A NEW APPROACH TO THE STABILITY PROBLEM OF PLATES SUBJECTED TO ARBITRARY COMPLEX LOADING

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SUMMARY. The pre-buckling and post-buckling deformation processes are assumed to be less complicated, i.e. processes of average curvature, the influence of complex loading on the stability of plates was analysed in [1]. In this paper eliminating this restriction, post-buckling process may be arbitrary complicated, a generalized expression for determining critical force is formulated by Bubnov-Galiorkin's method and loading parameter method.

1. PRE-BUCKLING PROCESS

Let consider a rectangular plate subjected to biaxial compressions of intensities \( p(t) \) and \( q(t) \). At any moment \( t \) there exists a membrane plane stress state in the plate

\[
\sigma_{11} = -p(t), \quad \sigma_{22} = -q(t), \quad \sigma_{12} = \sigma_{33} = \sigma_{32} = \sigma_{31} = 0,
\]

so that

\[
\sigma = \frac{1}{3}(p + q), \quad \sigma_u = (p^2 - pq + q^2)^{1/2}. \tag{1.1}
\]

The strain velocity tensor is determined from the following equations

\[
\dot{\varepsilon}_{11} = \frac{1}{\sigma_u/s} \left( \frac{1}{\phi'(s)} - \frac{1}{\phi(s)} \right) \frac{(p \dot{q} + q \dot{p} - \frac{1}{2} p \dot{q} - \frac{1}{2} q \dot{p}) (p - \frac{1}{2} q)}{p^2 - pq + q^2},
\]

\[
\dot{\varepsilon}_{22} = \frac{1}{\sigma_u/s} \left( \frac{1}{\phi'(s)} - \frac{1}{\phi(s)} \right) \frac{(p \dot{q} + q \dot{p} - \frac{1}{2} p \dot{q} - \frac{1}{2} q \dot{p}) (q - \frac{1}{2} p)}{p^2 - pq + q^2}, \tag{1.2}
\]

and the arc-length of deformation trajectory is evaluated from

\[
\frac{ds}{dt} = \sqrt{2(\dot{\varepsilon}_{11}^2 + \dot{\varepsilon}_{22}^2 + \dot{\varepsilon}_{11} \dot{\varepsilon}_{22})^{1/2}} = F(\varepsilon, p, q). \tag{1.3}
\]

2. POST-BUCKLING PROCESS

Suppose that external forces depend on a loading parameter \( t \). The parameter \( t \) increases and will reach some value \( t_k \). At this moment \( t_k \) a bifurcation of equilibrium states is assumed to appear: with an infinitesimal small increment of external force the plate is buckled and receives possible increments of deformation

\[
\delta \varepsilon_{ij} = \delta \varepsilon_{*ij} = Z \delta X_{ij},
\]

where
These post-buckling deformation processes may be arbitrary complicated, so that the stress-strain relations are defined by the elastoplastic deformation process theory [2]

\[
\delta \sigma_{ij} = \frac{2}{3} A (\delta \varepsilon_{ij} + \varepsilon_{ij} \delta \varepsilon_{kk}) + (P - A) \frac{\sigma_{kk} \delta \varepsilon_{kk}}{\sigma_u^2} \sigma_{ij},
\]

where

\[
A = \frac{\sigma_u f}{\sin \theta} = \frac{\sigma_u}{s} \left[ 1 + \frac{3G \sigma_u}{\sigma_u - 1} \right]^{\frac{1}{2}} \frac{1}{2} \left( 3G + \frac{\sigma_u}{s} - \frac{1}{2} \right) \left( 3G - \frac{\sigma_u}{s} \right) \cos \theta,
\]

\[
P = \frac{\psi}{\cos \theta} = \phi'(s) - \frac{3G - \phi t - \cos \theta}{2} = \frac{1}{2} \left( 3G + \phi' \right) - \frac{1}{2} \frac{3G - \phi t}{\cos \theta},
\]

with

\[
\cos \theta = \frac{\varepsilon_{ij} \delta \varepsilon_{ij}}{\sigma_u \delta s}, \quad \delta s = \frac{2}{\sqrt{3}} \left( \delta \varepsilon_{11}^2 + \delta \varepsilon_{22}^2 + \delta \varepsilon_{11} \delta \varepsilon_{22} + \delta \varepsilon_{12}^1 \right)^{1/2}.
\]

According to Ilyushin's approximate statement \( \delta T_{ij} = 0 \) and not accounting the unloading, we obtain

\[
\delta \varepsilon_{ij}^* = 0, \quad \cos \theta = -\frac{Z \sigma_u \delta \varepsilon_{ij}^*}{\sigma_u \delta s}, \quad \delta s = \frac{2}{\sqrt{3}} Z \left( \delta w_{11}^2 + \delta w_{22}^2 + \delta w_{12}^1 \right)^{1/2}.
\]

Hence the quantity \( \cos \theta \) does not depend on \( Z \), such that \( A \) and \( P \) do not depend on \( Z \) as well.

