T}$</td>
<td>$100{1 - \exp{-0.36130 \times \exp{-8.91424x^2}T}} \times \exp{-8.07624x^2}T}$</td>
</tr>
<tr>
<td>100</td>
<td>$100{1 - \exp{-0.50867 \times \exp{-(0.5x^2 + 25x^4)T}} \times \exp{-8.91424x^2}T}} \times \exp{-8.07624x^2}T}$</td>
<td>$100{1 - \exp{-0.67199 \times \exp{-8.91424x^2}T}} \times \exp{-8.07624x^2}T}$</td>
<td>$100{1 - \exp{-0.61060 \times \exp{-8.91424x^2}T}} \times \exp{-8.07624x^2}T}$</td>
</tr>
</tbody>
</table>
Table 2. Some numerical values ($\omega_0^2 = 1; \ h = 0.25; \ \sigma = 1; \ \varepsilon$ varies)

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$p_e^+$ (%)</th>
<th>$x$</th>
<th>$p_G^+$ (%)</th>
<th>$\text{Error}_G(%)$</th>
<th>$p_{LG}^+$ (%)</th>
<th>$\text{Error}_{LG}(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.2000</td>
<td>1.22140</td>
<td>17.89220</td>
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<tr>
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<td>1.40109</td>
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</table>

**Comments.** From the figures and from table 2, it is shown that: at a specific level of $p_e^+$, the extreme response $X(t)$ reduces when the nonlinearity increases. For larger values of $p_e^+$ (equal to small values of $X(t)$), one gets $p_{LG}^+$ more improved than $p_G^+$, especially with a strong nonlinearity. For specified small values of $p_e^+$, both $p_G^+$ and $p_{LG}^+$ have high relative errors though the absolute errors are not so high; however, one gets $p_G^+$ better than $p_{LG}^+$.

**Example 2.** Consider the case $\omega_0^2 = -1$ of the Duffing oscillator (3.1).

The formulas for calculating EMCR and AMCRs, then EEP and AEPs are quite the same as (3.2), (3.3), (3.4). Table 3 shows the terminal expressions corresponding with specific value of the parameters. Fig 5 - 8 show the EEP and AEPs. The graphic symbols are similar to the above-mentioned. Some numerical values given in Table 4.
Table 3. Expressions of the exceedance probabilities
\((\omega_0^2 = -1; \ h = 0.25; \ \sigma = 1; \ \varepsilon \ \text{varies})\)

<table>
<thead>
<tr>
<th>(\varepsilon)</th>
<th>(p_e^+)</th>
<th>(p_G^+)</th>
<th>(p_{LG}^+)</th>
</tr>
</thead>
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<tr>
<td>0.2</td>
<td>(100[1 - \exp{-0.02856x}] \times \exp{-0.21099x^2} T] \times \exp{-0.10216x^2} T]</td>
<td>(100[1 - \exp{-0.10338x}] \times \exp{-0.14387x^2} T]</td>
<td>(100[1 - \exp{-0.03713x}] \times \exp{-0.15756x^2} T]</td>
</tr>
<tr>
<td>1</td>
<td>(100[1 - \exp{-0.10216x}] \times \exp{-0.65139x^2} T] \times \exp{-0.05x^4} T]</td>
<td>(100[1 - \exp{-0.18166x}] \times \exp{-0.047504x^2} T]</td>
<td>(100[1 - \exp{-0.15756x}] \times \exp{-0.62589x^2} T]</td>
</tr>
<tr>
<td>10</td>
<td>(100[1 - \exp{-0.24688x}] \times \exp{-2.5x^2} T] \times \exp{-2.5x^2} T]</td>
<td>(100[1 - \exp{-0.35588x}] \times \exp{-0.033650x^2} T]</td>
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<tr>
<td>100</td>
<td>(100[1 - \exp{-0.47541x}] \times \exp{-8.41326x^2} T] \times \exp{-6.55996x^2} T]</td>
<td>(100[1 - \exp{-0.65290x}] \times \exp{-0.62589x^2} T]</td>
<td>(100[1 - \exp{-0.62589x}] \times \exp{-6.55996x^2} T]</td>
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</table>
Table 4. Some numerical values ($\omega_0^2 = -1; \ h = 0.25; \ \sigma = 1; \ \varepsilon \ \text{varies}$)

