HIGHER ORDER STOCHASTIC AVERAGING METHOD
FOR MECHANICAL SYSTEMS HAVING
TWO DEGREES OF FREEDOM

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ABSTRACT. Higher order stochastic averaging method is widely used for investigating single-degree-of-freedom nonlinear systems subjected to white and coloured random noises. In this paper the method is further developed for two-degree-of-freedom systems. An application to a system with cubic damping is considered and the second approximation solution to the Fokker-Planck (FP) equation is obtained.

1 Introduction

The stochastic averaging method was extended by Stratonovich (1963) and has a mathematically rigorous proof by Khasminskii (1963). At present, the stochastic averaging method (SAM) is widely used in different problems of stochastic mechanics, such as vibration, stability and reliability problems (see e.g. Ariaratnam and Tam, 1979; Bolotin, 1984; Roberts and Spanos, 1986; Zhu, 1988). It should be noted that principally only first order SAM has been applied in practice and usually to systems subject to white noise or wideband random processes. However, the effect of some nonlinear terms is lost during the first order averaging procedure. In order to overcome this insufficiency, different averaging procedures for obtaining approximate solutions have been developed (see e.g. Mitropolskii et al, 1992; Red-Horse and Spanos, 1992; Zhu and Lin, 1994; Zhu et al, 1997). Recently, a higher order averaging procedure using FP equation is developed in (Anh, 1993) and then applied to the systems having one degree of freedom under white noise and coloured noise excitations (Anh and Tinh, 1995; Tinh, 1999). In the present paper this procedure is further developed to lightly nonlinear systems subject to white noise excitations. An application to the system with cubic damping is considered.

2 Higher approximate solutions to FP equation

Consider the equations of motion of a mechanical system with two degrees of freedom

\[ \ddot{x}_1 + c_{11}x_1 + c_{12}x_2 = \varepsilon f_{11}(x, \dot{x}) + \varepsilon^2 f_{12}(x, \dot{x}) + \sqrt{\varepsilon} \sigma_1 \dot{\xi}(t), \]
\[ \ddot{x}_2 + c_{21}x_1 + c_{22}x_2 = \varepsilon f_{21}(x, \dot{x}) + \varepsilon^2 f_{22}(x, \dot{x}) + \sqrt{\varepsilon} \sigma_2 \dot{\xi}(t), \]

(2.1)

where \( c_{ij} \ (i, j = 1, 2) \), \( \sigma_1, \sigma_2 \) are constants \( \sigma_1 > 0, \sigma_2 > 0 \), \( \varepsilon \) is a small positive parameter, \( f_{ij} \ (i, j = 1, 2) \) are functions of \( x = (x_1, x_2) \) and \( \dot{x} = (\dot{x}_1, \dot{x}_2) \). The random excitation \( \dot{\xi}(t) \) is a Gaussian white noise process with unit intensity.
Suppose that the characteristic equation of the system (2.1)
\[ D(\lambda) = \begin{vmatrix} c_{11} - \lambda & c_{12} \\ c_{21} & c_{22} - \lambda \end{vmatrix} = 0, \]
has two distinct positive solutions \( \omega_1^2, \omega_2^2 \) \((0 < \omega_1^2 < \omega_2^2)\).

Now we introduce the principal coordinates \( u_1, u_2 \) from the primary coordinates \( x_1, x_2 \) by the relations
\[ x_1 = u_1 + u_2, \]
\[ x_2 = d_1 u_1 + d_2 u_2, \] \( (2.2) \)
where
\[ d_1 = \frac{\omega_1^2 - c_{11}}{c_{12}} = \frac{c_{21}}{\omega_1^2 - c_{22}}, \quad d_2 = \frac{\omega_2^2 - c_{11}}{c_{12}} = \frac{c_{21}}{\omega_2^2 - c_{22}}. \] \( (2.3) \)

Substituting (2.2) into (2.1) we have
\[ \ddot{u}_1 + \omega_1^2 u_1 = \varepsilon F_{11} + \varepsilon^2 F_{12} + \sqrt{\varepsilon} G_1 \dot{\xi}(t), \]
\[ \ddot{u}_2 + \omega_2^2 u_2 = \varepsilon F_{21} + \varepsilon^2 F_{22} + \sqrt{\varepsilon} G_2 \dot{\xi}(t), \] \( (2.4) \)

where
\[ F_{1j} = \frac{f_{2j} - d_2 f_{1j}}{d_1 - d_2}, \quad F_{2j} = \frac{d_1 f_{1j} - f_{2j}}{d_1 - d_2}, \quad G_1 = \frac{\sigma_2 - d_2 \sigma_1}{d_1 - d_2}, \quad G_2 = \frac{d_1 \sigma_1 - \sigma_2}{d_1 - d_2}, \quad (j = 1, 2). \] \( (2.5) \)

