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A FORM OF EQUATIONS OF MOTION OF A MECHANICAL SYSTEM IN QUASI-COORDINATES

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ABSTRACT. In [3, 4, 5] the form of equations of motion in holonomic coordinates has constructed. The equations obtained give an effective tool for investigating complicated systems.

In the present paper the form of equations of motion is written in quasi-coordinates. These equations are solved with respect to quasi-accelerations, which allow to define the motion of a holonomic and nonholonomic systems by a closed set of algebraic - differential equations. The reaction forces of constraints imposed on the system under consideration are calculated by means of a simple algorithm.

For illustrating the effectivness of this form of equations an example is considered.

1. Introduction

As known [1, 6, 7], in some cases the expression of kinetic energy of a mechanical system is written conveniently in quasi-velocities, for example, in the case of bodies moving about a fixed point. For nonholonomic systems the constraints are often written in quasi-velocities, for example, a rigid body rolls without sliding on a plane. In such a case, we can apply the Lagrange's equations with multipliers or the equations in quasi-coordinates. However, as shown in [6] these methods are very complicated.

In connection with this, for the method of Lagrange's multipliers it is necessary to eliminate undeterminate multipliers, but for the Lagrange's equations in quasi-coordinates we have to calculate complicated indices. These problems can be avoided by means of generalizing the equations obtained in [3, 5].

2. Equations of motion of a holonomic mechanical system in quasi-coordinates

Let us consider a holonomic mechanical system of n degrees of freedom. Denote by q_i the Lagrangian coordinates and by Q_i-generalized forces $(i = \overline{1,n})$.

Kinetic energy of the system under consideration is of the form

$$T = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{A}\, \dot{\mathbf{q}}, \tag{2.1}$$

where $\mathbf{A}$ is a quadratic, symmetric, and nonsingular matrix of n order, the elements of which depend only on Lagrangian coordinates, i.e., $\mathbf{A} = \mathbf{A}(\mathbf{q})$, but $\dot{\mathbf{q}}$ - the $n \times 1$ matrix of generalized velocities, that is

$$\dot{\mathbf{q}}^T = \left\|\dot{q}_1\ \dot{q}_2\ \ldots\ \dot{q}_n\right\|. \tag{2.2}$$

The letter T at the high right corner designates the transposition.

As known [3], the equations of motion of a holonomic system can be written in the form

$$\mathbf{A}\,\ddot{\mathbf{q}} = \mathbf{G} + \mathbf{Q}, \tag{2.3}$$

where $\mathbf{Q}$ is the $n \times 1$ matrix of generalized forces, i.e.,

$$\mathbf{Q}^T = \left\|\, Q_1\ Q_2\ \ldots\ Q_n \,\right\| \tag{2.4}$$

and the $n \times 1$ matrix $\mathbf{G}$ is determined only by the matrix of inertia $\mathbf{A}$ and $\ddot{\mathbf{q}}$ is the matrix of generalized accelerations

$$\ddot{\mathbf{q}}^T = \left\|\, \ddot{q}_1\ \ddot{q}_2\ \ldots\ \ddot{q}_n\right\|. \tag{2.5}$$

Let us introduce now quasi-coordinates σ_i $(i = \overline{1,n})$ of the form

$$\dot{\sigma}_i = \sum C_{ij}\dot{q}_j, \tag{2.6}$$

where: $C_{ij} = C_{ij}(q_1, q_2, \ldots, q_n)$ make quadratic and inversible matrix $\mathbf{C}$ of n order. The relations (2.6) can be written in the matrix form as follows:

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}\,\dot{\mathbf{q}}, \tag{2.7}$$

where $\dot{\boldsymbol{\sigma}}$ is an $n \times 1$ matrix.

Because of the nonsingular matrix $\mathbf{C}$, we have

$$\dot{\mathbf{q}} = \mathbf{C}^{\circ}\,\dot{\boldsymbol{\sigma}}, \tag{2.8}$$

where $\mathbf{C}^{\circ}$ is the inverse of the matrix $\mathbf{C}$, i.e., $\mathbf{C}\,\mathbf{C}^{\circ} = \mathbf{C}^{\circ}\,\mathbf{C} = \mathbf{E}$ ($\mathbf{E}$ - the identity matrix).

