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DYNAMIC ANALYSIS OF STEPPED COMPOSITE CYLINDRICAL SHELLS SURROUNDED BY PASTERNAK ELASTIC FOUNDATIONS BASED ON THE CONTINUOUS ELEMENT METHOD

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Abstract. This research presents a continuous element model for solving vibration problems of stepped composite cylindrical shells surrounded by Pasternak foundations with various boundary conditions. Based on the First Order Shear Deformation Theory (FSDT), the equations of motion of the circular cylindrical shell are introduced and the dynamic stiffness matrix is obtained for each segment of the uniform shell. The interesting assembly procedure of continuous element method (CEM) is adopted to analyze the dynamic behavior of the stepped composite cylindrical shell surrounded by an elastic foundation. Free vibrations and harmonic responses of different configurations of stepped composite cylindrical shells on elastic foundations are examined. Effects of structural parameters, stepped thickness and elastic foundations on the free vibration responses of stepped composite cylindrical shells are also presented. Comparisons with previously published results and finite element (FE) analyses show that the proposed technique saves data storage volume and calculating time, and is accurate and efficient for widening the studied frequency range.

Keywords: Stepped shell, vibration of cylindrical shell, composite shell, elastic foundation, Pasternak foundation, continuous element method, dynamic stiffness method.

1. INTRODUCTION

Stepped composite cylindrical shells are widely used in different engineering structures such as tubes, vessels, tanks etc. In some special applications, those components may be surrounded or placed in an elastic foundation such as soil medium. This elastic foundation is commonly approximated by a Winkler or Pasternak model. Therefore the investigation on free vibration of a stepped composite cylindrical shell surrounded by Pasternak elastic foundations is necessary because the elastic foundation modifies the vibration of the structure. The governing equations of laminated cylindrical or stepped cylindrical shells in contact with Pasternak elastic foundations become more complex than shells without contact with elastic foundations. Recently, scientists employed different methods and theories to study the dynamic behavior of those shells. Ahmed [1] studied the vibration of an elastic oval cylindrical shell with parabolically varying thickness and along of its circumference surrounded by Pasternak foundations based on the framework of the Flügge's shell theory, the transfer matrix approach and the Romberg integration method. Qu et al. used a Domain Decomposition Method (DDM) to evaluate the vibration of stepped circular cylindrical and conical shells [2] and laminated orthotropic conical shells on Pasternak foundation [3]. Sofiyev et al. applied Galerkin method to analyze laminated orthotropic shells surrounded by Pasternak foundation with different boundary conditions [4] and non-homogenous orthotropic shells surrounded by Pasternak elastic foundation under moving load [5]. Bagheziradeh et al. [6] considered the dynamic behavior of FGM cylindrical shells surrounded by elastic foundation based on the Higher-order Shear Deformation Theory (HSDT). Kim [7] has analyzed the free frequencies of FGM cylindrical shells surrounded by elastic foundations by using the Rayleigh–Ritz method. Dung and Nga [8] used the First Order Shear Deformation Theory (FSDT) and Galerkin method to study also FGM cylindrical shells surrounded by Pasternak foundations.

In this research, the efficient Continuous Element Method is developed to study the free vibrations of stepped composite cylindrical shells surrounded by Pasternak elastic foundations. The Continuous Element Method (CEM) or the Dynamic Stiffness Method (DSM) based on the closed form solution of the system of differential equations of the structure is developed to overcome the difficulties of current approaches like Finite Element Method (FEM) in dynamic of structures. Numerous Continuous Elements have been developed for metal and composite structures [9–11]. Casimir et al. [11] have succeeded in building the DSM for thick isotropic plate and shells of revolution. The CE models for composite cylindrical shells, conical shells and annular plates presented in works of Thinh and his co-workers [12–16] imposes a considerable advancement of the study on CEM for composite structures. Recently, the new research for thick laminated composite cylindrical and joined cylindrical-conical shells surrounded by Winkler elastic foundation by Cuong et al. [17] has emphasized the strong capacity of the CEM in assembling complex structures. However, these studies only evaluate thick laminated composite shells in case of being surrounded by Winkler elastic foundations. The effects of shear layer in model of the Pasternak elastic foundation on the frequency response of thick laminated composite shells haven't been investigated in previously published researches.

In the present work, the Continuous Element Method is extended to analyze the dynamic behavior of the stepped composite cylindrical shell surrounded by the Pasternak foundation. This element consists an important basic continuous element to compose and analyze more complex structures such as: joined cylindrical-conical shells, combined cylindrical-conical shell and annular plates, ring-stiffened shells and those structures surrounded by elastic foundations. When the effect of shearing layer stiffness k_p is considered in equations of motion of this type of structure, it requires more complicated manipulations and transformations for analyzing and solving the governing equations of the composite cylindrical shell in contact with Pasternak foundations. Based on the assembly procedure of single continuous elements, dynamic stiffness matrix of complex structure such as stepped shell surrounded by Pasternak foundation is established. In this paper, the influences of different parameters are studied in detail such as stepped thickness, geometrical ratios and elastic foundation stiffness. The achieved numerical results are compared to those calculated by the finite element method and by other researches in some singular cases. The efficiency and accuracy of the CE method for complex shells in contact with elastic foundations in medium and high frequencies as well as the saving in data storage and computed time are approved.