According to the averaging method the state coordinates \( (u_1, u_2) \) are to be transformed into the variables \( a = (a_1, a_2) \) and \( \varphi = (\varphi_1, \varphi_2) \) by the change
\[ u_j = a_j \cos \varphi_j, \quad (j = 1, 2) \]
\[ \dot{u}_j = -\omega_j a_j \sin \varphi_j. \] \( (2.6) \)

By using differentiation formula [8] the system of equations (2.4) is transformed into the following system of equations
\[ \dot{a}_j = \varepsilon A_{1j}(a, \varphi) + \varepsilon^2 A_{2j}(a, \varphi) - \sqrt{\varepsilon} G_j \frac{\sin \varphi_j}{\omega_j} \dot{\xi}(t), \]
\[ \dot{\varphi}_j = \omega_j + \varepsilon B_{1j}(a, \varphi) + \varepsilon^2 B_{2j}(a, \varphi) - \sqrt{\varepsilon} G_j \frac{\cos \varphi_j}{a_j \omega_j} \dot{\xi}(t), \quad (j = 1, 2), \] \( (2.7) \)

where it is denoted
\[ A_{1j}(a, \varphi) = -\frac{F_{1j}(a, \varphi)}{\omega_j} \sin \varphi_j + \frac{G_j^2 \cos^2 \varphi_j}{2a_j \omega_j^2}, \]
\[ B_{1j}(a, \varphi) = -\frac{F_{1j}(a, \varphi)}{a_j \omega_j} \cos \varphi_j - \frac{G_j^2 \cos \varphi_j \sin \varphi_j}{a_j \omega_j^2}, \quad (j = 1, 2), \]
\[ A_{2j}(a, \varphi) = -\frac{F_{2j}(a, \varphi)}{\omega_j} \sin \varphi_j, \quad B_{2j}(a, \varphi) = -\frac{F_{2j}(a, \varphi)}{a_j \omega_j} \cos \varphi_j, \] \( (2.8) \)
The Fokker-Planck (FP) equation for the stationary probability density function $W(a, \varphi)$ takes the form

$$\sum_{j=1}^{2} \omega_j \frac{\partial^2 W}{\partial \varphi_j^2} = -\epsilon[A_1, B_1]L[W] - \epsilon^2[A_2, B_2]L[W], \quad (2.9)$$

where the operators $[A_j, B_j]L[, j = 1, 2$ are defined as follows

$$[A_1, B_1]L[W] = \sum_{j=1}^{2} \left[ \frac{\partial}{\partial a_j} (A_{1j}W) + \frac{\partial}{\partial \varphi_j} (B_{1j}W) \right]$$

$$- \sum_{j=1}^{2} \sum_{s=0}^{1} \left\{ \frac{\partial^2}{\partial a_j \partial a_s} \left( \frac{G_j G_s \sin \varphi_j \sin \varphi_s W}{2 \omega_j \omega_s} \right) + \frac{\partial^2}{\partial \varphi_j \partial \varphi_s} \left( \frac{G_j G_s \cos \varphi_j \cos \varphi_s W}{2 \omega_j \omega_s} \right) \right\}, \quad (2.10)$$

$$[A_2, B_2]L[W] = \sum_{j=1}^{2} \left[ \frac{\partial}{\partial a_j} (A_{2j}W) + \frac{\partial}{\partial \varphi_j} (B_{2j}W) \right].$$

We seek the solution of (2.9) in the form

$$W(a, \varphi) = W_0(a, \varphi) + \epsilon W_1(a, \varphi) + \epsilon^2 W_2(a, \varphi) + \cdots \quad (2.11)$$

Substituting (2.11) into (2.9) and comparing the coefficients of like powers of $\epsilon$ we obtain

$$\epsilon^0 : \sum_{j=1}^{2} \omega_j \frac{\partial W_0}{\partial \varphi_j} = 0, \quad (2.12)$$

$$\epsilon^1 : \sum_{j=1}^{2} \omega_j \frac{\partial W_1}{\partial \varphi_j} = -[A_1, B_1]L[W_0], \quad (2.13)$$

$$\epsilon^2 : \sum_{j=1}^{2} \omega_j \frac{\partial W_2}{\partial \varphi_j} = -\{[A_2, B_2]L[W_0] + [A_1, B_1]L[W_1]\}, \quad (2.14)$$

From (2.12) we get

$$W_0 = W_0(a). \quad (2.15)$$

The arbitrary integration function $W_0(a)$ must be chosen from the condition for the function $W_1(a, \varphi)$ to be periodic to $\varphi$.