It is clear, that the elements of the matrix $\mathbf{C}^{\circ}$ depend only on generalized coordinates, i.e.,

$$\mathbf{C}^{\circ} = \mathbf{C}^{\circ}(\mathbf{q}), \tag{2.9}$$

where $\mathbf{C}^{\circ}$ is a quadratic matrix of n order too.

Derivating (2.8) we obtain

$$\ddot{\mathbf{q}} = \mathbf{C}^{\circ}\ddot{\boldsymbol{\sigma}} + \dot{\mathbf{C}}^{\circ}\dot{\boldsymbol{\sigma}}, \tag{2.10}$$

where $\dot{\mathbf{C}}^{\circ}$ is an $n \times n$ matrix, the elements of which are the derivation with respect to time of corresponding elements of the matrix $\mathbf{C}$.

Substituting (2.10) into (2.3) we obtain the equations described motion of the system under consideration

$$\mathbf{A}^{\circ}\ddot{\boldsymbol{\sigma}} = \mathbf{G}_0 + \mathbf{Q}_0 + \boldsymbol{\phi}_0, \tag{2.11}$$

where

$$\mathbf{A}^{\circ} = \mathbf{A}\,\mathbf{C}^{\circ}; \quad \boldsymbol{\phi}_0 = -\mathbf{A}\,\dot{\mathbf{C}}^{\circ}\dot{\boldsymbol{\sigma}}, \tag{2.12}$$

but $\mathbf{G}_0$, $\mathbf{Q}_0$ are just the matrices $\mathbf{G}$ and $\mathbf{Q}$ respectively, in which the generalized velocities are substituted by quasi-velocities from (2.8). This is realized on the matrix $\mathbf{C}$ too.

Because the matrices $\mathbf{A}$ and $\mathbf{C}^{\circ}$ depend only on Lagrangian coordinates, then the matrix $\mathbf{A}^{\circ}$ from (2.12) depends only on Lagrangian coordinates, i.e.,

$$\mathbf{A}^{\circ} = \mathbf{A}^{\circ}(\mathbf{q}). \tag{2.13}$$

From (2.12) it is possible to follow

$$\mathbf{A} = \mathbf{A}^{\circ}\,\mathbf{C}. \tag{2.14}$$

In other words, we can calculate the matrix of inertia $\mathbf{A}$ from the matrix $\mathbf{A}^{\circ}$.

This is necessary to notice, because in some cases the kinetic energy is written in quasi-velocities.

Let us consider the kinetic energy of the form

$$T = \frac{1}{2}\dot{\boldsymbol{\sigma}}^T\,\mathbf{A}^*\,\dot{\boldsymbol{\sigma}}. \tag{2.15}$$

By means of the formula (2.7), the expression of kinetic energy (2.15) can be written as follows:

$$T = \frac{1}{2}\dot{\mathbf{q}}^T\,\mathbf{C}^T\,\mathbf{A}^*\,\mathbf{C}\,\dot{\mathbf{q}}. \tag{2.16}$$

Comparing (2.16) with (2.1) we immediately draw:

$$\mathbf{A} = \mathbf{C}^T \mathbf{A}^* \mathbf{C}. \tag{2.17}$$

By means of (2.17) the matrix $\mathbf{A}^\circ$ in (2.11) takes the following form:

$$\mathbf{A}^\circ = \mathbf{A}\,\mathbf{C}^\circ = \mathbf{C}^T \mathbf{A}^* \mathbf{C}\,\mathbf{C}^\circ = \mathbf{C}^T \mathbf{A}^*. \tag{2.18}$$