The bending moments are of the form

\[
\delta M_{ij} = \int_{-h/2}^{h/2} \delta \sigma_{ij} Z dZ = \frac{h^2}{12} \left[ -\frac{2}{3} A (\delta w_{11}^1 + \delta w_{22}^1 + \delta w_{12}^1) + (A - P) \frac{\sigma_{ij} \sigma_{kk} \delta \varepsilon_{kk}}{\sigma_u^2} \right].
\]

Substituting \( \delta M_{ij} \) by (2.3), where \( A, P \) contain \( \delta w \), into the stability equation

\[
\frac{\partial^2 M_{ij}}{\partial x_l \partial x_j} + T_{ij} \delta X_{ij} = 0,
\]

we obtain

\[
\left[ A - \frac{3}{4} (A - P) \frac{p^2}{p^2 - pq + q^2} \right] \frac{\partial^4 \delta w}{\partial x^4} + 2 \left[ A - \frac{3}{4} (A - P) \frac{p^2}{p^2 - pq + q^2} \right] \frac{\partial^4 \delta w}{\partial x^2 \partial y^2} + \\
+ \left[ A - \frac{3}{4} (A - P) \frac{q^2}{p^2 - pq + q^2} \right] \frac{\partial^4 \delta w}{\partial y^4} + \frac{9p}{h^2} \frac{\partial^2 \delta w}{\partial x^2} + \frac{9p}{h^2} \frac{\partial^2 \delta w}{\partial y^2} + \\
+ \frac{1}{2} \frac{\partial^2 A}{\partial x^2} \left( \frac{\partial^2 \delta w}{\partial x^2} + \frac{\partial^2 \delta w}{\partial y^2} \right) + \frac{1}{2} \frac{\partial^2 A}{\partial y^2} \left( \frac{\partial^2 \delta w}{\partial x^2} + \frac{\partial^2 \delta w}{\partial y^2} \right) + \frac{\partial^2 A}{\partial x \partial y} \frac{\partial^2 \delta w}{\partial x \partial y} \]

\[
- \frac{3p}{4} \frac{\partial^2 \delta w}{\partial x^2} + q \frac{\partial^2 \delta w}{\partial y^2} \left[ p \left( \frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 P}{\partial x^2} \right) + q \left( \frac{\partial^2 A}{\partial y^2} - \frac{\partial^2 P}{\partial y^2} \right) \right] = 0.
\]

Satisfying kinematic boundary conditions with edges simply supported we can find a solution of the form
\[ \delta w = C \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \]

Notice that \( \cos \theta \) does not contain \( C \), so that \( A, P \) also do not contain \( C \). Substituting \( \delta w \) into the stability equation and applying Bubnov-Galiorkin's method [3]:

\[
\int_0^a \int_0^b X \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \, dx \, dy = 0, \tag{2.5}
\]

where \( X \) is the expression in the left hand side of the stability equation (2.4), give from (2.5) an equation for finding a critical force.

Consider the case of a square plate. In this case

\[
\delta w = C \sin \frac{\pi x}{a} \sin \frac{\pi y}{a}, \tag{2.6}
\]

\[
\cos \theta = \frac{p + q}{(p^2 - pq + q^2)^{1/2} \sqrt{3} \left( 3 + \ctg^2 \frac{\pi x}{a} \ctg^2 \frac{\pi y}{a} \right)^{1/2}}. \tag{2.7}
\]