Thus, we get from (2.13)

$$\langle [A_1, B_1]L[W_0(a)] \rangle = 0, \quad (2.16)$$
where \( \langle . \rangle \) is the averaging operator with respect to \( \varphi \)

\[
\langle . \rangle = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} (.) d\varphi_1 d\varphi_2.
\]  

(2.17)

Substituting (2.10) into (2.16) yields

\[
\sum_{j=1}^{2} \left\{ \frac{\partial}{\partial a_j} \left( (A_{1j}) W_0(a) \right) - \frac{G_j^2}{\omega_j^2} \frac{\partial^2}{\partial a_j^2} W_0(a) \right\} = 0.
\]

(2.18)

The second term \( W_1(a, \varphi) \) in (2.11) is determined from (2.13), using Fourier expansion

\[
[A_1, B_1] L[W_0(a)] = W_0(a) \sum_{k_1} \sum_{k_2} C_{k_1k_2}(a) \exp[i(k_1\varphi_1 + k_2\varphi_2)],
\]

(2.19)

where

\[
C_{k_0k_1}(a) = \frac{1}{i(2\pi)^2 W_0(a)} \int_0^{2\pi} \int_0^{2\pi} \left[ A_1, B_1 \right] L[W_0(a)] \exp[-i(k_0\varphi_0 + k_1\varphi_1)] d\varphi_0 d\varphi_1.
\]

(2.20)

Sustituting (2.19) into (2.13) yields

\[
W_1(a, \varphi) = W_0(a) \left[ W_{10}(a) + \sum_{k_1} \sum_{k_2} \frac{C_{k_1k_2}(a)}{k_1\omega_1 + k_2\omega_2} \exp[i(k_1\varphi_1 + k_2\varphi_2)] \right],
\]

(2.21)

where

\[
k_1\omega_1 + k_2\omega_2 \neq 0.
\]

(2.22)

The arbitrary integration function \( W_{10}(a) \) must be chosen from the condition for the function \( W_2(a, \varphi) \) to be periodic to \( \varphi \). Similarly, we can find the third term \( W_2(a, \varphi) \) in (2.11).

### 3 Application

In order to illustrate the procedure proposed we consider the system with cubic damping whose equation of motion takes the form

\[
\begin{align*}
\dot{x}_1 + c_{11} x_1 - c_{22} x_2 &= -2\varepsilon(h_{11} \dot{x}_1 - h_{12} \dot{x}_2) - \varepsilon^2 \beta \dot{x}_1^2 + \sqrt{\varepsilon} \sigma_1 \xi(t), \\
\dot{x}_2 - c_{22} x_1 + c_{22} x_2 &= -2\varepsilon h_{12} (\dot{x}_2 - \dot{x}_1) + \sqrt{\varepsilon} \sigma_2 \xi(t).
\end{align*}
\]

(3.1)

The physical model of this system is represented in Fig. 1. Where \( k_{11}, k_{12}, h_{11}, h_{12}, \beta, \sigma_1, \sigma_2 \) are positive constants, \( m_1 = m_2 = 1 \) and

\[
\begin{align*}
c_{11} &= k_{11} + k_{12}, & c_{22} &= k_{12}, & h_{11} &= h_1 + h_{12}, \\
c_{11} &= k_{11} + k_{12}, & c_{22} &= k_{12}, & h_{11} &= h_1 + h_{12},
\end{align*}
\]

(3.2)

\[
R_1 = -\varepsilon^2 \beta \dot{x}_1^2 + \sqrt{\varepsilon} \sigma_1 \xi(t), & \quad R_2 = \sqrt{\varepsilon} \sigma_2 \xi(t).
\]

(3.3)
In this case we have

\[ f_{11} = -2h_{11} \ddot{x}_1 + 2h_{12} \ddot{x}_2, \quad f_{12} = -\beta \ddot{x}_1, \]
\[ f_{21} = 2h_{21} \ddot{x}_1 - 2h_{12} \ddot{x}_2, \quad f_{22} = 0, \]
\[ \omega_{1,2}^2 = \frac{c_{11} + c_{22} + \sqrt{(c_{11} - c_{22})^2 + 4c_{22}^2}}{2}, \]
\[ d_{1,2} = \frac{c_{11} - c_{22}}{c_{22}} \pm \frac{\sqrt{\Delta}}{2c_{22}}, \]
\[ d_1 - d_2 = \frac{\sqrt{\Delta}}{c_{22}}, \quad \Delta = (c_{11} - c_{22})^2 + 4c_{22}^2. \]