In accordance with (2.18) the matrix $\boldsymbol{\phi}_0$ in (2.11) will be now

$$\boldsymbol{\phi}_0 = -\mathbf{A}\,\dot{\mathbf{C}}^0 \dot{\boldsymbol{\sigma}} = -\mathbf{A}\,\dot{\mathbf{C}}^\circ\,\mathbf{C}\,\dot{\mathbf{q}} = \mathbf{A}\,\mathbf{C}^\circ\,\dot{\mathbf{C}}\,\dot{\mathbf{q}} = \mathbf{C}^T \mathbf{A}^*\,\dot{\mathbf{C}}\,\dot{\mathbf{q}}. \tag{2.19}$$

Thus, in the case of the kinetic energy (2.15) the equations of motion of the system under consideration are written in the form (2.11), in which the matrix $\mathbf{A}^\circ$ takes the form (2.18), but $\boldsymbol{\phi}_0$ - (2.19).

In such a way it is possible to avoid the calculation of the inverse of the matrix $\mathbf{C}$ in the establishment of equations of motion.

It is easy to see that although the expression of kinetic energy written in quasi-velocities, for example, by (2.15), the equations of motion can be written in Lagrangian coordinates too.

Indeed, the equations (2.3) in consideration (2.17) are written in the form

$$\mathbf{C}^T \mathbf{A}^* \mathbf{C}\,\ddot{\mathbf{q}} = \mathbf{G} + \mathbf{Q}. \tag{2.20}$$

3. Equations of motion of a nonholonomic system in quasi-coordinates

Let us consider a mechanical system, the position of which is described by n Lagrangian coordinates ($i = \overline{1,n}$). Suppose that the constraints imposed on the system under consideration are of the form

$$\sum_{i=1}^{n} b_{ri}\dot{q}_i = 0, \quad r = \overline{1,s}, \tag{3.1}$$

which can be written in the matrix form

$$\mathbf{b}\,\dot{\mathbf{q}} = 0, \tag{3.2}$$

where $\mathbf{b}$ is an $s \times n$ matrix, the coefficients of which depend only on Lagrangian coordinates, i.e., $\mathbf{b} = \mathbf{b}(\mathbf{q})$. Let introduce the quasi-coordinates of the form (2.7).

As known [2, 5] the equations of motion of the system considered can be written as follows

$$\mathbf{A}\,\dot{\mathbf{q}} = \mathbf{G} + \mathbf{Q} + \mathbf{R}, \tag{3.3}$$

where $\mathbf{R}$ is the matrix of reaction forces of the constraints (3.1).

As shown in [5], it is possible to eliminate the reaction forces $\mathbf{R}$ from the equation (3.3) by means of the condition of ideallity of the constraints (3.1). By means of such a way we obtain

$$\mathbf{D}\,\mathbf{A}\,\ddot{\mathbf{q}} = \mathbf{D}\,\mathbf{G} + \mathbf{D}\,\mathbf{Q}, \tag{3.4}$$

where $\mathbf{D}$ is the transpose of the matrix, the elements of which are the coefficients in the expression of accelerations in term of independent accelerations in consideration of the constraints (3.1).

Substituting (2.10) into (3.4) we have

$$\mathbf{D}^*\ddot{\boldsymbol{\sigma}} = \mathbf{G}^* + \mathbf{Q}^* + \boldsymbol{\phi}^*, \tag{3.5}$$

where

$$\mathbf{D}^* = \mathbf{D}\,\mathbf{A}\,\mathbf{C}^\circ, \tag{3.6}$$

$$\mathbf{G}^* = \mathbf{D}\,\mathbf{G}_0,\ \mathbf{Q}^* = \mathbf{D}\,\mathbf{Q}_0,\ \boldsymbol{\phi}^* = \mathbf{D}\boldsymbol{\phi}_0 = -D\,\mathbf{A}\,\dot{\mathbf{C}}^\circ\dot{\boldsymbol{\sigma}}, \tag{3.7}$$

where $\mathbf{G}_0$, $\mathbf{Q}_0$, $\boldsymbol{\phi}_0$ take the same meaning as in (2.11). The equations (3.5) and (3.2) describe the motion of the system under consideration.