2. GOVERNING EQUATIONS OF LAMINATE COMPOSITE CYLINDRICAL SHELL SURROUNDED BY ELASTIC FOUNDATION

2.1. Force resultants-displacement relationships

Consider a composite cylindrical shell of length L, thickness h and radius R (see Fig. 1). The shell consists of a finite number of layers which are perfectly bonded together. This shell is surrounded by a Winkler elastic foundation having a foundation stiffness k_w or by a Pasternak foundation with the foundation stiffness k_w and shear layer stiffness k_p. Such shell is the basic continuous shell element to contribute a stepped cylindrical shell surrounded by two above types of elastic foundations.

Following the Reissner–Mindlin assumption, the displacement components are assumed to be

$$\begin{aligned} u(x,\theta,z,t) &= u_0(x,\theta,t) + z\varphi_x(x,\theta,t), \\ v(x,\theta,z,t) &= v_0(x,\theta,t) + z\varphi_\theta(x,\theta,t), \\ w(x,\theta,z,t) &= w_0(x,\theta,t), \end{aligned} \tag{1}$$

where u, v, w: Displacement components in the x, θ and z directions, respectively.

u_0, v_0, w_0: Displacements of a point at the median radius of the shell.

ϕ_x, ϕ_θ: Rotations of a transverse normal about the θ-axis and x-axis, respectively.

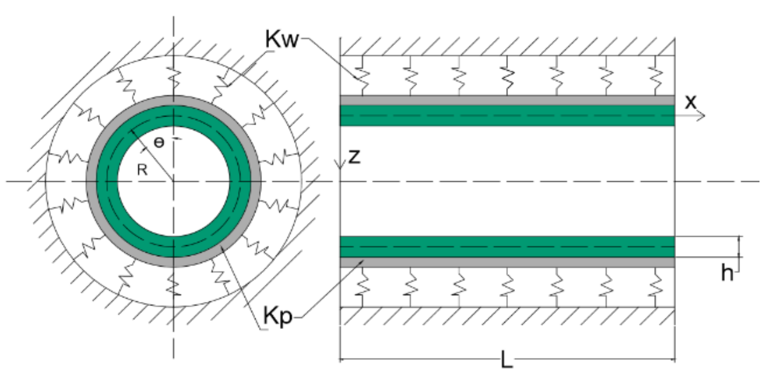

Fig. 1. Composite cylindrical shell surrounded by Pasternak elastic foundation

The strain-displacement relations of cylindrical shell by using Reissner-Mindlin theory can be written as [2]

$$\begin{aligned}
&\varepsilon_x = \frac{\partial u_0}{\partial x} + z\frac{\partial \phi_x}{\partial x}, \quad \varepsilon_\theta = \frac{1}{R}\frac{\partial v_0}{\partial \theta} + \frac{z}{R}\frac{\partial \phi_\theta}{\partial \theta} + \frac{w_0}{R},\\
&\gamma_{x\theta} = \frac{1}{R}\frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{\partial x} + z\left(\frac{1}{R}\frac{\partial \phi_x}{\partial \theta} + \frac{\partial \phi_\theta}{\partial x}\right),\\
&\gamma_{xz} = \phi_x + \frac{\partial w_0}{\partial x}, \quad \gamma_{\theta z} = \phi_\theta + \frac{1}{R}\frac{\partial w_0}{\partial \theta} - \frac{v_0}{R}.
\end{aligned} \tag{2}$$

For a laminate composite shell of total thickness h composed by N orthotropic layers, the stress-strain relations of the k^{th} layer are written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_{\theta z} \\ \sigma_{xz} \\ \sigma_{x\theta} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & 0 & 0 & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & 0 & 0 & \overline{Q}_{26} \\ 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0 \\ 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} & 0 \\ \overline{Q}_{16} & \overline{Q}_{26} & 0 & 0 & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \gamma_{\theta z} \\ \gamma_{xz} \\ \gamma_{x\theta} \end{Bmatrix}, \tag{3}$$

where $\overline{Q}_{ij}$ are the transformed stiffnesses and Q_{ij} are the lamina stiffnesses referred to principal material coordinates of the k^{th} lamina. The plane stress-reduced stiffnesses are calculated as [4]

$$\begin{aligned}
&Q_{11} = \frac{E_1}{1 - v_{12}v_{21}}, \quad Q_{12} = \frac{v_{12}E_2}{1 - v_{12}v_{21}}, \quad Q_{66} = G_{12},\\
&Q_{22} = \frac{E_2}{1 - v_{12}v_{21}}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13},
\end{aligned} \tag{4}$$

here $E_i, G_{ij}, v_{12}, v_{21}$: Elastic constants of the k^{th} layer.

The stress and moment resultants of laminate composite cylindrical shell are given by [15]

$$\{N_x, N_\theta, N_{x\theta}, N_{\theta x}, Q_x, Q_\theta\} = \sum_k^N \int_{z_k}^{z_{k+1}} \left\{\sigma_x\left(1+\frac{z}{R}\right), \sigma_\theta, \sigma_{x\theta}\left(1+\frac{z}{R}\right), \sigma_{x\theta}, \sigma_{\theta z}\left(1+\frac{z}{R}\right), \sigma_{xz}\right\} dz, \tag{5}$$

$$(M_x, M_\theta, M_{x\theta}, M_{\theta x}) = \sum_k^N \int_{z_k}^{z_{k+1}} \left\{\sigma_x\left(1+\frac{z}{R}\right), \sigma_\theta, \sigma_{x\theta}\left(1+\frac{z}{R}\right), \sigma_{x\theta}\right\} zdz, \tag{6}$$

and the laminate stiffness coefficients (A_{ij}, B_{ij}, D_{ij}), are defined by

$$\begin{aligned}
&A_{ij} = \sum_{k=1}^N \bar{Q}_{ij}^k (z_{k+1} - z_k)(i,j = 1,2,6), \quad A_{ij} = \sum_{k=1}^N \bar{Q}_{ij}^k (z_{k+1} - z_k)(i,j = 4,5)\\
&B_{ij} = \frac{1}{2}\sum_{k=1}^N \bar{Q}_{ij}^k (z_{k+1}^2 - z_k^2)(i,j = 1,2,6), \quad D_{ij} = \frac{1}{3}\sum_{k=1}^N \bar{Q}_{ij}^k (z_{k+1}^3 - z_k^3)(i,j = 1,2,6),
\end{aligned} \tag{7}$$

where z_{k-1} and z_k are the boundaries of the k^{th} layer.