Substituting (3.3) into (2.5) yields

\[ F_{11} = p_{11} \ddot{x}_1 + p_{12} \ddot{x}_2, \quad F_{12} = p_{13} \ddot{x}_1, \]
\[ F_{21} = p_{21} \ddot{x}_1 + p_{22} \ddot{x}_2, \quad F_{22} = p_{23} \ddot{x}_1, \]

where

\[ p_{11} = \frac{2(h_{12} + d_2 h_{11})}{d_1 - d_2}, \quad p_{12} = -\frac{2h_{12}(1 + d_2)}{d_1 - d_2}, \quad p_{13} = \frac{d_2 \beta}{d_1 - d_2}, \]
\[ p_{21} = \frac{2(d_2 h_{11} + h_{12})}{d_1 - d_2}, \quad p_{22} = \frac{2h_{12}(1 + d_2)}{d_1 - d_2}, \quad p_{23} = -\frac{d_1 \beta}{d_1 - d_2}. \]

From (2.8), using (2.2), (2.6) and (3.4), after calculations we obtain

\[ \langle A_{11} \rangle = \frac{p_{11} + p_{13} d_1}{2} a_1 + \frac{G_1^2}{4\omega_1^2 a_1}, \quad \langle A_{12} \rangle = \frac{p_{21} + p_{23} d_2}{2} a_2 + \frac{G_2^2}{4\omega_2^2 a_2}, \]
\[ \langle A_{21} \rangle = \frac{3p_{13}}{8}(\omega_1^2 a_1^2 + 2\omega_2^2 a_2^2), \quad \langle A_{22} \rangle = \frac{3p_{23}}{8}(2\omega_1^2 a_1^2 a_2 + \omega_2^2 a_2^2). \]
Substituting (3.6) into (2.18) we have

\[ W_0(a) = C a_1 a_2 \exp \left( \frac{\omega_1^2}{G_1}(p_{11} + p_{12}d_1) a_1^2 + \frac{\omega_2^2}{G_2}(p_{21} + p_{22}d_2) a_2^2 \right). \]  

(3.8)

The parameters in the expression of \( W_0(a) \) must satisfy the inequalities

\[ \gamma_1 = \frac{\omega_1^2}{G_1}(p_{11} + p_{12}d_1) < 0, \quad \gamma_2 = \frac{\omega_2^2}{G_2}(p_{21} + p_{22}d_2) < 0. \]  

(3.9)

Substituting (3.8) into (2.13), using (2.6), (2.8) and (3.5), after some calculations we obtain

\[ W_{11}(a, \varphi) = \frac{p_{11} + p_{12}d_1}{2\omega_1} \sin 2\varphi_1 + \frac{p_{21} + p_{22}d_2}{2\omega_2} \sin 2\varphi_2 \]

\[ + \frac{1}{\omega_1^2 - \omega_2^2} (\omega_2 s_{12} a_1 a_2 - \frac{G_1 G_2}{\omega_1 a_1 a_2}) \sin \varphi_1 \cos \varphi_2 \]

\[ + \frac{1}{\omega_1^2 - \omega_2^2} (\frac{G_1 G_2}{\omega_2 a_1 a_2} - \omega_1 s_{12} a_1 a_2) \cos \varphi_1 \sin \varphi_2, \]

(3.10)

where

\[ s_{12} = \frac{2\omega_1 \omega_2}{G_1 G_2} \left[ G_2^2(p_{11} + d_1p_{12})(p_{11} + d_2p_{12}) + G_1^2(p_{21} + d_1p_{22})(p_{21} + d_2p_{22}) \right. \]

\[ - 2\omega_1 \omega_2(p_{11} + d_1p_{12})(p_{21} + d_2p_{22}) \].

