NON-LINEAR OSCILLATIONS OF PENDULUM WITH VERTICALLY MOVING MULTIFREQUENCY SUPPORT POINT

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In this paper the problem of nonlinear oscillations of the pendulum with moving two frequency support point [1] is extended to the case of vertical multifrequency motion of the support point by means of the asymptotic method of nonlinear oscillations [2]. The stability conditions of the equilibrium positions of the pendulum are delivered.

§1. EQUATION OF MOTION

Let us consider a mathematical pendulum of length \( \ell \), mass \( m \), whose support point A moves vertically by the law:

\[ OA = Z_A = a_1 \sin(\omega_1 t + \alpha_1) + a_2 \sin(\omega_2 t + \alpha_2) + \ldots + a_N \sin(\omega_N t + \alpha_N), \]

where \( a_i, \omega_i, \alpha_i \) are constants.

The equation of motion of the pendulum is:

\[ m \ddot{\theta} + h \dot{\theta} + m \ell (g - Z_A) \sin \theta = 0, \]

where overdots denote the derivatives relatively to time.

Dividing equation (1.1) by \( m \ell^2 \omega^2 \) and introducing the notations:

\[ \omega_0^2 = \frac{g}{\ell}, \quad H = \frac{h}{m \ell^2}, \quad \tau = \omega t, \quad \tau = \frac{d}{d \tau}, \quad \epsilon_i = \frac{a_i}{a}, \quad \nu_i = \frac{\omega_i}{\omega}, \]

\[ \tag{1.2} \]
where \( \omega \) is a positive number which characterises the average value of \( \omega_1, \omega_2, \ldots, \omega_N \) and \( a \) is average value of \( a_1, a_2, \ldots, a_N \), and \( h \) is coefficient of friction, we can write (1.1) in the form:

\[
\theta'' + \frac{H}{\omega} \theta' + \left( \frac{\omega_0^2}{\omega} + \frac{a}{\ell} \left[ e_1 \nu_1^2 \sin(\nu_1 \tau + \alpha_1) + e_2 \nu_2^2 \sin(\nu_2 \tau + \alpha_2) + \cdots + e_N \nu_N^2 \sin(\nu_N \tau + \alpha_N) \right] \right) \sin \theta = 0
\]

(1.3)

It is supposed that \( a/\ell, \omega_0/\omega \) and \( H/\omega_0 \) are small quantities of \( \varepsilon \) - degree, where \( \varepsilon \) is a small positive parameter, so that

\[
\varepsilon = \frac{a}{\ell}, \quad \omega_0 = \kappa \varepsilon, \quad \frac{H}{\omega} = 2\varepsilon^2.
\]

(1.4)

We have

\[
\theta'' + 2\varepsilon^2 \theta' + \left( \varepsilon^2 k^2 + \varepsilon \left[ e_1 \nu_1^2 \sin(\nu_1 \tau + \alpha_1) + e_2 \nu_2^2 \sin(\nu_2 \tau + \alpha_2) + \cdots + e_N \nu_N^2 \sin(\nu_N \tau + \alpha_N) \right] \right) \sin \theta = 0.
\]

(1.5)

Using the new variables \( \varphi \) and \( \Omega \) instead of \( \theta \) and \( \theta' \) by formulae [2]

\[
\theta = \varphi + \varepsilon \left[ e_1 \sin(\nu_1 \tau + \alpha_1) + e_2 \sin(\nu_2 \tau + \alpha_2) + \cdots + e_N \sin(\nu_N \tau + \alpha_N) \right] \sin \varphi,
\]

\[
\theta' = \varepsilon \Omega + \varepsilon \left[ e_1 \nu_1 \cos(\nu_1 \tau + \alpha_1) + e_2 \nu_2 \cos(\nu_2 \tau + \alpha_2) + \cdots + e_N \nu_N \cos(\nu_N \tau + \alpha_N) \right] \sin \varphi,
\]