For cross-ply composite laminated cylindrical shells, the forces and moment resultants are determined by [15]

$$
\begin{aligned}
N_x &= A_{11}\frac{\partial u_0}{\partial x} + \frac{A_{12}}{R}\left(\frac{\partial v_0}{\partial \theta} + w_0\right) + B_{11}\frac{\partial \varphi_x}{\partial x} + \frac{B_{12}}{R}\frac{\partial \varphi_\theta}{\partial \theta},\\
N_\theta &= A_{12}\frac{\partial u_0}{\partial x} + \frac{A_{22}}{R}\left(\frac{\partial v_0}{\partial \theta} + w_0\right) + B_{12}\frac{\partial \varphi_x}{\partial x} + \frac{B_{22}}{R}\frac{\partial \varphi_\theta}{\partial \theta},\\
N_{x\theta} &= A_{66}\left(\frac{\partial v_0}{\partial x} + \frac{1}{R}\frac{\partial u_0}{\partial \theta}\right) + B_{66}\left(\frac{1}{R}\frac{\partial \varphi_x}{\partial \theta} + \frac{\partial \varphi_\theta}{\partial x}\right),\\
M_x &= B_{11}\frac{\partial u_0}{\partial x} + \frac{B_{12}}{R}\left(\frac{\partial v_0}{\partial \theta} + w_0\right) + D_{11}\frac{\partial \varphi_x}{\partial x} + \frac{D_{12}}{R}\frac{\partial \varphi_\theta}{\partial \theta},\\
M_\theta &= B_{12}\frac{\partial u_0}{\partial x} + \frac{B_{22}}{R}\left(\frac{\partial v_0}{\partial \theta} + w_0\right) + D_{12}\frac{\partial \varphi_x}{\partial x} + \frac{D_{22}}{R}\frac{\partial \varphi_\theta}{\partial \theta},\\
M_{x\theta} &= B_{66}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{R\partial \theta}\right) + D_{66}\left(\frac{1}{R}\frac{\partial \varphi_x}{\partial \theta} + \frac{\partial \varphi_\theta}{\partial x}\right),\\
Q_\theta &= kA_{44}\left(\frac{-v_0}{R} + \frac{1}{R}\frac{\partial w_0}{\partial \theta} + \varphi_\theta\right), \quad Q_x = kA_{55}\left(\frac{\partial w_0}{\partial x} + \varphi_x\right),
\end{aligned}
\tag{8}
$$

where k is the shear correction factor $(k = 5/6)$.

2.2. Equations of motion of laminated composite cylindrical shell surrounded by Pasternak foundation

The load-displacement relationship of the Pasternak foundation is assumed to be: $p_0 = k_w w - k_p \Delta w$ where p_0 is the force per unit area, k_w is the Winkler foundation stiffness and k_p is the shearing layer stiffness of the Pasternak foundation, and Δ is the Laplace operator. Therefore, the equations of motion for cross-ply laminated composite cylindrical shell surrounded by Pasternak foundation are expressed as

$$
\begin{aligned}
&\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{\theta x}}{\partial \theta} = I_0\frac{\partial^2 u_0}{\partial t^2} + I_1\frac{\partial^2 \phi_x}{\partial t^2},\\
&\frac{\partial N_{x\theta}}{\partial x} + \frac{\partial N_\theta}{R\partial \theta} + \frac{Q_\theta}{R} = I_0\frac{\partial^2 v_0}{\partial t^2} + I_1\frac{\partial^2 \phi_\theta}{\partial t^2},\\
&\frac{\partial Q_x}{\partial x} + \frac{\partial Q_\theta}{R\partial \theta} - \frac{N_\theta}{R} - k_w w + k_p\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial \theta^2}\right) = I_0\frac{\partial^2 w_0}{\partial t^2},\\
&\frac{\partial M_x}{\partial x} + \frac{\partial M_{\theta x}}{R\partial \theta} - Q_x = I_1\frac{\partial^2 u_0}{\partial t^2} + I_2\frac{\partial^2 \phi_x}{\partial t^2},\\
&\frac{\partial M_{x\theta}}{\partial x} + \frac{\partial M_\theta}{R\partial \theta} - Q_\theta = I_1\frac{\partial^2 v_0}{\partial t^2} + I_2\frac{\partial^2 \phi_\theta}{\partial t^2},
\end{aligned}
\tag{9}
$$

$I_i = \sum\limits_{k=1}^{N} \int\limits_{z_k}^{z_{k+1}} \rho^{(k)} z^i dz (i = 0, 1, 2)$ and $\rho^{(k)}$ is the material mass density of the k^{th} layer.

