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SHORT COMMUNICATION

DUAL APPROACH TO AVERAGED VALUES OF FUNCTIONS: A FORM FOR WEIGHTING COEFFICIENT

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Abstract. Averaged values play major roles in the study of dynamic processes. The use of those values allows transforming varying processes to some constant characteristics that are much easier to be investigated. In order to extend the use of averaged values one may apply the dual approach which suggests a consideration of two different aspects of a problem in question. In this short communication the main idea of the dual conception is further extended to suggest a new form for weighting coefficient and then a new averaged value of functions. This new averaged value depends on the parameter s and contains the classical averaged value when $s = 0$. In the example of Duffing oscillator it is shown that the parameter s can be chosen as $s = n/(2\pi)$ and for $n = 4$ one gets the solution that is much accurate than the conventional one obtained by the classical criterion of equivalent linearization.

Keywords: Weighting coefficient, averaged value, dual approach, Duffing oscillator.

1. INTRODUCTION

Averaged values play major roles in the study of dynamic processes. The use of those values allows transforming varying processes to some constant characteristics that are much easier to be investigated. In order to extend the use of averaged values one may apply the dual approach recently proposed and developed in [1–3]. One of significant advantages of the dual conception is its consideration of two different aspects of a problem in question which allows investigation to be more appropriate. In this short communication the main idea of the dual conception is further extended to suggest a new form for weighting coefficient and then a new averaged value of functions. Duffing oscillator is investigated for application.

2. WEIGHTING COEFFICIENTS FOR WEIGHTED AVERAGED VALUES

In our life for a given data set the most common statistic is the arithmetic mean. The concept of average of a data set can be extended to functions. The conventional average value (CAV) of an integrable deterministic function $x(t)$ on a domain D: $(0, d)$ is a constant value defined by

$$< x(t) > = \frac{1}{d} \int_0^d x(t)\,\mathrm{d}t. \tag{1}$$

In many cases when the function $x(\omega t)$ is periodic with period $2\pi/\omega$ the value d is taken as $2\pi/\omega$ and it leads to the averaged value of $x(t)$over one period

$$< x(\omega t) > = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} x(\omega t)\,\mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} x(\tau)\,\mathrm{d}\tau, \tag{2}$$

where

$$\tau = \omega t, \tag{3}$$

is the new variable or "new time". Averaged values play surely major roles in the past and at present, however, the definition (1) has some deficiencies, for example, if (1) or (2) are equal zero, the information about $x(t)$ is lost. For all harmonic functions $\cos(n\omega t)$ and $\sin(n\omega t)$ this observation is true. The dual approach to averaged values may be a possible way to suggest an alternative choice for the conventional average value, namely the constant coefficient $1/d$ in (1) can be extended to a weighting coefficient as function $h(t)$. Thus one gets so-called weighted average value

$$W(x(t)) \; = \int_0^d h(t) x(t)\,\mathrm{d}t, \tag{4}$$

where the condition of normalization is satisfied

$$\int_0^d h(t)\,\mathrm{d}t = 1. \tag{5}$$

In the following we will introduce some basic weighting coefficients and a way to build new weighting coefficients from basic ones.

2.1. Basic optimistic weighting coefficients

Basic optimistic weighting coefficients are increasing functions of t and denoted as $O(t)$. Examples are αt^{β} and $\alpha e^{\beta t}$, α, $\beta > 0$.

2.2. Basic pessimistic weighting coefficients

Basic pessimistic weighting coefficients are decreasing functions of t and denoted as $P(t)$. Examples are αt^{β} and $\alpha e^{\beta t}$, $\alpha < 0, \beta > 0$; or $\alpha > 0, \beta < 0$.

2.3. Neutral weighting coefficients

Basic neutral weighting coefficients, denoted as $N(t)$ are constants.

An arbitrary weighting coefficient $h(t)$ can be obtained as summation and/or product of basic weighting coefficients. Example is

$$h(t) = \sum_{i=1}^{n} (A_i O_i(t) + B_i P_i(t) + C_i O_i(t) P_i(t)) + N(t), \tag{6}$$

where A_i, B_i, C_i are constant.