The matrices $\mathbf{G}^*$, $\mathbf{Q}^*$ and $\boldsymbol{\phi}^*$ are the $k \times 1$ matrices, but $\mathbf{D}^*$ - the $k \times n$ matrix.

It is important that among quasi-velocities (2.6) we can choose the left expressions of the constraint equations (3.1). In other words, we can take

$$C_{rj} = b_{rj} \quad r = \overline{k+1, n};\ j = \overline{1, n};\ k = n - s. \tag{3.8}$$

In such a case the constraint equations (3.1) take the following form

$$\sum_{j=1}^{n} C_{rj}\,\dot{q}_j = 0, \quad r = \overline{k+1, n};\ k = n - s. \tag{3.9}$$

This is equivalent to

$$\dot{\sigma}_r = 0 \quad r = \overline{k+1, n},\ k = n - s. \tag{3.10}$$

Equations of motion of the considered system can be written in the form (3.5), in which the quasi-velocities and quasi-accelerations takes zero-values, i.e., $\ddot{\sigma}_r = \dot{\sigma}_r = 0$ $(r = \overline{k+1,n})$.

Equations of motion of the system under consideration will be now

$$\overline{\mathbf{D}}^* \ddot{\bar{\boldsymbol{\sigma}}} = \overline{\mathbf{G}}^* + \overline{\mathbf{Q}}^* + \bar{\boldsymbol{\phi}}^*, \tag{3.11}$$

where $\ddot{\bar{\boldsymbol{\sigma}}}$ is $k \times 1$ matrix, which is obtained from $\ddot{\boldsymbol{\sigma}}$, after striking out s last row, but $\overline{\mathbf{D}}^*$ is just the matrix $\mathbf{D}^*$ in which the s last columns are striked out.

It is noticed that for applying the equations (3.5) or (3.11), it is necessary to calculate the inverse of the matrix $\mathbf{C}$. As above mentioned, this can be avoided by means of using (2.18) and (2.19), i.e.

$$\mathbf{D}^* = \mathbf{D}\,\mathbf{C}^T\,\mathbf{A}^*, \tag{3.12}$$

$$\boldsymbol{\phi}^* = \mathbf{D}\,\mathbf{C}^T\,\mathbf{A}^*\,\dot{\mathbf{C}}\,\dot{\mathbf{q}}. \tag{3.13}$$

Note

1) In the case of the constraint equations written in quasi-velocities, for calculating the matrix $\mathbf{D}$, it is uneccessary to rewrite these equations in the form of Lagrangian velocities.

Suppose that the constraint equations are written in the form

$$\sum_{j=1}^{n} C_{rj}\,\dot{\sigma}_j = 0, \quad r = \overline{k+1,n}. \tag{3.14}$$

As shown above, we can choose k independent quasi-velocities $\dot{\sigma}_\nu$ $(\nu = \overline{1,k})$, but r dependent quasi-velocities have the form

$$\dot{\sigma}_r = \sum_{j=1}^{n} C_{rj}\,\dot{q}_j, \quad r = \overline{k+1,n}. \tag{3.15}$$

In accordance with (3.14) we define the $n \times k$ matrix $\mathbf{D}_\sigma$ so that

$$\dot{\boldsymbol{\sigma}} = \mathbf{D}_\sigma \dot{\boldsymbol{\sigma}}_0, \tag{3.16}$$

where

$$\dot{\boldsymbol{\sigma}}_0^T = \left\| \dot{\sigma}_1\ \dot{\sigma}_2\ \ldots\ \dot{\sigma}_k \right\| \tag{3.17}$$

and

$$\dot{\boldsymbol{\sigma}}^T = \left\| \dot{\sigma}_1\ \dot{\sigma}_2\ \ldots\ \dot{\sigma}_k\ \ldots\ \dot{\sigma}_n \right\|. \tag{3.18}$$

Assume that the independent quasi-velocities are expressed through independent Lagrangian velocities, that is

$$\dot\sigma_\xi = \sum_{\nu=1}^{k} \ell_{\xi\nu}\dot q_\nu, \quad \xi = \overline{1,k}. \tag{3.19}$$