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$p^+_e(%)$</th>
<th>$x$</th>
<th>$p^+_G(%)$</th>
<th>Error$_G(%)$</th>
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<th>Error$_L(%)$</th>
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</table>

Comments. Numerical results show that: at a specific level of $p^+_e$, the influence of nonlinearity over the extreme response $X(t)$ is similar to the example 1. At $\varepsilon = 0.2$ (weakly nonlinearity) a maximum value of $p^+_e$ exists: $\text{Max}(p^+_e) \approx 25\%$ at $x \approx 2.4$. $\text{Error}_G$ and $\text{Error}_{L_G}$ both are higher than that of the case $\omega_0^2 = 1$ respectively. For weakly nonlinearity $\varepsilon = 0.2$ and strong nonlinearity $\varepsilon = 100$, $p^+_G$ is generally better than $p^+_L$; for medium nonlinearity $\varepsilon = 1-10$ one gets $p^+_L$ more improved than $p^+_G$ at larger values of $p^+_e$ and the contrary.

Example 3. Consider the oscillator of nonlinear stiffness and damping with Gaussian white noise excitation:

$$\ddot{x} + 4h(\frac{x^2}{2} + \omega_0^2 x^2 + \varepsilon \frac{x^4}{4}) + \omega_0^2 x + \varepsilon x^3 = \sigma w(t). \quad (3.5)$$

Similarly, the calculation procedures to obtain the mean up-crossing rates (EMCR and AMCR) to be done in [12]. The results are:
\[ \nu_e(x) = \int_0^\infty \int_0^\infty \exp \left\{ -\frac{4h}{\sigma^2} \left( \frac{1}{2} \dot{x}^2 + \frac{\omega_0^2}{2} x^2 + \frac{\varepsilon}{4} x^4 \right)^2 \right\} d\dot{x} 

\]

Using (2.5), one receives the formula for estimation of the exact exceedance probability \( p_e^+ \). The formulas for calculation of the AMCRs of Caughey and LOMSEC are quite the same as the above Duffing example (3.3), (3.4). (2.9) is used for estimating the approximate exceedance probability \( p_G^+ \) and \( p_{LG}^+ \).

The terminal expressions corresponding with the specific value of the parameters are given in table 5. Fig 9 - 10 show the EEP and AEPs. Numerical values are given in table 6.

**Table 5. Expressions of the exceedance probabilities \( (w_0^2 = 1; \sigma = 1; h, \varepsilon \) varies)**

<table>
<thead>
<tr>
<th>( h, \varepsilon )</th>
<th>( p_e^+ )</th>
<th>( p_G^+ )</th>
<th>( p_{LG}^+ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h = 0.1 )</td>
<td>( 100[1 - \exp{-0.11378} \times \int_0^\infty \dot{x} \exp{-0.4} \times \exp{-1.12135x^2}T] \times \exp{-0.56990x^2}T] \times (0.5\dot{x}^2 + 0.5x^2 + 0.0025x^4)^2 d\dot{x}T] \times \exp{-0.16022} \times \exp{-0.16072} \times \int_0^\infty \dot{x} \exp{-0.4} \times \exp{-1.12135x^2}T] \times \exp{-0.56990x^2}T] \times (0.5\dot{x}^2 + 0.5x^2 + 0.0025x^4)^2 d\dot{x}T] \times \exp{-0.16822} \times \exp{-0.16551}</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( \varepsilon = 0.01 \) | \( \int_0^\infty \dot{x} \exp\{-0.4\} \times \exp\{-0.12135x^2\}T] \times \exp\{-0.56990x^2\}T] \times (0.5\dot{x}^2 + 0.5x^2 + 0.0025x^4)^2 d\dot{x}T] \times \exp\{-0.16022\} \times \exp\{-0.16072\} \times \int_0^\infty \dot{x} \exp\{-0.4\} \times \exp\{-1.12135x^2\}T] \times \exp\{-0.56990x^2\}T] \times (0.5\dot{x}^2 + 0.5x^2 + 0.0025x^4)^2 d\dot{x}T] \times \exp\{-0.16822\} \times \exp\{-0.16551\} | | |