(1.6)

one can transform the equation (1.5) into a system of two equations of the first order:

\[
\varphi' = \varepsilon \Omega - \varepsilon^2 \Omega I_1 \cos \varphi,
\]

\[
\Omega' = -\varepsilon (\Omega I_2 \cos \varphi + k^2 \sin \varphi + I_1 I_3 \sin \varphi \cos \varphi) + \varepsilon^2 \left(\Omega I_1 I_2 \cos^2 \varphi \right.
\]

\[
- 2\lambda \Omega - 2\lambda I_2 \sin \varphi - k^2 I_1 \sin \varphi \cos \varphi + \frac{1}{2} I_1^2 I_3 \sin^3 \varphi),
\]

(1.7)

where the terms with degree of smallness higher than two are neglected.

\[
I_1 = \sum_{i=1}^{N} e_i \sin(\nu_i \tau + \alpha_i), \quad I_2 = \sum_{i=1}^{N} e_i \nu_i \cos(\nu_i \tau + \alpha_i), \quad I_3 = \sum_{i=1}^{N} e_i \nu_i^2 \sin(\nu_i \tau + \alpha_i)
\]

(1.8)

§2. APPROXIMATE SOLUTION IN THE NON-RESONANCE CASE

Let us consider the non-resonance case when the frequencies \( \nu_1, \nu_2, \ldots, \nu_N \) are linearly independent, so that between them there is no relation of the form

\[
n_1 \nu_1 + n_2 \nu_2 + \cdots + n_N \nu_N = 0,
\]

(2.1)

where \( n_i \) are integers, \( n_1^2 + n_2^2 + \cdots + n_N^2 \neq 0 \). Since the variables \( \varphi \) and \( \Omega \) are slowly varying in \( \tau \), in the first approximation one can replace the right hand sides of (1.7) by their average values. We have in the first approximation

\[
\varphi = \varphi_1, \quad \Omega = \Omega_1,
\]

(2.2)

where \( \varphi_1 \) and \( \Omega_1 \) satisfy the averaged equations:
\[ \varphi_i = \varepsilon \Omega_i, \quad \Omega_i = -e \left[ k^2 \sin \varphi_i + \frac{1}{4} \left( e_1^2 \nu_1^2 + e_2^2 \nu_2^2 + \cdots + e_N^2 \nu_N^2 \right) \sin 2 \varphi_i \right]. \]  

(2.3)

The stationary solutions of (2.3) are

1) \[ \Omega_1 = 0, \quad \varphi_1 = 0, \]  

(downward vertical position of equilibrium: \( \theta = \dot{\theta} = 0 \))

(2.4)

2) \[ \Omega_1 = 0, \quad \varphi_1 = \pi, \]  

(upward vertical position of equilibrium: \( \theta = \pi, \dot{\theta} = 0 \))

(2.5)

3) \[ \Omega_1 = 0, \quad \varphi_1 = \varphi_0, \]  

determined by the relation

\[ k^2 + \frac{1}{2} \left( e_1^2 \nu_1^2 + e_2^2 \nu_2^2 + \cdots + e_N^2 \nu_N^2 \right) \cos \varphi_0 = 0. \]  

(2.7)

This solution exists if

\[ k^2 \leq \frac{1}{2} \left( e_1^2 \nu_1^2 + e_2^2 \nu_2^2 + \cdots + e_N^2 \nu_N^2 \right). \]  

(2.8)

To study the stability of the stationary solutions we use the variational equations. Introducing the variations:

\[ \delta \Omega = \Omega_i, \quad \delta \varphi = \varphi_1 - \varphi^*, \]  

(2.9)

where \( \varphi^* = 0, \varphi^* = \pi \) and \( \varphi^* = \varphi_0 \) for the cases (2.4), (2.5) and (2.6) respectively, we have

\[ \frac{d}{d\tau} \delta \varphi = \varepsilon \delta \Omega, \]  

\[ \frac{d}{d\tau} \delta \Omega = -e \left[ k^2 \cos \varphi^* + \frac{1}{2} \left( e_1^2 \nu_1^2 + e_2^2 \nu_2^2 + \cdots + e_N^2 \nu_N^2 \right) \cos 2 \varphi^* \right] \delta \varphi, \]  

