SENSITIVITY ANALYSIS FOR A PROBLEM OF OPTIMAL STRUCTURE DESIGN

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SUMMARY. On the basis of theory of structure sensitivity analysis, the gradient-projection method and the finite element method, a detailed and effective algorithm of determination of the sensitivity-vector for the optimal structure design problem is proposed. Numerical results illustrating the program corresponding to this algorithm have been given for some plane frames.

§1. PROBLEM OF OPTIMAL STRUCTURE DESIGN

The problem of optimal elastic structure design here is formulated as follows [1]:

Determine design variable vector \( \mathbf{b} \in \mathbb{R}^n \) to minimize objective function \( \psi_0(z, \xi, \mathbf{b}) \) (it can be the weight of the structure) satisfying the following state equations and function constraints:

1. The equations of equilibrium for a structure (static and dynamic):

\[
\begin{align*}
\mathbf{h}(z, \mathbf{b}) &= \mathbf{0}, \\
\mathbf{K}(\mathbf{b})\mathbf{y} &= \xi \mathbf{M}(\mathbf{b})\mathbf{y}
\end{align*}
\]

2. Function constraints:

\[
\psi_j(z, \xi, \mathbf{b}) \geq 0, \quad j = 1, m
\]

(they can be constraints on displacement, stress or natural frequency and design variables). where

\( \mathbf{K}(\mathbf{b}) \) - stiffness matrix of the system.

\( \mathbf{M}(\mathbf{b}) \) - mass matrix

\( z \in \mathbb{R}^n \) - displacement vector

\( \xi \in \mathbb{R}^n \) - natural frequency of vibration of system

\( \mathbf{y} \in \mathbb{R}^n \) - eigenvector

\( z, \xi, \mathbf{y} \) all they are state variables.

Note that, if the structure is a truss-system of \( m \) elements, then the weight function has the form:

\[
\psi_0 = \sum_{i=1}^{m} \xi_i L_i A_i
\]

where

\( L_i \) - the length of \( i \)-th truss element

\( A_i \) - its cross area

\( \xi_i \) - material density of \( i \)-element

The design variables \( \mathbf{b} \) (they can be \( L_i \) or \( A_i \)) are chosen and changed by a designer, but state variables \( z, \xi, \mathbf{y} \) or stresses of a structure depend on equilibrium conditions and correlations between displacements and stresses. Therefore a designer can not change state variables directly.
There are many methods for solving the above problem [2, 3, 4]. For almost iterative optimal structure methods it requires to know gradient values, which are received in result of sensitivity analysis. The essence of this analysis is presented in the following section.

§2. SENSITIVITY ANALYSIS OF STRUCTURE

Here the influence of project variation is investigated by approximating nonlinear function of the problem with linear expression corresponding to considering variables.

The change of the objective function $\psi_0(x, \xi, b)$ and of the function constrains $\psi_j(x, \xi, b), \ j = 1, m$ corresponding to small changes of variables can be written in the form:

$$\frac{\partial \psi_0}{\partial z}[x_0, \xi_0, 0] \delta z + \frac{\partial \psi_0}{\partial \xi}[x_0, \xi_0, 0] \delta \xi + \frac{\partial \psi_0}{\partial b}[x_0, \xi_0, 0] \delta b,$$

$$\frac{\partial \psi_j}{\partial z}[x_0, \xi_0, 0] \delta z + \frac{\partial \psi_j}{\partial \xi}[x_0, \xi_0, 0] \delta \xi + \frac{\partial \psi_j}{\partial b}[x_0, \xi_0, 0] \delta b.$$  \hspace{1cm} (2.1)

Because equation of equilibrium $h(x_0, 0) = 0$ is also true in the case of increasing displacement $z$ and design variable $b$ with a small value, we have:

$$h(x_0 + \delta z, b_0 + \delta b) = 0$$

it follows:

$$\frac{\partial h}{\partial x}(x_0, b_0) \delta x + \frac{\partial h}{\partial b}(x_0, b_0) \delta b = 0$$  \hspace{1cm} (2.2)

This equation can be considered as the condition to determine $\delta z$ as a function of $\delta b$. Then with the notation

$$J = \frac{\partial h_0}{\partial x}(x_0, 0)$$

equation (2.2) has the form:

$$J\delta z = -\frac{\partial h}{\partial b} \delta b$$  \hspace{1cm} (2.3)

On the other hand, if we consider column vectors $\lambda^i$ as a solution of conjugate equation:

$$J^T \lambda^i = \frac{\partial \psi_i^T}{\partial z}, \ 0 \leq i \leq m$$  \hspace{1cm} (2.4)

from (2.3) and (2.4) we have a relation between $\delta b$ and $\delta z$:

$$-\lambda^T \frac{\partial h}{\partial b} \delta b = \frac{\partial \psi_i}{\partial z} \delta z.$$  \hspace{1cm} (2.5)