\( h = 1 \) | \( 100[1 - \exp\{-0.37257\} \times \int_0^\infty \dot{x} \exp\{-0.4\} \times \exp\{-2.56187x^2\}T] \times \exp\{-2.09899x^2\}T] \times (0.5\dot{x}^2 + 0.5x^2 + 0.0025x^4)^2 d\dot{x}T] \times \exp\{-0.16822\} \times \exp\{-0.16551\} | | |

\( \varepsilon = 0.2 \) | \( \int_0^\infty \dot{x} \exp\{-4\} \times \exp\{-2.56187x^2\}T] \times \exp\{-2.09899x^2\}T] \times (0.5\dot{x}^2 + 0.5x^2 + 0.0025x^4)^2 d\dot{x}T] \times \exp\{-0.16822\} \times \exp\{-0.16551\} | | |

Fig. 9

Fig. 10
Table 6. Some numerical values ($\omega_0^2 = 1; \sigma = 1; h, \varepsilon$ varies)

<table>
<thead>
<tr>
<th>$h, \varepsilon$</th>
<th>$P_e^+(%)$</th>
<th>$x$</th>
<th>$P_G^+(%)$</th>
<th>Error$_G$(%)</th>
<th>$P_{LG}^+(%)$</th>
<th>Error$_{LG}$(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 0.1$</td>
<td>21.23780</td>
<td>1.22474</td>
<td>8.55215</td>
<td>-59.732</td>
<td>18.54220</td>
<td>-12.693</td>
</tr>
<tr>
<td>$\varepsilon = 0.01$</td>
<td>16.07240</td>
<td>1.41421</td>
<td>4.97521</td>
<td>-69.045</td>
<td>14.29290</td>
<td>-11.072</td>
</tr>
<tr>
<td></td>
<td>11.59350</td>
<td>1.58114</td>
<td>2.87103</td>
<td>-75.236</td>
<td>10.95180</td>
<td>-5.535</td>
</tr>
<tr>
<td></td>
<td>5.18161</td>
<td>1.87083</td>
<td>0.94470</td>
<td>-81.768</td>
<td>6.34977</td>
<td>22.544</td>
</tr>
<tr>
<td></td>
<td>1.04236</td>
<td>2.23607</td>
<td>0.17639</td>
<td>-83.078</td>
<td>2.75184</td>
<td>164.001</td>
</tr>
<tr>
<td>$h = 1$</td>
<td>20.28710</td>
<td>0.70711</td>
<td>13.08010</td>
<td>-35.525</td>
<td>15.95710</td>
<td>-21.344</td>
</tr>
<tr>
<td>$\varepsilon = 0.2$</td>
<td>14.97660</td>
<td>0.80623</td>
<td>9.10420</td>
<td>-39.211</td>
<td>11.91670</td>
<td>-20.431</td>
</tr>
<tr>
<td></td>
<td>10.40120</td>
<td>0.89443</td>
<td>6.29326</td>
<td>-39.495</td>
<td>8.84557</td>
<td>-14.956</td>
</tr>
<tr>
<td></td>
<td>5.76053</td>
<td>1.00000</td>
<td>3.81913</td>
<td>-33.702</td>
<td>5.90495</td>
<td>2.507</td>
</tr>
<tr>
<td></td>
<td>1.18175</td>
<td>1.18322</td>
<td>1.38778</td>
<td>17.434</td>
<td>2.59441</td>
<td>119.540</td>
</tr>
</tbody>
</table>

Comments. We also get the rule that at a specific level of $P_e^+$, the extreme response $X(t)$ reduces when the nonlinearity increases. In any case of the nonlinearity we get $P_{LG}^+$ more improved than $P_G^+$ at larger and medium values of $P_e^+$ and the contrary at smaller values of $P_e^+$.