According to (2.6), the equation (2.4) reduces to

\[
X = \left[ \frac{\pi^4}{a^4} \left( 4 - \frac{3}{4} \frac{(p + q)^2}{p^2 - pq + q^2} \right) A + \frac{3 \pi^4}{a^4} \frac{(p + q)^2}{p^2 - pq + q^2} P - \frac{9 \pi^2}{a^2 h^2} (p + q) + \frac{3 \lambda^2}{4a^2} \left( -2 + \frac{p(p + q)}{p^2 - pq + q^2} \right) \frac{\partial^2 A}{\partial x^2} + \frac{3 \pi^2}{4a^2} \left( -2 + \frac{g(p + q)}{p^2 - pq + q^2} \right) \frac{\partial^2 A}{\partial y^2} \right] \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \frac{\pi x}{a} \frac{\pi y}{a} \cos \frac{\pi x}{a} \cos \frac{\pi y}{a} = 0. \tag{2.8}
\]

The expressions of \( A \) and \( P \) from (2.2) and (2.7) are written in the form

\[
A = A_1(s, p, q) + A_2(s, q, p) f(x, y), \tag{2.9}
\]

where

\[
A_1 = \frac{1}{2} \left( 3G + \frac{\sigma_u}{s} \right), \quad A_2 = \frac{1}{2} \left( 3G - \frac{\sigma_u}{s} \right) \frac{p + q}{\sigma_u},
\]

\[
\sigma_u = (p^2 - pq + q^2)^{1/2}, \quad f(x, y) = \frac{\sqrt{3}}{2} \left( 3 + \ctg^2 \frac{\pi x}{a} \ctg^2 \frac{\pi y}{a} \right)^{-1/2},
\]

and

\[
P = P_1(s, p, q) + P_2(s, q, p) g(x, y), \tag{2.10}
\]

where

\[
P_1 = \frac{1}{2} \left( 3G + \phi' \right), \quad P_2 = \frac{1}{2} \left( 3G - \phi' \right) \frac{\sigma_u}{p + q}, \quad g(x, y) = \frac{2}{\sqrt{3}} \left( 3 + \ctg^2 \frac{\pi x}{a} \ctg^2 \frac{\pi y}{a} \right)^{1/2} = \frac{1}{f(x, y)}.
\]

Hence

\[
\frac{\partial^2 A}{\partial x^2} = A_2 f_{xx}, \quad \frac{\partial^2 A}{\partial y^2} = A_2 f_{yy}, \quad \frac{\partial^2 A}{\partial x \partial y} = A_2 f_{xy}, \quad \frac{\partial^2 P}{\partial x^2} = P_2 g_{xx}, \quad \frac{\partial^2 P}{\partial y^2} = P_2 g_{yy}.
\]
Applying Bubnov-Gačorkin's method

\[ \int_0^a \int_0^a X \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \, dx \, dy = 0, \]

we obtain

\[
\frac{\pi^2}{4} \left( 4 - \frac{3}{4} \frac{(p + q)^2}{p^2 - pq + q^2} \right) A_1 + \frac{3\pi^2}{16} \frac{(p + q)^2}{p^2 - pq + q^2} P_1 + \frac{\pi^2}{a^2} \frac{3}{4} \left( 4 - \frac{3}{4} \frac{(p + q)^2}{p^2 - pq + q^2} \right) C_1 A_2 + \\
\frac{3\pi^2}{4a^2} \frac{(p + q)^2}{p^2 - pq + q^2} C_6 P_2 - \frac{3}{4} \frac{q(p + q)}{p^2 - pq + q^2} C_7 P_2 = 0.
\]

(2.11)

where \(C_i(i = 1 \pm 7)\) are constants, evaluated by the following integrals

\[
C_1 = \int_0^a \int_0^a f(x, y) \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{a} \, dx \, dy; \quad C_2 = \int_0^a \int_0^a f''(x, y) \sin^2 \frac{\pi y}{a} \sin^2 \frac{\pi x}{a} \, dx \, dy;
\]

\[
C_3 = \int_0^a \int_0^a f''(x, y) \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{a} \, dx \, dy; \quad C_4 = \int_0^a \int_0^a f''(x, y) \sin \frac{\pi x}{a} \cos \frac{\pi y}{a} \sin \frac{\pi y}{a} \cos \frac{\pi x}{a} \, dx \, dy;
\]

\[
C_5 = \int_0^a \int_0^a g(x, y) \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{a} \, dx \, dy; \quad C_6 = \int_0^a \int_0^a g''(x, y) \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{a} \, dx \, dy.
\]