(2.10)

\[ \frac{d^2}{d\tau^2} \delta \varphi = e^2 \left[ k^2 \cos \varphi^* + \frac{1}{2} \left( e_1^2 \nu_1^2 + e_2^2 \nu_2^2 + \cdots + e_N^2 \nu_N^2 \right) \cos 2 \varphi^* \right] \delta \varphi = 0. \]

The necessary stability condition of the stationary solution is:

\[ k^2 \cos \varphi^* + \frac{1}{2} \left( e_1^2 \nu_1^2 + e_2^2 \nu_2^2 + \cdots + e_N^2 \nu_N^2 \right) \cos 2 \varphi^* > 0. \]  

(2.11)

Namely,

1. For \( \varphi^* = 0 \) the condition (2.11) is always satisfied.
2. For \( \varphi^* = \pi \) the necessary stability condition becomes:

\[ k^2 < \frac{1}{2} \left( e_1^2 \nu_1^2 + e_2^2 \nu_2^2 + \cdots + e_N^2 \nu_N^2 \right) \]  

(2.12)

or taking into account (1.2), (1.4):

\[ \omega_0^2 < \frac{1}{2} \left[ \left( \frac{a_1}{l} \right)^2 \omega_1^2 + \cdots + \left( \frac{a_N}{l} \right)^2 \omega_N^2 \right]. \]  

(2.13)

3. For \( \varphi^* = \varphi_0 \) the necessary stability condition is
\[ \omega_0^2 > \frac{1}{2} \left[ \left( \frac{a_1}{\ell} \right)^2 \omega_1^2 + \cdots + \left( \frac{a_N}{\ell} \right)^2 \omega_N^2 \right]. \]  

(2.14)

which is in contrary with (2.8).

Since the characteristic equation of (2.10) has zero real parts, to solve completely the stability problem of stationary solution it is necessary to consider the higher approximation.

The refinement of the first approximation of solutions of (1.7) is:

\[ \varphi = \varphi_1, \]

\[ \Omega = \Omega_1 - \varepsilon \{ \Omega_1 \cos \varphi_1 + \sum_{j=1}^{N} \varepsilon_j \sin \xi_j - \frac{1}{8} \sin 2\varphi_1 \sum_{j=1}^{N} \nu_j \varepsilon_j^2 \sin 2\xi_j + \]

\[ + \frac{1}{4} \sin 2\varphi_1 \sum_{i=1}^{N} \sum_{j=1}^{N} \varepsilon_i \varepsilon_j \nu_i^2 \left[ \frac{1}{\nu_i - \nu_j} \sin (\xi_i - \xi_j) - \frac{1}{\nu_i + \nu_j} \sin (\xi_i + \xi_j) \right] \}, \]

(2.15)

where \( \xi_i = \nu_i \tau + \alpha_i \).

Substituting (2.15) into (1.7) and averaging their right hand sides we obtain the equations of the second approximation

\[ \varphi' = \varepsilon \Omega_1, \]

\[ \Omega_1 = -\varepsilon \left[ k^2 \sin \varphi_1 + \frac{1}{4} (e_1^2 \nu_1^2 + \cdots + e_N^2 \nu_N^2) \sin 2\varphi_1 \right] - 2e^2 \lambda \Omega_1, \]

(2.16)

The stationary solutions of these equations are as before (2.4), (2.5) and (2.6), but the variational equation for them is of the form:

\[ \frac{d^2}{d\tau^2} \delta \varphi_1 + 2e^2 \lambda \frac{d}{d\tau} \delta \varphi_1 + \varepsilon^2 \left[ k^2 \cos \varphi_1^* + \frac{1}{2} (e_1^2 \nu_1^2 + \cdots + e_N^2 \nu_N^2) \cos 2\varphi_1^* \right] \delta \varphi_1 = 0 \]