3. DYNAMIC STIFFNESS MATRIX FOR COMPOSITE CYLINDRICAL SHELLS SURROUNDED BY ELASTIC FOUNDATION

The chosen state-vector is $y^T = \{u_0, v_0, w_0, \varphi_x, \varphi_\theta, N_x, N_{x\theta}, Q_x, M_x, M_{x\theta}\}^T$. Next, the Fourrier series expansion for state variables is written as:

$$\begin{aligned}
&\{u_0(x,\theta,t), w_0(x,\theta,t), \varphi_x(x,\theta,t), N_x(x,\theta,t), Q_x(x,\theta,t), M_x(x,\theta,t)\}^T \\
&\quad = \sum_{m=1}^{\infty} \{u_m(x), w_m(x), \varphi_{xm}(x), N_{x_m}(x), Q_{xm}(x), M_{x_m}(x)\}^T \cos m\theta e^{i\omega t}, \\
&\{v_0(x,\theta,t), \varphi_\theta(x,\theta,t), N_{x\theta}(x,\theta,t), M_{x\theta}(x,\theta,t)\}^T \\
&\quad = \sum_{m=1}^{\infty} \{v_m(x), \varphi_{\theta m}(x), N_{x\theta m}(x), M_{x\theta m}(x)\}^T \sin m\theta e^{i\omega t},
\end{aligned} \tag{10}$$

where m is the number of circumferential wave.

Substituting (10) in Eqs. (8) and (9), a system of ordinary differential equations in the x-coordinate for the m^{th} mode will be obtained. Then it can be expressed in the matrix form for each circumferential mode m as

$$\frac{d\mathbf{y}_m}{dx} = \mathbf{A}_m \mathbf{y}_m \text{ with } \mathbf{A}_m \text{ is a } 10 \times 10 \text{ matrix.} \tag{11}$$

The dynamic transfer matrix $\mathbf{T}_m$ is evaluated as

$$T_m(\omega) = e^{\int_0^L A_m(\omega) dx} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}. \tag{12}$$

Finally, the dynamic stiffness matrix $\mathbf{K}_m(\omega)$ for cylindrical shell is determined by [13]

$$K_m(\omega) = \begin{bmatrix} T_{12}^{-1} T_{11} & -T_{12}^{-1} \\ T_{21} - T_{22} T_{12}^{-1} T_{11} & T_{22} T_{12}^{-1} \end{bmatrix}. \tag{13}$$

4. CONTINUOUS ELEMENT FOR STEPPED COMPOSITE CYLINDRICAL SHELLS

Let's investigate a stepped cylindrical shell (SCS) including four segments shown in Fig. 2. The SCS consists of four lengths L_1, L_2, L_3, L_4 and four step thicknesses h_1, h_2, h_3, h_4. Let the co-ordinate system be chosen as shown in Fig. 2; θ is the circumferential co-ordinate. R is the radius of the SCS at the origin O. u, v and w are the displacement components in the x, θ and normal directions, respectively.

The dynamic stiffness matrix $\mathbf{K}_m(\omega)$ for the above stepped composite cylindrical shells surrounded by elastic foundation will be constructed by assembling the DSM of many sections having different constant thickness and lengths. First, the shell is divided in to four elements. It is necessary to build four separate dynamic stiffness matrices $\mathbf{K}_{seg1}$, $\mathbf{K}_{seg2}$, $\mathbf{K}_{seg3}$ and $\mathbf{K}_{seg4}$ for these segments. Then, Fig. 3 describes the assembly procedure for constructing the DSM for the stepped cylindrical shells. The natural frequencies of the studied structure will be determined from this matrix by using the method detailed in [12].

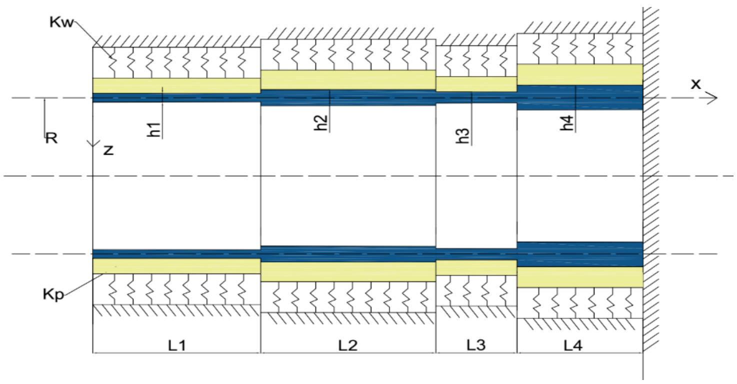

Fig. 2. Geometry of composite stepped cylindrical shells surrounded by elastic foundations

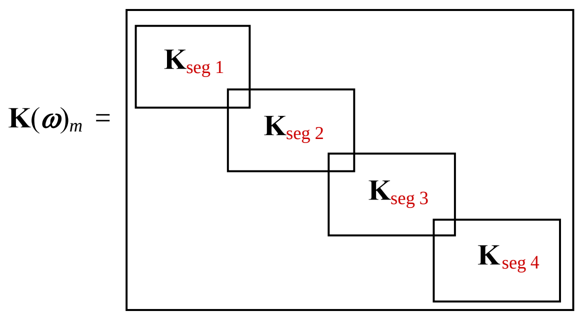

Fig. 3. Construction of the dynamic stiffness matrix for composite stepped cylindrical shells surrounded by elastic foundations

5. RESULTS AND DISCUSSIONS

5.1. Validation of the present model

The proposed continuous element model will be validated by comparison with solutions available in the literature and with finite element results.