Similarly, a variation $\delta \xi$ depends on $\delta b$ [1] as follows:

$$\delta \xi = \ell^T \delta b$$

where

$$\ell^T = \left[ \frac{\partial}{\partial b} (y^T K(b)y) - \xi \frac{\partial}{\partial b} (y^T M(b)y) \right] \delta b$$  \hspace{1cm} (2.6)

Substituting (2.5) and (2.6) into (2.1) we get equations

$$\delta \psi_0 = \ell^T \delta b$$

$$\delta \psi = \ell^T \delta b$$
where vector $\xi_0$ and column-vector $\xi_i$ of matrix $\ell$ at point $z_0$, $\xi_0$, $b_0$ according to formula

$$\ell^i = \frac{\partial \psi_i^T}{\partial b} - \frac{\partial h^T}{\partial b} \chi_i + \frac{\partial \psi_i}{\partial \xi} \ell_i.$$  \hspace{1cm} (2.7)

Components of the vector $\ell_i$ are called sensitivity coefficients of the constraint-function $\psi_i$ corresponding to design variable $b$. These vectors give derivatives of the object function and constraint-function ($\ell^i_j$ is derivative of $\psi_i$ with respect to $j$-th design variable).

These components $\ell^i_j$ are needed for designers because they present the influence of design variable changes on object function or constraint-function. If $\ell^i_j > 0$ then increasing $b_j$ follows increasing $\psi_i$. If $\ell^i_j < 0$ then increasing $b_j$ follows decreasing $\psi_i$. Moreover, order of a value of the different sensitivity coefficients $l^i_j$ informs designer that what design variable has great or small influence on $\psi_i$.

Remark that an elastic structure, best of all, is simulated by the finite element methods. When equation (1.1) has the linear form:

$$h(b, z) = K(b)z - S(b) = 0$$  \hspace{1cm} (2.8)

$S(b)$ - matrix of external loads and the Jacobian of this equation is expressed as:

$$J = \frac{\partial h}{\partial x} = K$$

Because $K$ is a symmetric matrix, equation (2.4) takes the form:

$$K\ell^i = \frac{\partial \psi_i}{\partial x}$$  \hspace{1cm} (2.9)

This equation has the same form as the equation (2.8) does. The difficulty in obtaining $\ell^i$ lies in derivation $\partial h(b, z)/\partial b$ or $(\partial K(b)z - S(b))/\partial b$. In this work it is formed by derivation with respect to design variables for each element stiffness matrix and sum the results by an algorithm of the finite element method.

### §3. CONNECTION BETWEEN SENSITIVITY VECTOR $\ell$ AND SOLUTION OF OPTIMAL PROBLEM

Now the optimal problem (1.1) can be reduced to the following problem [1]: To find $\delta b$ for minimum $\delta \psi_0 = \ell^0 T \delta b$, satisfying conditions:

1. $\delta b^T W \delta b \leq \beta^2$, where $\beta$ - small parameter; $W$ - weight matrix
2. linear constraint

$$\delta \psi = \frac{\partial \psi}{\partial b} \delta b = \ell^j \delta b = \left\{ \begin{array}{ll} = \Delta \psi_j, & \text{when } j = 1, ..., n \\ \leq \Delta \psi_j, & \text{when } j > n; \quad \psi_j(b_0) \geq -\varepsilon \end{array} \right.$$  \hspace{1cm}

where $\Delta \psi_j = -\psi_j(b_0)$ is the limit of the change of the constraint-functions for $e$-active constraint [1]. On the basic theorems about the existence of a solution of nonlinear programming problems it is shown in [1] that there exists a vector-multiplier $\mu$ and a scalar $\gamma \geq 0$ satisfying the following equations:

$$\ell_0 + \ell \mu + 2\gamma W \delta b = 0$$

$$\mu_i (\delta \psi_i - \Delta \psi_i) = 0 \quad i \geq n$$

$$\gamma (\delta b^T W \delta b - \beta^2) = 0$$

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From these equations we have formulae determining $\delta b, \mu, \gamma$ by the sensitivity vector,

$$\delta b = -\frac{1}{2} \delta b_1 + \delta b_2$$  \hspace{1cm} (3.1)

where

$$\delta b_1 = W^{-1}[\xi_0 + \ell \mu_1], \hspace{0.5cm} \delta b_2 = -W^{-1} \ell \mu_2$$

$\mu_1$ and $\mu_2$ are solutions of the equations

$$M_{\psi \phi} \mu_1 = -M_{\psi \psi_0}, \hspace{0.5cm} M_{\psi \phi} \mu_2 = -\Delta \psi$$

where

$$M_{\psi \phi} = \ell^T W^{-1} \ell, \hspace{0.5cm} M_{\psi \psi_0} = \ell^T W^{-1} \ell_0.$$  

If all $\mu_i$ corresponding to $\psi_i(b_0)$ - are positive when $b_0$ is a solution satisfying Kunatake condition. If exist some $\mu_i < 0$ corresponding to $\psi_i(b_0) \geq \epsilon$ for $i > 0$, then one can receive a better solution by excluding the corresponding constraint $\psi_i(b)$ and it is necessary to calculate $\ell, M_{\psi \phi}, M_{\psi \psi_0}, \psi_i, \psi_0$, once more and the process will be stopped when all $\mu_i > 0$ and $\delta b$ will be calculated by (3.1).