In this short communication we will consider only ω-periodic functions $x(\omega t)$. A special form of weighting coefficient is introduced

$$h(t) = s^2 \omega^2 t e^{-s\omega t},\ s > 0. \tag{7}$$

It is seen that the weighting coefficient (7), obtained as a product of the optimistic weighting coefficient t and the pessimistic weighting coefficient $e^{-s\omega t}$, has one maximal value at $t_{\max} = 1/(s\omega)$, and then decreases to zero as $t \to \infty$ (see Fig. 1). If one requires that the time $t_{\max}$ is equal to $T/n = 2\pi/(n\omega)$ where n is a natural number or zero, one gets $s = n/(2\pi)$. So the meaning of s can be specified as follows: for $n = 1$, $s = 1/(2\pi)$ the weighting coefficient (7) has maximal value after one period, and for $n = 4$, $s = 4/(2\pi)$ the weighting coefficient (7) has maximal value after quarter period, and for $n = 0$, $s = 0$ the weighting coefficient (7) has maximal value at infinity. This case corresponds to the conventional averaged value.

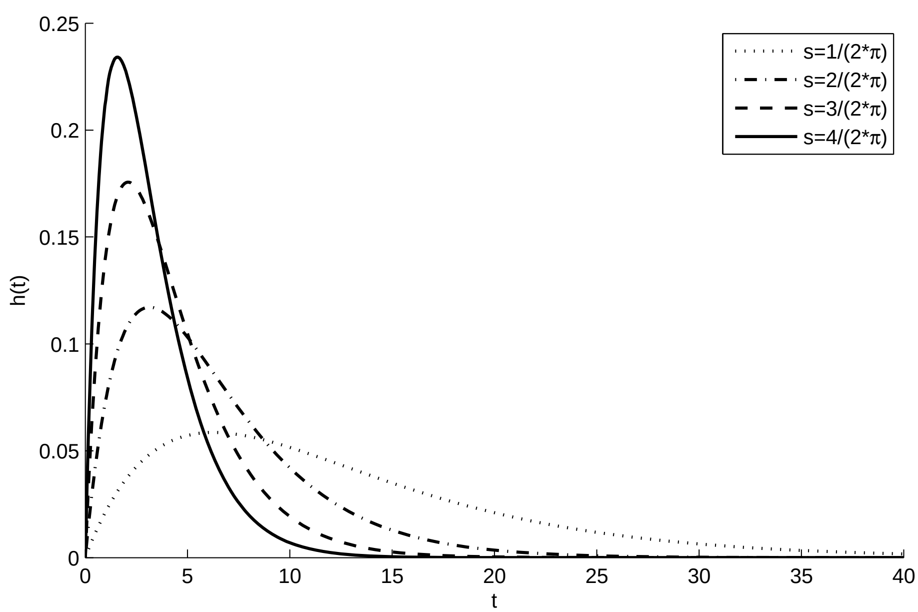

Fig. 1. Plot of $h(t)$, (7)

Based on the weighting coefficient (7) a new weighted average value is proposed

$$W(x(\omega t)) = \int_0^\infty s^2\omega^2 t e^{-s\omega t} x(\omega t)\,\mathrm{d}t = \int_0^\infty s^2 \tau e^{-s\tau} x(\tau)\,\mathrm{d}\tau, \tag{8}$$

which is a linear operator. From Laplace transformation Table one gets, for example,

$$W(\cos n\omega t) = \int_0^\infty s^2\omega^2 t e^{-s\omega t}\cos(n\omega t)\,\mathrm{d}t = \int_0^\infty s^2\tau e^{-s\tau}\cos(n\tau)\,\mathrm{d}\tau = s^2\frac{s^2-n^2}{(s^2+n^2)^2}, \tag{9}$$

$$W(\sin n\omega t) = \int_0^\infty s^2\omega^2 t e^{-s\omega t}\sin(n\omega t)\,\mathrm{d}t = \int_0^\infty s^2\tau e^{-s\tau}\sin(n\tau)d\tau = s^2\frac{2sn}{(s^2+n^2)^2}. \tag{10}$$

As ω-periodic functions $x(\omega t)$ can be expanded into Fourier series, hence one can easy calculate (8) by using (9) and (10).