(2.17)

which is different from (2.9) by the appearance of the term \( 2e^2 \lambda \frac{d}{d\tau} \delta \varphi_1 \) (\( \varphi^* = 0, \pi, \varphi_0 \)). The necessary and sufficient stability condition for the stationary solutions (2.4), (2.6) is:

1. The solution \( \Omega_1 = 0, \varphi_1 = 0 \) is always stable
2. The solution \( \Omega_1 = 0, \varphi_1 = \pi \) is stable if

\[ k^2 < \frac{1}{2} (e_1^2 \nu_1^2 + \cdots + e_N^2 \nu_N^2) \]

or if

\[ \omega_0^2 < \frac{1}{2} \left[ \left( \frac{a_1}{\ell} \right)^2 \omega_1^2 + \cdots + \left( \frac{a_N}{\ell} \right)^2 \omega_N^2 \right]. \]

(2.19)

3. The solution \( \Omega_1 = 0, \varphi_1 = \varphi_0 \) is always unstable.

§3. THE RESONANCE CASE

It is supposed that between the frequency \( \nu_j \) and \( \nu_1 \) there exists a relation

\[ \omega_j = 2\nu_1, \quad j \neq 1. \]

(3.1)

The other resonance cases can be examined by an analogous way. Now, the equations of the refinement of the first approximation remain the same as (2.15). Substituting them into (1.7) and averaging on \( \tau \) we obtain the following equations of the second approximation.
\[ \phi_1 = \epsilon \Omega_1, \]

\[ \Omega_1 = -\epsilon (k^2 \sin \phi_1 + \frac{1}{2} \sin \phi_1 \cos \phi_1 \sum_{i=1}^{N} \epsilon_i^2 \nu_i^2) + \epsilon^2 \left( -2\lambda \Omega_1 + \frac{\epsilon_i^2 \epsilon_j \nu_i (\nu_i^2 + \nu_j^2)}{4(\nu_j - \nu_i)} \sin(\alpha_j - 2\alpha_1) \sin \phi_1 \cos^2 \phi_1 - \frac{1}{8} \epsilon_i^2 \epsilon_j (2\nu_i^2 + \nu_j^2) \sin(\alpha_j - 2\alpha_1) \sin^2 \phi_1 \right). \] \tag{3.2}

The stationary solutions of equations (3.2) are

1) \[ \Omega_1 = 0, \quad \phi_1 = 0, \] \tag{3.3}

2) \[ \Omega_1 = 0, \quad \phi_1 = \pi, \] \tag{3.4}

3) \[ \Omega_1 = 0, \quad \phi_1 = \phi_{10} \] \tag{3.5}

where \( \phi_{10} \) is determined from

\[ k^2 + \frac{1}{2} \cos \phi_{10} \sum_{i=1}^{N} \epsilon_i^2 \nu_i^2 + \epsilon \left( \frac{\epsilon_i^2 \epsilon_j \nu_j (\nu_j^2 + \nu_i^2)}{4(\nu_j - \nu_i)} \sin(\alpha_j - 2\alpha_1) \cos^2 \phi_{10} - \frac{1}{8} \epsilon_i^2 \epsilon_j (2\nu_i^2 + \nu_j^2) \sin(\alpha_j - 2\alpha_1) \sin^2 \phi_{10} \right) = 0, \] \tag{3.6}

This solution exists if

\[ k^2 \leq \frac{1}{2} (\epsilon_1^2 \nu_1^2 + \epsilon_2^2 \nu_2^2 + \cdots + \epsilon_N^2 \nu_N^2). \] \tag{3.7}