The relations (3.19) can be written in the matrix form

$$\dot{\boldsymbol\sigma}_0 = \mathbf{L}\,\dot{\mathbf{q}}_0, \tag{3.20}$$

where $\mathbf{L}$ is the $k \times k$ matrix of the elements $\ell_{\xi\nu}$ $(\xi, \nu = \overline{1,k})$, but $\dot{\mathbf{q}}_0$ is the $k \times 1$ matrix of independent Lagrangian velocities, i.e.,

$$\dot{\mathbf{q}}_0^T = \left\| \, \dot q_1 \; \dot q_2 \; \ldots \; \dot q_k \, \right\|. \tag{3.21}$$

After some transformations, we obtain

$$\mathbf{D} = \mathbf{L}^T\, \mathbf{D}_\sigma^T\, \mathbf{C}^{\circ T}. \tag{3.22}$$

2) It is possible to obtain the equations of motions in Lagrangian coordinates, although the expression of the kinetic energy is written in quasi-velocities.

Indeed, from (2.18) we have

$$\mathbf{A} = \mathbf{C}^T\, \mathbf{A}^*\, \mathbf{C}. \tag{3.23}$$

By means of (3.4) we write the equations of motions of the system under consideration in the following form

$$\mathbf{D}\,\mathbf{C}^T\,\mathbf{A}^*\,\mathbf{C}\,\ddot{\mathbf{q}} = \mathbf{D}\,\mathbf{G} + \mathbf{D}\,\mathbf{Q}. \tag{3.24}$$

Example. Write equations of motion of a uniform sphere of mass M and radius a, which rolls without sliding on a fixed horizontal plane [6].

Assume that the $Oxyz$ principal axes are in the translated motion together the mass center O. Denote by A, B, C the moments of inertia of the sphere about the principal axes. Because of the uniformity of the sphere then

$$A = B = C = \frac{2}{5}Ma^2.$$

The kinetic energy of the sphere is calculated by the formula

$$T = \frac{1}{2}M(\dot x_0^2 + \dot y_0^2) + \frac{1}{2}\,\frac{2}{5}Ma^2(\omega_1^2 + \omega_2^2 + \omega_3^2), \tag{3.25}$$

where ω_1, ω_2 and ω_3 are the projects of angular velocity of the sphere on the fixed axes.

The condition of rolling without sliding is written as follows [6]

$$\begin{aligned} \dot{x}_0 - a(s\psi\dot{\theta} + s\theta C\psi\dot{\varphi}) &= 0, \\ \dot{y}_0 + a(C\psi\dot{\theta} + s\theta s\psi\dot{\varphi}) &= 0, \end{aligned} \tag{3.26}$$

here and bellow we use following symbols

$$s\psi \equiv \sin\psi, \quad C\psi \equiv \cos\psi, \quad s\theta \equiv \sin\theta, \quad C\theta \equiv \cos\theta, \ldots$$

Let us choose the quasi-velocities $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ and σ_5 in the form

$$\begin{aligned} &\dot{\sigma}_1 = \omega_1, \quad \dot{\sigma}_2 = \omega_2, \quad \dot{\sigma}_3 \equiv \omega_3, \\ &\dot{\sigma}_4 = \dot{x}_0 - a(s\psi\dot{\theta} + s\theta C_\psi\,\dot{\varphi}) \equiv \omega_4, \\ &\dot{\sigma}_5 = \dot{y}_0 + a(C\psi\dot{\theta} + s\theta s\psi\,\dot{\varphi}) \equiv \omega_5. \end{aligned} \tag{3.27}$$

By means of (3.27) the expression of the kinetic energy (3.25) is written in the form

$$T = \frac{1}{2}M\Big[\frac{7}{5}a^2(\omega_1^2 + \omega_2^2) + \frac{2}{5}a^2\omega_3^2 + 2a\omega_2\omega_4 - 2a\omega_1\omega_5)\Big]. \tag{3.28}$$