Substituting the expressions of \(A_1, A_2, P_1, P_2\) into the equation (2.11), from which we get the formula for determining a critical force

\[
i^2 \equiv \frac{9a^2}{\lambda^2} = \frac{4}{p + q} \left( \frac{\pi^2}{2} (3G + \frac{\sigma_u}{s}) + \frac{3\pi^2}{32} (p + q) \right) \frac{(p + q)^2}{p^2 - pq + q^2} + \\
\frac{1}{2} \left( 3G - \frac{\sigma_u}{s} \right) \frac{p + q}{\sigma_u} \frac{\pi^2}{a^2} \left( 4 - \frac{3}{4} \frac{(p + q)^2}{p^2 - pq + q^2} \right) C_1 + \frac{3}{4} \left( 4 - \frac{3}{4} \frac{(p + q)^2}{p^2 - pq + q^2} \right) C_2 + \\
\frac{3}{4} \left( 4 - \frac{3}{4} \frac{(p + q)^2}{p^2 - pq + q^2} \right) C_3 + C_4 + \frac{1}{2} (3G - \phi') \frac{3\pi^2}{p + q} \frac{(p + q)^2}{4a^2} \frac{p + q}{p^2 - pq + q^2} C_5 - \\
\frac{3}{4} \frac{p(p + q)}{p^2 - pq + q^2} C_6 - \frac{3}{4} \frac{q(p + q)}{p^2 - pq + q^2} C_7 = H(s, p, q),
\]

(2.12)

where \(\sigma_u = (p^2 - pq + q^2)^{1/2}\)

Now, suppose that \(p \equiv p(t), q \equiv q(t)\) are known as functions of loading parameter \(t\). The equations (1.3) and (2.12) are satisfied simultaneously, from that we can determine a critical value \(t_k\) of the loading parameter. Then the critical forces are as follows

\[
p_{th} = p(t_k); \quad q_{th} = q(t_k).
\]
In particular case when post-buckling process is a process of average curvature, we have

\[ A = \frac{\sigma_u}{s}, \quad P = \phi'(s) \]

from (2.2) to get

\[ \cos \theta = -\frac{P + q}{\sigma_u} f(x, y) = 1, \]

the functions \( f(x, y) \) and \( g(x, y) \) must be constant. Hence we obtain

\[ -\frac{p + q}{\sigma_u} C_4 = \frac{a^2}{4}, \quad -\frac{\sigma_u}{p + q} C_5 = \frac{a^2}{4}, \quad C_2 = C_3 = C_4 = C_6 = C_7 = 0. \]

Finally, from (2.12) we get the known expression of the critical force in [1]

\[ t^2 = \frac{9a^2}{k^2} = \frac{4}{p + q} \left[ \pi^2 \frac{\sigma_u}{s} + \frac{3\pi^2}{16} \left( \phi' - \frac{\sigma_u}{s} \right) \frac{(p + q)^2}{p^2 - pq + q^2} \right], \]

or

\[ t^2 = \frac{9a^2}{k^2} = \frac{3G\pi^2 \varphi_N}{p + q} \left[ 4 - \frac{3}{4} \left( 1 - \frac{\varphi_1}{\varphi_N} \right) \frac{(p + q)^2}{p^2 - pq + q^2} \right], \]

where

\[ \varphi_N = \frac{\sigma_u/s}{3G} = \left( \frac{p^2 - pq + q^2}{3G} \right)^{1/2}, \quad \varphi_1 = \frac{\phi'(s)}{3G}. \]

**CONCLUSIONS**

1. A typical example on the application of the general elastoplastic deformation process theory in the stability problem of plates is given.
2. Establishing a method for formulating an expression of critical force in the general case without any restriction on the arbitrary complex loading, this approach has a practical meaning in engineering calculation.
3. Received result in [1] is a particular case of the generalized expression (2.12) shown in this paper.

**REFERENCES**


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**MỘT CÁCH TIẾP CẦN MÔI BÀI TOÁN ƠN ĐỊNH CỦA BẢN CHIẾU TAI PHỤC TAP BẤT KỲ**

Gồm đa vị trí thiết hàn chê lên quá trình biến dạng sau khi mặt ổn định, ở đây có thể xem đó là quá trình phức tạp bất kỳ. Bằng phương pháp Bubnov - Galerkin và phương pháp tham số tài đã thiết lập được công thức hiện xắc định lý tối hàn tổng quát hơ tư kết quả nhận được trước đây.