3. APPLICATION TO NONLINEAR EQUATIONS

For illustration of possible uses of the proposed weighting averaged formula (8) consider the following equation of Duffing oscillator

$$\ddot{z}(t) + \gamma z^3(t) = 0,\ z(0) = a,\ \dot{z}(0) = 0. \tag{11}$$

Let $x(t)$ is a solution of the linear equation

$$\ddot{x}(t) + kx(t) = 0,\ x(0) = a,\ \dot{x}(0) = 0. \tag{12}$$

The equation error is to be

$$e(x(t)) = \gamma x^3(t) - kx(t).$$

The value of k can be determined from a minimum condition, for example,

$$W(e^2(x(t))) \to \min_k \tag{13}$$

where W is the weighting averaging operator (8). Thus, from (13) one gets the approximate frequency of nonlinear response of (11) as follow

$$\omega^2 = k = \gamma\frac{W(x^4(t))}{W(x^2(t))} = \gamma\frac{W(a^4\cos^4(\omega t))}{W(a^2\cos^2(\omega t))}$$
$$= \gamma a^2\frac{\int_0^\infty s^2\omega^2 t e^{-s\omega t}\cos^4(\omega t)\,\mathrm{d}t}{\int_0^\infty s^2\omega^2 t e^{-s\omega t}\cos^2(\omega t)\,\mathrm{d}t} = \gamma a^2\frac{\int_0^\infty s^2\tau e^{-s\tau}\cos^4(\tau)\,\mathrm{d}\tau}{\int_0^\infty s^2\tau e^{-s\tau}\cos^2(\tau)\,\mathrm{d}\tau}. \tag{14}$$

One has

$$\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t,\ \cos^4 t = \frac{3}{8} + \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t. \tag{15}$$

Using (9), (15) one gets from (14)

$$\omega^2 = k = \gamma a^2\frac{A(s)}{B(s)}, \tag{16}$$

where

$$A(s) = \frac{3}{8} + \frac{1}{2}s^2 \frac{s^2 - 4}{(s^2+4)^2} + \frac{1}{8}s^2 \frac{s^2 - 16}{(s^2+16)^2},$$
$$B(s) = \frac{1}{2} + \frac{1}{2}s^2 \frac{s^2 - 4}{(s^2+4)^2}. \quad (17)$$

Substituting $s = n/(2\pi)$ into (16), (17) yields approximate frequencies ω_a as shown in Tab. 1. It is seen that for $n = 0$ (corresponding to $s = 0$) one gets $\omega_a = 0.866\sqrt{\gamma}a$

Table 1. Approximate frequencies of response Eq. (11)

n	ω_a	ω_{exact}
0	$0.866\sqrt{\gamma}a$	$0.847\sqrt{\gamma}a$
1	$0.865\sqrt{\gamma}a$	$0.847\sqrt{\gamma}a$
2	$0.862\sqrt{\gamma}a$	$0.847\sqrt{\gamma}a$
3	$0.856\sqrt{\gamma}a$	$0.847\sqrt{\gamma}a$
4	$0.851\sqrt{\gamma}a$	$0.847\sqrt{\gamma}a$

which is equal to the value obtained by the conventional averaging operator. For $n = 4$, (corresponding to $s = 4/(2\pi)$) the approximate frequency $\omega_a = 0.851\sqrt{\gamma}a$ is much close to the exact one.

4. CONCLUSION

In this short communication the main idea of the dual conception is further extended to suggest a new form for weighting coefficient and then a new averaged value of function. This new averaged value depends on the parameter s and contains the classical averaged value when $s = 0$. In the example of Duffing oscillator it is shown that the parameter s can be chosen as $s = n/(2\pi)$ and for $n = 4$ one gets the solution that is much accurate than the conventional one obtained by the classical criterion of equivalent linearization. On the other hand, the parameter s can be ignored by using the global-local approach as shown in [4]. Full paper is undergoing investigation.

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