Because of the Euler formula (see, for example, [1, 6, 7])

$$\begin{aligned} \omega_1 &= C\psi\,\dot{\theta} + s\theta s\psi\,\dot{\varphi}, \\ \omega_2 &= s\psi\,\dot{\theta} + s\theta C\psi\,\dot{\varphi}, \\ \omega_3 &= C\theta\,\dot{\varphi} + \dot{\psi}, \end{aligned} \tag{3.29}$$

the transformation matrix $\mathbf{C}$ takes the form

$$\mathbf{C} = \begin{Vmatrix} C\psi & s\theta s\psi & 0 & 0 & 0 \\ s\psi & -s\theta C\psi & 0 & 0 & 0 \\ 0 & \omega & 1 & 0 & 0 \\ -as\psi & as\theta C\psi & 0 & 1 & 0 \\ aC\psi & as\theta s\psi & 0 & 0 & 1 \end{Vmatrix}. \tag{3.30}$$

From here we calculate the translated and derivated matrices

$$\mathbf{C}^T = \begin{Vmatrix} C\psi & s\psi & 0 & -as\psi & aC\psi \\ s\theta s\psi & -s\theta C\psi & C\theta & as\theta C\psi & as\theta s\psi \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}, \tag{3.31}$$

$$
\dot{\mathbf{C}} = \left\| \begin{array}{ccccc} -s\psi\,\dot{\psi} & (C\theta s\psi\dot{\theta} + s\theta C\psi\dot{\psi}) & 0 & 0 & 0 \\ C\psi\,\dot{\psi} & (-C\theta C\psi\dot{\theta} + s\theta s\psi\dot{\psi}) & 0 & 0 & 0 \\ 0 & -s\theta\dot{\theta} & 0 & 0 & 0 \\ -aC\psi\dot{\psi} & a(C\theta C\psi\dot{\theta} - s\theta s\psi\dot{\psi}) & 0 & 0 & 0 \\ -as\psi\dot{\psi} & a(C\theta s\psi\dot{\theta} + s\theta C\psi\dot{\psi}) & 0 & 0 & 0 \end{array} \right\| . \tag{3.32}
$$

In accordance with (3.28) the matrix $\mathbf{A}^*$ has the form

$$
\mathbf{A}^* = \left\| \begin{array}{ccccc} \frac{7}{5}Ma^2 & 0 & 0 & 0 & -Ma \\ 0 & \frac{7}{5}Ma^2 & 0 & Ma^2 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 & 0 & 0 \\ 0 & Ma & 0 & M & 0 \\ -Ma & 0 & 0 & 0 & M \end{array} \right\| . \tag{3.33}
$$

By (2.17) we calculate the matrix of inertia $\mathbf{A}$

$$
\mathbf{A} = \mathbf{C}^T\,\mathbf{A}^*\,\mathbf{C} = \left\| \begin{array}{ccccc} \frac{2}{5}Ma^2 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{5}Ma^2 & \frac{2}{5}Ma^2C\theta & 0 & 0 \\ 0 & \frac{2}{5}Ma^2C\theta & \frac{2}{5}Ma^2 & 0 & 0 \\ 0 & 0 & 0 & M & 0 \\ 0 & 0 & 0 & 0 & M \end{array} \right\| . \tag{3.34}
$$

In accordance with the matrix (3.34) we draw the 5×1 matrix $\mathbf{G}$

$$
\mathbf{G}^T = \left\| -\frac{2}{5}Ma^2C\theta\dot{\varphi}\dot{\psi} \quad \frac{2}{5}Ma^2 s\theta\dot{\psi}\dot{\theta} \quad \frac{2}{5}Ma^2 s\theta\dot{\varphi}\dot{\theta} \quad 0 \quad 0 \right\| . \tag{3.35}
$$