First, to check the accuracy of the present method on the vibration analysis of cylindrical shells with Pasternak foundations, Tab. 1 illustrates the natural frequencies for the Simply Supported-Simply Supported (S-S) isotropic cylindrical shell surrounded by different Pasternak foundations. The result obtained by Wu and Qu [3] using a Domain Decomposition Method are compared with our solutions. Two values of foundation stiffnesses $k_w = 10^7$ N/m^3 and $k_p = 10^7$ N/m are in consideration. The material properties of the cylindrical shell are: $E = 200$ GPa, $\upsilon = 0.32, \rho = 7916$ kg/m^3. Small errors are found between our results and those of [3] varying from 0.09% to 6.58% with Winkler foundation stiffness and from 0.96% to 7.05% with shearing layer stiffness. Next, Tab. 2 has shown the natural frequencies of the Clamped-Free (C-F) isotropic two-step cylindrical shell with $h_1/R = 0.01, R = 1$ m, $h_2/h_1 = 0.5, L_1/L = 0.5, n = 1$, $E = 210$ GPa, $\rho = 7916$ kg/m^3, $\upsilon = 0.3$. Natural frequencies computed by CEM are compared with

Table 1. Natural frequency of cylindrical shell with S-S boundary conditions: $h/R = 0.002$, $R = 1$ m, $m = 1$, $L = 6.27$ m, $E = 200$ GPa, $\upsilon = 0.32$, $\rho = 7916$ kg/m^3

n	$k_w = 10^7$ N/m^3			$k_p = 10^7$ N/m		
	Ref. [3] A	CEM B	Errors (%) $\mid$ (A − B)*100/A $\mid$	Ref. [3] A	CEM B	Errors (%) $\mid$ (A − B)*100/A $\mid$
1	138.862	148	6.58	145.225	140	3.60
2	307.983	312	1.30	320.462	337	5.17
3	470.106	472	0.40	492.276	527	7.05
4	589.166	592	0.48	629.010	623	0.96
5	664.330	666	0.25	726.611	757	4.18
6	709.355	710	0.09	797.847	819	2.65
7	736.961	738	0.14	854.794	846	1.03
8	754.741	754	0.10	904.850	884	2.30
9	766.778	766	0.10	951.928	937	1.57
10	775.291	776	0.09	998.038	980	1.81

Table 2. Natural frequency for a C-F two-stepped cylindrical shell ($h_1/R = 0.01$, $R = 1$ m, $h_2/h_1 = 0.5$, $L_1/L = 0.5$, $n = 1$, $E = 210$ GPa, $\rho = 7916$ kg/m^3, $\upsilon = 0.3$)

m	CEM A	Ref. [2] B	Errors (%) $\mid$ (A − B)*100/B $\mid$
1	563	551.7441	2.04
2	346	354.4554	2.39
3	228	238.1008	4.24
4	172	168.4163	2.13
5	129	127.5367	1.15
6	113	107.0208	5.59
7	107	102.3002	4.59
8	114	109.1313	4.46

those of Qu [2] and obtained errors vary from 1.15% to 5.59%. Therefore, this continuous element model is reliable and effective to study composite cylindrical shells surrounded by Pasternak elastic foundations.

5.2. Harmonic responses

Next, the advantages of CEM will be demonstrated by an analysis on harmonic responses performed for a cross-ply composite cylindrical shell having the following properties: $L_1 : L_2 : L_3 : L_4 = 1 : 1 : 1 : 1$, $h_1 : h_2 : h_3 : h_4 = 1 : 2 : 3 : 4$, $h/R = 0.02$, $L = 4R$, $h_1 = h = 0.02$ m, layer scheme [90°/0°/90°/0°] and surrounded by Winkler elastic foundation $k_w = 2 \times 10^6$ N/m^3. Two types of boundary conditions are considered: free-clamped (F-C) and free-free (F-F). The material properties of the each layer are: $E_1 = 138.9$ GPa, $E_2 = E_3 = 8.96$ GPa, $G_{12} = G_{13} = 7.1$ GPa, $G_{23} = 3.45$ GPa; $v_{12} = 0.3$, $\rho = 1645$ kg/m^3 (Material 1). The same cylindrical shell was modeled in finite element industrial software (Ansys) with SHELL99 element. Fig. 4 and Fig. 5 illustrates the variation of harmonic response curves calculated by CEM and by FEM (Ansys) with two different meshing ways ($24 \times 8, 60 \times 20$ elements).

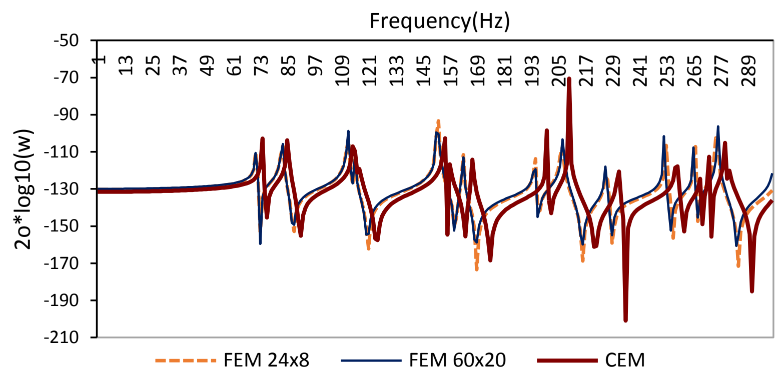

Fig. 4. Comparison of harmonic response of F-F stepped composite cylindrical shell surrounded by Winkler elastic foundation by CEM and by FEM ($L_1 : L_2 : L_3 : L_4 = 1 : 1 : 1 : 1$, $h_1 : h_2 : h_3 : h_4 = 1 : 2 : 3 : 4$, $h/R = 0.02, L = 4R$, $h_1 = h = 0.02$ m, layer scheme $[0^\circ/90^\circ/0^\circ/90^\circ]$, $k_w = 2 \times 10^6$ N/m^3, Material 1)