Thus, clearly that the sensitivity vector of the constraint-functions $\psi_i$ corresponding to design variables has important role in calculating variables $\delta b$ in the optimum design of the structures.

§4. ALGORITHM FOR DETERMINATION OF SENSITIVITY VECTORS
NUMERICAL EXAMPLES AND CONCLUSION

As the result of above investigation, the effective algorithm for obtaining sensitivity vectors is presented, and consists of the following steps:

1. Chose engineering design variable $b_0$ for project.
2. Solve equilibrium equations:

$$h(z) = K(b)z - S(b) = 0$$

or:

$$K(b)y = \xi M(b)y,$$

finding $z, y, \xi$, corresponding to $b_0$.
3. Check if the obtained values $z, y, \xi$, satisfy the conditions of the constraints. If they don't, then establish corresponding constraint-function vector $[\psi]$.
4. Solve equation $K\lambda_i = \partial \psi_i^T / \partial z$ to find $\lambda_i$.
5. Calculate derivatives $\partial h / \partial b, \partial \psi_i / \partial b, \partial \psi_i / \partial \xi$ and $\ell_i$ corresponding to constraint $\psi_i$ by formula (8). Two numerical examples are given, which illustrate the algorithm and its effectiveness.

Example 1. Consider the truss system of ten elements (see fig. 1.) with the parameters:

$E = 10^7 N/m^2, \varepsilon = 0.1 N/m^3$, the critical displacement $z = 2m$, critical stress $\sigma _c = \pm 25.10^3 N/m^2$.

This system is subjected to external loads: $P_1 = 10^5 N, P_2 = 10^5 N, P_3 = 15.10^3 N, P_4 = 13.10^3 N$.

The results (see table 1) show that the sensitivity vectors $\ell_1, \ell_2$ are corresponding to displacements in the direction of the axis $Oy$ at the points 1 and 4 (their values are greater than critical displacements). Vectors $\ell_3, \ell_4, \ell_5$ are corresponding to the stresses in the elements 5, 6, 7 (there the stresses are greater than critical value). The column vector $\ell_5$ shows that, in order to decrease the stress in the element 5 one must increase the cross areas of the elements 3-7, 9 and decrease the cross area of the element 1, 2, 8 and 10.

Note that, if the external loads $P_3$ and $P_4$ are absent, our results for sensitivity vectors coincide with the results given in [1], that shows exactness of the program.
Example 2. Consider the truss system of 29 elements (see fig. 2.) with the parameters: $E = 10^7 N/m^2$, $\varepsilon = 0.1 N/m^3$, the critical displacement $z = 0.026 m$, the critical stress $\sigma_c = 26.10^3 N/m^2$. This system is subjected to external loads: $P_1 = 500 N$, $P_2 = 1000 N$, $P_3 = 1200 N$, $P_4 = 1400 N$, $P_5 = 1600 N$.

The results (see table 2) show that sensitivity vectors $\ell_1$ are corresponding to the displacement in the direction of the axis Ox at the points 16 (these values are greater than critical displacement). The vectors $\ell_2$, $\ell_3$, $\ell_4$, $\ell_5$ are corresponding to the stresses in the elements 1, 4, 5, 10 (those stresses are greater than critical value). Column vector $\ell_2$ shows that, in order to decrease the stress in element 1 one must increase the cross areas of the elements 1, 3-5, 9, 10, 11, 13, 14, 16, 19, 20 and decrease the cross area of the element 2, 6.

In the tables 1 and 2 $N$ is an index of the beam, the second column is the cross area of the beam. To decrease the stress or displacement of the beam corresponding $\ell_1$ need to be interested.
in the value of the column vector component $\ell_4$. If the value of vector component $\ell_4$ is negative then the cross area of corresponding beam must be increased. If the value of vector component $\ell_4$ is positive then cross area of corresponding beam must be decreased. The order of value $\ell_4$ in the tables informs us about the change level of the cross area.

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CONCLUSION

Sensitivity analysis have a great role in solving the optimal design problem. With the help of above mentioned algorithm and program we obtain the sensitivity vectors for solving optimal design problems.

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PHÂN TÍCH ĐỒ NHẢY CÁM TRONG BÀI TOÁN THIẾT KẾ TỐI ƯU

Trên cơ sở lý thuyết phân tích đồ nhảy cám của cấu trúc, phương pháp chiều gradient và phương pháp phân từ hữu hạn, một thuật toán chỉ tiết và hiệu quả để xác định véc to nhảy cám trong bài toán thiết kế tối ưu đã được đưa ra. Các kết quả số minh họa cho chương trình trong ứng với thuật toán trên đã được thực hiện với một số dân phương.