Example 4. Consider the oscillator with nonlinear damping following $x, \dot{x}$ under Gaussian white noise excitation, which obtained from (3.5) with $\varepsilon = 0$:

$$\ddot{x} + 4h \left( \frac{\dot{x}^2}{2} + \omega_0^2 \frac{x^2}{2} \right) \dot{x} + \omega_0^2 x = \sigma w(t).$$

(3.7)

Similarly, we have:

$$\nu_e(x) = \frac{\int_0^\infty \dot{x} \exp \left\{ -\frac{4h}{\sigma^2} \left( \frac{\dot{x}^2}{2} + \frac{\omega_0^2}{2} x^2 \right) \right\} d\dot{x}}{\int_0^\infty \int_0^\infty x \exp \left\{ -\frac{4h}{\sigma^2} \left( \frac{\dot{x}^2}{2} + \frac{\omega_0^2}{2} x^2 \right) \right\} dx d\dot{x}}.$$

(3.8)

The calculation of the AMCRs of Caughey and LOMSEC, plus the exact exceedance probability $P_e^+$ and the approximate exceedance probability $P_G^+$, $P_{LG}^+$ are the same as the discussions in example 3.

The terminal expressions corresponding with specific value of the parameters are given in table 7. Fig 11 - 12 show the EEP and AEPs. Numerical values are given in table 8.
Table 7. Expressions of the exceedance probabilities ($\omega_0^2 = 1; \sigma = 1; h$ varies)

<table>
<thead>
<tr>
<th>$h$</th>
<th>$p_e^+$</th>
<th>$p_G^+$</th>
<th>$p_{LG}^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 0.25$</td>
<td>$100[1 - \exp(-0.17959) \int_0^\infty \frac{\hat{x}}{x} \exp{-0.5\hat{x}^2 } d\hat{x}T} + 0.5\hat{x}^2 } d\hat{x}T}</td>
<td>$100[1 - \exp(-0.15915) \int_0^\infty \frac{\hat{x}}{x} \exp{-0.87058\hat{x}^2 } d\hat{x}T}</td>
<td>$100[1 - \exp(-0.15915) \int_0^\infty \frac{\hat{x}}{x} \exp{-0.87058\hat{x}^2 } d\hat{x}T}</td>
</tr>
<tr>
<td>$h = 1$</td>
<td>$100[1 - \exp(-0.35917) \int_0^\infty \frac{\hat{x}}{x} \exp{-2\hat{x}^2 } d\hat{x}T} + 0.5\hat{x}^2 } d\hat{x}T}</td>
<td>$100[1 - \exp(-0.15915) \int_0^\infty \frac{\hat{x}}{x} \exp{-1.82862\hat{x}^2 } d\hat{x}T}</td>
<td>$100[1 - \exp(-0.15915) \int_0^\infty \frac{\hat{x}}{x} \exp{-1.82862\hat{x}^2 } d\hat{x}T}</td>
</tr>
</tbody>
</table>

Fig. 11  Fig. 12

Table 8. Some numerical values ($\omega_0^2 = 1; \sigma = 1; h$ varies)