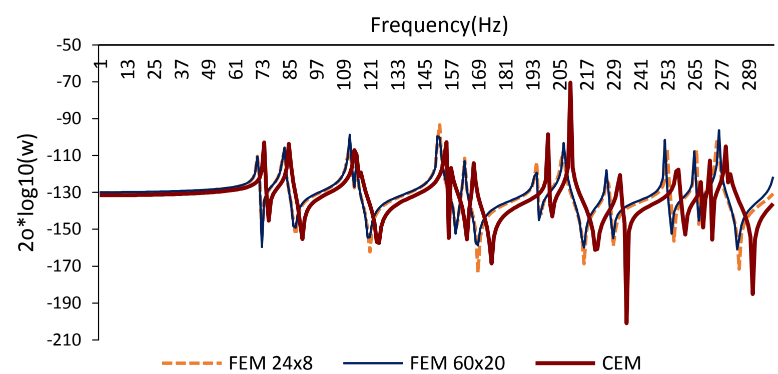

Fig. 5. Comparison of harmonic response of F-C stepped composite cylindrical shell surrounded by Winkler elastic foundation by CEM and by FEM ($L_1 : L_2 : L_3 : L_4 = 1 : 1 : 1 : 1$, $h_1 : h_2 : h_3 : h_4 = 1 : 2 : 3 : 4$, $h/R = 0.02, L = 4R$, $h_1 = h = 0.02$ m, layer scheme $[0^\circ/90^\circ/0^\circ/90^\circ]$, $k_w = 2 \times 10^6$ N/m^3, Material 1)

For a complex structure such as SCS, the CE model demonstrates advantageous in comparison with other approaches, especially approximate method like FEM. The minimum number of discretization elements leading to a smaller size of dynamic stiffness matrix and the exact formulations used in CEM give faster and more accurate results compare to those of FEM. It is observed through Fig. 4 and Fig. 5 that CE model offers more precise solutions because FE results converge to those of CE if a finer meshing is used. This characteristic is well illustrated in the two graphics as harmonic response curves of the 60×20 mesh is closer to CE curves than the 24×8 mesh for both boundary conditions. It is also necessary to remark that natural frequency values of CEM coincide to those of FEM in low frequencies (the same picks are found) but more discrepancies are noticed when frequencies raise. Such phenomenon confirms the difficulties of using FEM in the medium and high frequency analysis and CEM consists an interesting and robust way to overcome those challenges.

In addition, the CEM can save computational time compared to FEM. For the F-C stepped shell surrounded by elastic foundation, our model required 5 minutes to plot the harmonic response while FEM models needed much more calculating time: 120 minutes for 24×8 mesh and 180 minutes for 60×20 mesh. Therefore, the CE model uses a minimum meshing of the structure yielding to a very small size of system of equations to resolve so it accelerates the speed of calculation.

In conclusion, through these examples, CEM demonstrates its performance when dealing with complex structures and with a higher frequency range. The high precision, saving in computing time and data volume storage as well as the capacity of application in medium and high frequency range are main advantages of our formulation as confirmed in previous researches.

5.3. Influences of shell parameters

In this section, the influences of various shell parameters and elastic foundations on the dynamic behavior of the stepped composite cylindrical shells will be studied such as stepped thickness configurations, thickness-to-radius ratio, boundary conditions. Different stiffness values of k_w, k_p are also taken into account.

5.3.1. *Effect of shell parameters*

First, the effect of stepped thickness on free vibration of stepped composite cylindrical shells will be analyzed in detail. The considered stepped composite cylindrical shell is subjected to the free-clamped boundary condition and has the following dimensions: $L_1 : L_2 : L_3 : L_4 = 1 : 1 : 1 : 1$, $h/R = 0.02, L = 4R, h_1 = h = 0.02$ m, layer scheme $[90^\circ/0^\circ/90^\circ/0^\circ]$, Material 1 is used and the Winkler elastic foundation stiffness is $k_w = 2 \times 10^6$ N/m^3. Here, four different configurations of stepped thickness are examined: $h_1 : h_2 : h_3 : h_4 = 1 : 2 : 3 : 4/2 : 2 : 3 : 4/2 : 3 : 3 : 4/4 : 4 : 4 : 4/4 : 3 : 2 : 1$ and results are summarized in Fig. 6. It can be seen that except the first three modes, the augmentation of the stepped thickness leads to the raise of natural frequencies of all other circumferential modes (m). In addition, the effect of the thickness of segments on the first mode is minimal.