From (3.26) we calculate the 3×5 matrix $\mathbf{D}$

$$
\mathbf{D} = \left\| \begin{array}{ccccc} 1 & 0 & 0 & as\psi & -aC\psi \\ 0 & 1 & 0 & -as\theta C\psi & -as\theta s\psi \\ 0 & 0 & 1 & 0 & 0 \end{array} \right\| . \tag{3.36}
$$

The 3×5 matrix $\mathbf{D}^*$ is calculated by (3.12), but the 3×1 matrix $\boldsymbol{\phi}^*$ - by (3.13), they are

$$
\mathbf{D}^* = \mathbf{D}\,\mathbf{C}^T\,\mathbf{A}^* =
$$

$$
\left\| \begin{array}{ccccc} \frac{7}{5}Ma^2C\psi & \frac{7}{5}Ma^2s\psi & 0 & Mas\psi & -MaC\psi \\ \frac{7}{5}Ma^2s\theta s\psi & -\frac{7}{5}Ma^2s\theta C\psi & \frac{2}{5}Ma^2C\theta & 0 & 0 \\ 0 & 0 & \frac{2}{5}Ma^2 & -Mas\theta C\psi & -Mas\theta s\psi \end{array} \right\| , \tag{3.37}
$$

$$\boldsymbol{\phi}^* = \mathbf{D}\,\mathbf{C}^T\,\mathbf{A}^*\,\dot{\mathbf{C}}\,\dot{\mathbf{q}} = \left\| \begin{matrix} \frac{2}{5}Ma^2 s\theta\dot{\varphi}\dot{\psi} \\ -\frac{2}{5}Ma^2 s\theta\dot{\psi}\dot{\theta} \\ -\frac{2}{5}Ma^2 s\theta\dot{\theta}\dot{\varphi} \end{matrix} \right\| . \tag{3.38}$$

As shown in [6], the 5×1 matrix of generalized forces corresponding to Lagrangian coordinates are

$$\mathbf{Q}^T = \| \, m_\theta \quad m_\varphi \quad m_\psi \quad V_1 \quad V_2 \, \| \, . \tag{3.39}$$

Following (3.7) in accordance with (3.35) and (3.39) we have

$$\mathbf{G}^* + \mathbf{Q}^* = \mathbf{D}(\mathbf{G} + \mathbf{Q}) = \left\| \begin{matrix} -\frac{2}{5}Ma^2 s\theta\dot{\varphi}\dot{\psi} + m_\theta - a(V_1 s\psi - V_2 C\psi) \\ \frac{2}{5}Ma^2 s\theta\dot{\psi}\dot{\theta} + m_\varphi - as\theta(V_1 C\psi + V_2 s\psi) \\ \frac{2}{5}Ma^2 s\theta\dot{\varphi}\dot{\theta} + m\psi \end{matrix} \right\| . \tag{3.40}$$

Now, equations of motion of the system by (3.5) take the form

$$\begin{aligned} &\frac{7}{5}Ma^2 C\psi\dot{\omega}_1 + \frac{7}{5}Ma^2 s\psi\dot{\omega}_2 = M - \theta + a(V_1 s\psi - V_2 C\psi), \\ &\frac{7}{5}Ma^2 s\theta s\psi\dot{\omega}_1 - \frac{7}{5}Ma^2 s\theta C\psi\dot{\omega}_2 + \frac{2}{5}Ma^2 C\theta\dot{\omega}_3 = m_\varphi - a(V_1 C\psi + V_2 s\psi)s\theta, \\ &\frac{2}{5}Ma^2\dot{\omega}_3 = m_\psi. \end{aligned} \tag{3.41}$$

These equations together with constraint equations (3.26) describe the motion of the system under consideration.

It is easy to see that after some simple transformations the equations (3.41) coincide with the equations obtained in [6] (see [6] pp. 374).