<table>
<thead>
<tr>
<th>$h$</th>
<th>$p_e^+(%)$</th>
<th>$x$</th>
<th>$p_G^+(%)$</th>
<th>Error$_G$(%)</th>
<th>$p_{LG}^+(%)$</th>
<th>Error$_{LG}$(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 0.25$</td>
<td>20.46300</td>
<td>1.00000</td>
<td>16.10830</td>
<td>-21.281</td>
<td>18.11980</td>
<td>-11.451</td>
</tr>
<tr>
<td>10.37180</td>
<td>1.30384</td>
<td>8.35266</td>
<td>-19.468</td>
<td>10.29890</td>
<td>-0.703</td>
<td></td>
</tr>
<tr>
<td>5.18964</td>
<td>1.50000</td>
<td>4.90776</td>
<td>-5.432</td>
<td>6.51164</td>
<td>25.474</td>
<td></td>
</tr>
<tr>
<td>1.02397</td>
<td>1.80278</td>
<td>1.83424</td>
<td>79.130</td>
<td>2.77993</td>
<td>171.485</td>
<td></td>
</tr>
<tr>
<td>$h = 1$</td>
<td>20.46250</td>
<td>0.70711</td>
<td>16.10830</td>
<td>-21.279</td>
<td>17.41640</td>
<td>-14.886</td>
</tr>
<tr>
<td>4.83868</td>
<td>1.07238</td>
<td>4.67409</td>
<td>-3.402</td>
<td>5.66300</td>
<td>17.036</td>
<td></td>
</tr>
<tr>
<td>1.12292</td>
<td>1.26491</td>
<td>1.92738</td>
<td>71.640</td>
<td>2.52770</td>
<td>125.101</td>
<td></td>
</tr>
</tbody>
</table>

Comments. Comments for this case are the same as for the above in example 3.

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4. Conclusions

Through the analysis of the exact exceedance probability of the nonlinear systems considered, some important characters of the exceedance probability, especially the influence of nonlinearity over the exceedance probability are investigated:

- It is obvious that the exceedance probability is generally a contra-variant function versus the extreme response ($p^+$ goes down when $X(t)$ increases). However it is a type of system whose exceedance probability contains a maximum-peak at a value of the extreme response $X(t)$; which is not as small as the case of Duffing with $\omega_0^2 = -1$.
- At a specified level of $p^+$, the extreme response $X(t)$ reduces when the nonlinearity increases; in other words, the nonlinearity effect causes a reduction of the exceedance probability.

The obtained result shows the applied possibility of the proposed linearization techniques for estimating the approximate exceedance probability:

- In a specified large domain of $p^+$, it is usually to get $p_{LG}^+$ more improved than $p_G^+$. When $p^+$ goes down at a specified smaller value, both $p_G^+$ and $p_{LG}^+$ have rather high relative errors though the absolute errors are not so high, however one gets $p_G^+$ better than $p_{LG}^+$.
- The influence of the nonlinearity effect over the degree of accuracy of the approximate exceedance probability $p_G^+$ and $p_{LG}^+$ is rather complicated and in general it is only possible to obtain individual answers for each type of system.

In short, according to design purposes ($p^+$ required is large or small) as well as basing upon each specific system, we apply either the Caughey or the LOMSEC for estimating the approximate exceedance probability.

The results and comments in this paper are supplementary to the previous researches on the approximate exceedance probability through the linearization using the Caughey. Further more, this is the first research conducted on the analysis of the approximate exceedance probability using the LOMSEC linearization. The above tendency of research can be enlarged for other nonlinear systems to aim at discovering more natures of the exceedance probability. The expansion also can be used for multi-degree of freedom randomly nonlinear systems.

REFERENCES


Received September 14, 2000

PHÂN TÍCH XÁC SUẤT VUỘT CỦA ĐÁP ÚNG CHUYỂN VỊ CỦA CÁC CÁU TRỤC PHI TUYỂN NGẪU NH叕N

Bài báo trình bày việc tính toán xác suất vượt chính xác của các đáp ứng một số hệ phụ tuyến ngẫu nhiên chịu kích động ổn định mà có hàm mặt độ xác suất chính xác có thể tìm được. Tiếp theo, các xác suất vượt gần đúng được xác định trên cơ sở phân tích các hệ phụ tuyến tính hóa trong đường dùng phương pháp Caughey truyền thông và "tiêu chuẩn sai số bình phương trung bình khí vực" (LOMSEC). Các số bán xác suất vượt gần đúng đối với xác suất vượt chính xác được đưa ra. Kết quả nhận được cho ra một số tính chất quan trọng của xác suất vượt, ảnh hưởng của tính phụ tuyến đối với xác suất vượt. Việc đánh giá khả năng áp dụng các tiêu chuẩn tuyến tính hóa Caughey và LOMSEC để tính xác suất vượt gần đúng được thực hiện.