Next, Fig. 7 shows the influence of the h/R ratio on natural frequencies of a free-clamped stepped composite cylindrical shell. The shell parameters are: $L_1 : L_2 : L_3 : L_4 = 1 : 1 : 1 : 1$, $L = 2$ m, $h_1 : h_2 : h_3 : h_4 = 1 : 2 : 3 : 4$, $h = h_1 = 0.02$ m, $L = 2$ m, layer configuration $[90^\circ/0^\circ/90^\circ/0^\circ]$, Material 1 is used and the elastic foundation stiffness is

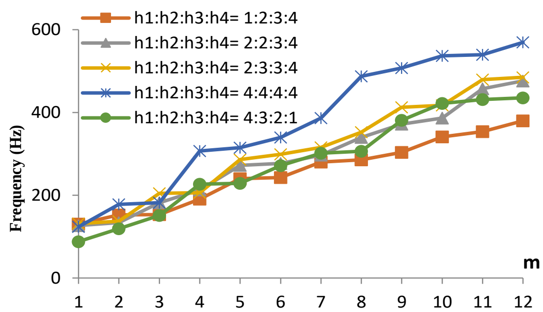

Fig. 6. Effect of stepped thickness on the vibration of the F-C stepped composite cylindrical shell

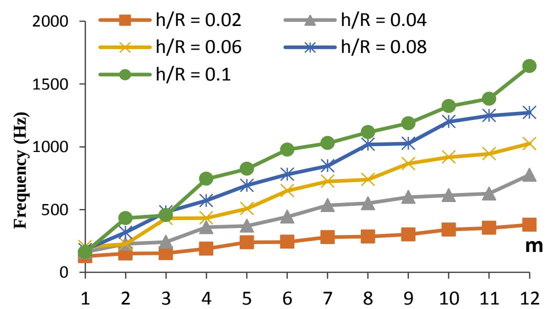

Fig. 7. Influence of h/R ratio on the vibration of the F-C stepped composite cylindrical shell

$k_w = 15 \times 10^4$ N/m^3. Five values of h/R ratio $(0.02, 0.04, 0.06, 0.08, 0.1)$ have been chosen for the analysis. It is shown that when the circumferential mode $m = 1$ and the axial mode $n = 1$, the natural frequencies take the same value for all h/R ratios. As the h/R ratio increases, the shell become thicker and thus natural frequencies raise more rapidly with each value of m. Moreover, the h/R ratio has more effect on the natural frequency with the higher circumferential modes.

5.3.2. *Influences of elastic foundations*

It is necessary to examine the effects of different types of elastic foundations on the free vibration of the stepped composite cylindrical shell. Consider now the above mentioned stepped composite cylindrical shell surrounded by a Winkler foundation with various values of foundation stiffness $k_w (h_1 : h_2 : h_3 : h_4 = 1 : 2 : 3 : 4, h/R = 0.02)$. Five different values of $k_w (0, 15 \times 10^4, 2 \times 10^6, 2 \times 10^7, 2 \times 10^8$ N/m$^3)$ are taken for the study and results are illustrated in Fig. 8. It is easy to remark that when $k_w < 2 \times 10^6$ N/m^3 the effects of Winkler foundation stiffness on natural frequency are not obvious. The effects

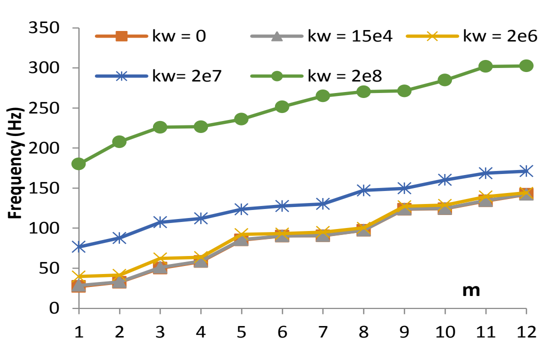

Fig. 8. Influence of Winkler foundations on natural frequencies of the F-C stepped composite cylindrical shell

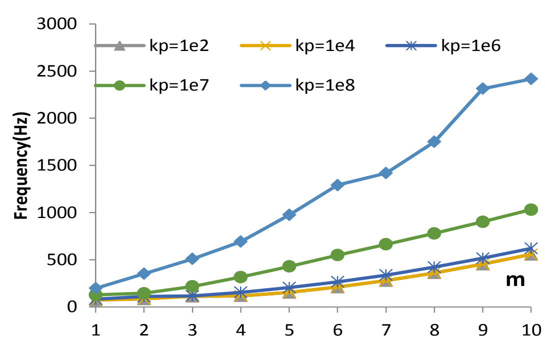

Fig. 9. Influence of Pasternak foundations on natural frequencies of the F-C stepped composite cylindrical shell

of Winkler foundation are only significant if $k_w \geq 2 \times 10^7$ N/m^3 as the stiffness of the structure becomes important.

Effect of Pasternak foundations has been investigated in the next test case. The same structure rests on a Pasternak foundation with $k_w = 2 \times 10^6$ N/m^3 and various values of shear stiffness are chosen: $k_p = 0, 10^2, 10^4, 10^6, 10^7, 10^8$ (N/m). Fig. 9 presents the variation of natural frequencies of the studied structure with respect to many values of k_p. It is observed from this figure that natural frequencies increase rapidly as $k_p > 10^7$ N/m. When m increases, the influence of Pasternak foundation on natural frequency becomes larger. With $k_p \leq 10^4$ N/m, Pasternak foundations have almost no effects on the natural frequencies of the shell.