The stability of the stationary solutions (3.3)-(3.5) is studied by using the variational equation of (3.2) which has the following form

\[ \frac{d^2}{dr^2} \delta \phi_1 + 2\lambda \epsilon \frac{d}{dr} \delta \phi_1 + \epsilon^2 \left( k^2 \cos \phi_1^* + \frac{1}{2} (\epsilon_1^2 \nu_1^2 + \cdots + \epsilon_N^2 \nu_N^2) \cos 2\phi_1^* + \frac{\epsilon}{16} \left( \frac{\epsilon_i^2 \epsilon_j \nu_j (\nu_j^2 + \nu_i^2)}{\nu_j - \nu_i} \sin(\alpha_j - 2\alpha_1) \left[ \cos \phi_1^* + 3 \cos 3\phi_1^* \right] - \frac{\epsilon}{32} \epsilon_i^2 \epsilon_j (2\nu_i^2 + \nu_j^2) \sin(\alpha_j - 2\alpha_1) \left[ 3 \cos \phi_1^* - 3 \cos 3\phi_1^* \right] \right) \delta \phi_1 = 0 \] \tag{3.8}

where \( \delta \phi_1 = \phi_1 - \phi_{10}, \quad \phi_1^* = 0, \pi, \phi_{10} \).

From (3.7) it follows:

1) For \( \phi_1^* = 0 \) and for small \( \epsilon \), the solution \( \Omega_1 = 0, \phi_1 = 0 \) (downward vertical position) is stable if:

\[ k^2 + \frac{1}{2} (\epsilon_1^2 \nu_1^2 + \cdots + \epsilon_N^2 \nu_N^2) > 0 \] \tag{3.9}

This condition is always satisfied.

2) For \( \phi_1^* = \pi \) and for small \( \epsilon \), the solution \( \Omega_1 = 0, \phi_1 = \pi \) (upward vertical position) is stable if

\[ k^2 < \frac{1}{4} (\epsilon_1^2 \nu_1^2 + \cdots + \epsilon_N^2 \nu_N^2). \] \tag{3.10}
3) For \( \varphi_1 = \varphi_{10} \) and for small \( \varepsilon \), the solution \( \Omega = 0, \varphi = \varphi_{10} \) is stable if

\[
k^2 \cos \varphi_{10} + \frac{1}{2} \left( c_1^2 \nu_1^2 + \cdots + c_N^2 \nu_N^2 \right) \cos 2\varphi_{10} > 0.
\]

Taking into account the relation (3.6):

\[
\cos \varphi_{10} \approx -\frac{2k^2}{\left( c_1^2 \nu_1^2 + \cdots + c_N^2 \nu_N^2 \right)}
\]

one can write the stability condition in the form

\[
k^2 > \frac{1}{2} \left( c_1^2 \nu_1^2 + \cdots + c_N^2 \nu_N^2 \right),
\]

which is in the contrary with (3.7).

Thus, with small \( \varepsilon \) and for \( k^2 > \frac{1}{2} \left( c_1^2 \nu_1^2 + \cdots + c_N^2 \nu_N^2 \right) \) the stability conditions (3.9), (3.10) are similar to those in the non-resonance case.

\section*{§4. CONCLUSION}

1. The downward position of equilibrium \( (\Omega = 0, \varphi = 0) \) of the pendulum is always stable. The middle equilibrium position \( (\Omega = 0, \varphi = \varphi_0) \) of the pendulum is unstable. The upward equilibrium position \( (\Omega = 0, \varphi = \pi) \) of the pendulum is stable if there exists the condition (2.9) or (3.10).

2. The presence of vertically moving components of the support point intensifies the stability of downward and upward positions of equilibrium of the pendulum (see (3.9), (3.10)).

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\section*{REFERENCE}


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DAO DONG PHI TUYEN CUA CON LAC

C0 DIEM TREO DI CHUYEN THANG DUNG DA TAN

Trong bai bao, cac ket qua cua [1] duoc mot trong cho trong hop diem treo cua con lac thuc hien dao dong thang dung da tan. De giai quyet tren bai toan vee su on dinh cua cac vi tri can bang cua con lac da phai xet den xap xai cap cao. Ket qua nghieng cua cho thay:

1. Vi tri can bang phia duoi cua con lac luon luon on dinh. Vi tri can bang trung gian khong on dinh. Vi tri can bang phia tren on dinh nen dien kiem (3.10) duoc thoa man.

2. Su xuat hien cac thanh phan dao dong da tan cua diem treo cua con lac da lam tang tinh on dinh cua cac vi tri can bang phia duoi va phia tren (xem (3.9), (3.10)).