It is noticed that applying the equations (3.24) we obtain the equations of motion of the system considered in the form

$$\begin{aligned} &\frac{2}{5}Ma^2\ddot{\theta} + Ma(\ddot{x}_0 s\psi - \ddot{y}_0 C\psi) = -\frac{2}{5}Ma^2 s\theta\dot{\varphi}\dot{\psi} + M_\theta - a(V_1 s\psi - V_2 C\psi), \\ &\frac{2}{5}Ma^2(\ddot{\varphi} + C\theta\ddot{\psi}) - Ma(\ddot{x}_0 C\psi + \ddot{y}_0 s\psi) = \frac{2}{5}Ma^2 s\theta\dot{\psi}\dot{\theta} + m_\varphi - as\theta(V_1 C\psi + V_2 s\psi), \\ &\frac{2}{5}Ma^2(\ddot{\psi} + C\theta\ddot{\varphi}) = \frac{2}{5}Ma^2 s\theta\dot{\varphi}\dot{\theta} + m_\psi. \end{aligned} \tag{3.42}$$

These equations are equivalent to the ones (3.41). However, equations (3.42) coincide completely with the equations obtained in [6] (see [6] pp. 375).

Note.

In order to determine the reaction forces of constraints, it is possible to apply the equations (3.3), in which accelerations, velocities and coordinates are known after integrating the equations of motion, that is

$$\mathbf{R} = \mathbf{A}(t)\,\ddot{\mathbf{q}}(t) - \mathbf{G}(t) - \mathbf{Q}(t). \tag{3.43}$$

4. Conclusions

Up to now only the Lagrange's equations with multipliers or the Lagrange's of a mechanical system equations in quasi-coordinates are used for writing equations of motion when the expression of kinetic energy or constraint equations are written in quasi-coordinates. In such a case there are many difficulties, especially for systems with large dimension [6, 7].

In present work the form of equations of motion is written by parameters of the system, which are given in quasi-coordinates. It is important that the equations obtained have been written in the matrix form. This is convienient for the system of large dimension.

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REFERENCES

1. Dobronravov V. V. The base of nonholonomic mechanics, "Vysha skola" Moscow, 1970 (in Russian).
2. Do Sanh. On the motion of constrained mechanical systems, the thesic of Science Doctor, Hanoi, 1984.
3. Do Sanh. A form of equations of motion of a mechanical system, Journal of Mechanics, NCNST of Vietnam, No 3, 1995.
4. Do Sanh. A form of equations of a constrained mechanical system, Proceedings of International Conference on Applied Dynamics, Hanoi, 1995.
5. Do Sanh. A new form of equations of motion of a mechanical system, Journal of Mechanics, NCNST of Vietnam, T. XIX, No 3, 1997 (in Vietnamese).
6. Lurie A. I. Analytical Mechanics, 1961, GIFML (in Russian)

7. Nejmark J. I., Fufajew N. A. Dynamics of nonholonomic systems, Izd. Nauka, 1967 (in Russian).

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MỘT DẠNG PHƯƠNG TRÌNH CHUYỂN ĐỘNG CỦA HỆ CƠ HỌC TRONG Á TỌA ĐỘ (TỌA ĐỘ KHÔNG HÔLÔNÔM)

Trước đây đối với các hệ cơ học phức tạp, đặc biệt khi biểu thức động năng của hệ hoặc các phương trình liên kết được viết trong á tọa độ để viết phương trình chuyển động của những hệ như vậy phải dùng hoặc phương trình Lagrange với nhân tử hoặc phương trình Lagrange trong á tọa độ.

Theo các con đường như vậy sẽ gặp phải nhiều khó khăn, ví dụ, khi sử dụng phương trình Lagrange dạng nhân tử thì phải khử các nhân tử, còn khi dùng phương trình Lagrange trong á tọa độ phải tính khá nhiều hệ số phức tạp [6].

Trong bài báo đã đưa ra một dạng phương trình rất thích hợp cho các trường hợp đã nêu trên. Sử dụng trực tiếp các thông số của hệ cho trong á tọa độ và điều quan trọng là dạng phương trình được viết trong dạng ma trận. Điều này rất thuận tiện cho việc lập tự động các phương trình chuyển động, ví dụ trong hướng symbolic, đặc biệt đối với các hệ có thứ nguyên lớn.

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