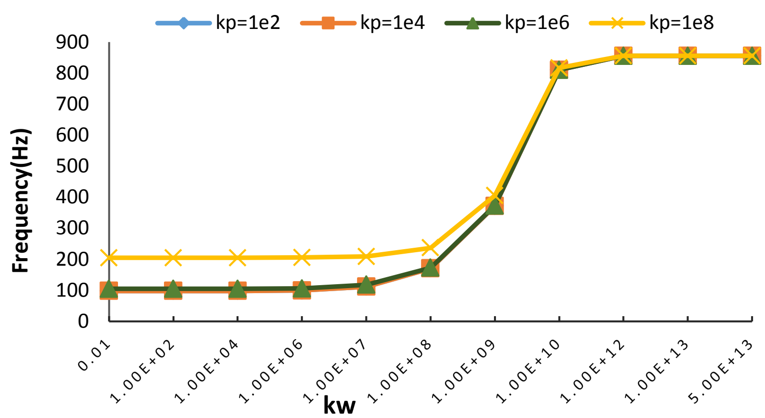

Fig. 10. Influences of both Winkler stiffness and Pasternak stiffness to natural frequencies of the S-S stepped composite cylindrical shell

Next, effects of both Winkler stiffness and Pasternak stiffness on natural frequencies of the S-S stepped composite cylindrical shell will be studied. The values of Winkler stiffness and Pasternak stiffness are $k_w = 10^{-2}, 10^2, 10^4, 10^6, 10^7, 10^8, 10^9, 10^{10}, 10^{12}, 10^{13}, 5.10^{13}$ N/m^3 and $k_p = 10^2, 10^4, 10^6, 10^8$ N/m, respectively. From Fig. 10, it can be seen that the effect of Winkler stiffness and Pasternak stiffness on natural frequencies is important only on a certain range (k_w from 10^6 to 10^{12} N/m^3, k_p from 10^4 to 10^8 N/m). When k_w reaches to the limit value $k_w = 10^{12}$ N/m^3, Pasternak stiffness values have less effect on natural frequencies.

6. CONCLUSIONS

In this paper, a Continuous Element model for stepped laminated composite cylindrical shells surrounded by Winkler and Pasternak elastic foundations has been successfully constructed. The effect of the Pasternak elastic foundation on uniform and stepped cylindrical shells has been well integrated into the presented element. Very good agreements are noticed between the results obtained by our approach and those of other methods. Various numerical results have confirmed that Continuous element model increases the speed of calculation and economies the storage capacity of computers by using a minimum meshing. The effects of various parameters on vibration behavior of the shell are investigated. From the above results, it can be concluded that:

1. The ratio thickness-to-radius has larger effect on natural frequencies when m increases $(m > 1)$.

2. The stiffness parameters of the elastic foundation have significant effect on the vibration of the stepped cross-ply composite cylindrical shell. As the stiffness parameters of the elastic foundation are greater, the frequencies are higher.

3. For the stepped cross-ply composite cylindrical shell surrounded by elastic foundation, the effect of Winkler stiffness and Pasternak stiffness on natural frequency is noticeable in a certain range. When the Winkler stiffness reaches limited value (as $k_w = 10^{12}$ N/m^3), the influence of shearing layer elastic stiffness parameter in natural frequency is hardly recognized.

The developed continuous element model with its powerful assembling procedure can be expanded to study more complex shell structures such as: joined cylindrical-conical shells, combined cylindrical-conical shell and annular plates, ring-stiffened shells and those structures surrounded by elastic foundations or in contact with fluid.

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APPENDIX

Matrice $\mathbf{A}_m =$

$$
\left[\begin{array}{cccc}
0 & mc_4 & c_4 & 0 \\
\dfrac{m}{R} & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & mc_2 & c_2 & 0 \\
0 & 0 & 0 & \dfrac{m}{R} \\
-I_0\omega^2 & 0 & 0 & -I_1\omega^2 \\
0 & -I_0\omega^2 + m^2c_6 + \dfrac{kA_{44}}{R^2} & m\left(c_6 + \dfrac{kA_{44}}{R^2}\right) & 0 \\
0 & c_{11}m\left(c_6 + \dfrac{kA_{44}}{R^2} + k_pc^2\right) & c_{11}\left[c_6 - I_0\omega^2 + \dfrac{mkA_{44}}{R^2} + k_w + k_pc^2 + \dfrac{k_pm^2}{R^2}\right] + \dfrac{k_pm^2}{R^2} & 0 \\
-I_1\omega^2 & 0 & 0 & -I_2\omega^2 \\
0 & m^2c_8 - \dfrac{kA_{44}}{R} & m^2c_8 - \dfrac{kA_{44}}{R} & 0
\end{array}\right.
$$

$$
\left.\begin{array}{cccccc}
mc_5 & \dfrac{D_{11}}{c_1} & 0 & 0 & -\dfrac{B_{11}}{c_1} & 0 \\
0 & 0 & -\dfrac{D_{66}}{c_{10}} & 0 & 0 & \dfrac{B_{66}}{c_{10}} \\
0 & 0 & 0 & \dfrac{1}{kA_{55}} & 0 & 0 \\
mc_3 & -\dfrac{B_{11}}{c_1} & 0 & 0 & \dfrac{A_{11}}{c_1} & 0 \\
0 & 0 & \dfrac{B_{66}}{c_{10}} & 0 & 0 & -\dfrac{A_{66}}{c_{10}} \\
0 & 0 & -\dfrac{m}{R} & 0 & 0 & 0 \\
m^2c_7 - \dfrac{kA_{44}}{R} - I_1\omega^2 & -mc_4 & 0 & 0 & -mc_2 & 0 \\
c_{11}m\left(c_7 - \dfrac{kA_{44}}{R} + k_pc_3\right) & -c_{11}\left(c_4 + \dfrac{k_pB_{11}}{c_1}\right) & 0 & 0 & -c_{11}\left(c_2 + \dfrac{k_pA_{11}}{c_1}\right) & 0 \\
0 & 0 & 0 & 1 & 0 & -\dfrac{m}{R} \\
-I_2\omega^2 + kA_{44} + c_9m^2 & -mc_5 & 0 & 0 & -mc_4 & 0
\end{array}\right]
$$