(3.11)

Substituting (3.8) and (3.11) into (2.14) we have the equation for the arbitrary function \( W_{10}(a) \) in the form

\[ \sum_{j=1}^{2} \left\{ \frac{\partial}{\partial a_j} [(A_{1j}) W_0(a) W_{10}(a)] - \frac{G_j}{4\omega_j^2} \frac{\partial^2}{\partial a_j^2} [W_0(a) W_{10}(a)] \right\} = - \sum_{j=1}^{2} \frac{\partial}{\partial a_j} [\langle A_{2j} > W_0(a) \rangle]. \]

(3.12)

From (3.12), using (3.6) and (3.7) we have

\[ W_{10}(a) = \alpha_1 a_1^2 + \alpha_2 a_2^2 + \alpha_{12} a_1 a_2 + \alpha_{11} a_1^4 + \alpha_{22} a_2^4, \]

(3.13)

where

\[ \alpha_1 = -\frac{3\omega_1^2 d_1}{2(d_1 - d_2) \gamma_1}, \quad \alpha_2 = -\frac{3\omega_2^2 d_2}{2(d_1 - d_2) \gamma_2}, \]

\[ \alpha_{11} = \frac{3\omega_1^2 d_1(d_1 - d_2)}{8(\sigma_2 - d_2\sigma_1)^2}, \quad \alpha_{22} = -\frac{3\omega_2^2 d_1(d_1 - d_2)}{8(\sigma_2 - d_2\sigma_1)^2}, \]

\[ \alpha_{12} = \frac{3\omega_1^2 \omega_2^2 (d_2 G_2^2 \gamma_1 - d_1 G_1^2 \gamma_1)}{2G_1^2 G_2^2(\gamma_1 + \gamma_2)}. \]

(3.14)

Thus, the second order approximate solution of the FP equation (2.9) for the system (3.1) takes the form

\[ W(a, \varphi) = W_0(a) \{1 + \varepsilon [W_{10}(a) + W_{11}(a, \varphi)]\}. \]

(3.15)
where $W_0(a), W_1(a)$ and $W_{10}(a)$ are defined in (3.8), (3.10) and (3.13), respectively. It is seen from (3.13) and (3.1) that the effect of the nonlinear term $\varepsilon^2 \beta z^3$ is shown in (3.13) and (3.14).

The approximate mean squares $E[x_i^2]$ and $E[x_2^2]$ are to be found

$$E[x_1^2] = \int_0^{2\pi} \int_0^{2\pi} \int_0^{\infty} \int_0^{\infty} x_1^2 W(a, \varphi) da_1 da_2 d\varphi_1 d\varphi_2, \quad (i = 1, 2). \quad (3.16)$$

Substituting (3.15) and $x_i$ in (2.2), (2.6) into (3.16), after calculations we have

$$E[x_1^2] = -\frac{71 + \gamma_2}{\gamma_1 \gamma_2} + \frac{2}{\gamma_1 \gamma_2} \left[ \alpha_1 \gamma_1 \gamma_2 + \alpha_2 \gamma_2 \right] - 3\alpha_1 \gamma_1 \gamma_2 (\gamma_1 + \gamma_2)
+ 2\alpha_1 (\gamma_1^3 + \gamma_2 (\gamma_1 - 2\gamma_1 \gamma_2 - 6\gamma_2^3) + 2\alpha_2 (\gamma_1 + \gamma_2 \gamma_1 - 2\gamma_2 \gamma_1 - 6\gamma_1^3)\),
$$

$$E[x_2^2] = -\frac{d_1^2 \gamma_2 + d_2^2 \gamma_1}{\gamma_1 \gamma_2} + \frac{2}{\gamma_1 \gamma_2} \left[ \alpha_1 d_1 \gamma_1 \gamma_2 + \alpha_2 d_2 \gamma_2 \gamma_1 - 3\alpha_1 \gamma_1 \gamma_2 (d_1^2 \gamma_1 + d_2^2 \gamma_2) \right]
+ 2\alpha_1 (d_1^2 \gamma_1^3 + d_1^3 \gamma_2 - 2d_1 d_2 \gamma_1 \gamma_2 - 6\gamma_2^3) + 2\alpha_2 (d_2^2 \gamma_2^3 + d_2^3 \gamma_1 - 2d_1 d_2 \gamma_1 \gamma_2 - 6\gamma_1^3). \quad (3.17)$$

In the case $\beta = 0$ (linear system) we have

$$E[x_1^2] = -\frac{71 + \gamma_2}{\gamma_1 \gamma_2}, \quad E[x_2^2] = -\frac{d_1^2 \gamma_1 + d_2^2 \gamma_2}{\gamma_1 \gamma_2}. \quad (3.18)$$

4 Conclusion

For many years the stochastic averaging method has been a very useful tool for investigating non-linear vibration systems subject to white noise and coloured noise excitations. In this paper, the higher order stochastic averaging method is applied to non-linear systems with two degrees of freedom subject to white noise excitations. The application to the system with cubic damping is considered and shows the effect of the non-linear term to the mean square response of the system.

Acknowledgement. Support from the Council for natural sciences of Vietnam is gratefully acknowledged.

References


Received May 14, 2003

PHƯƠNG PHÁP TRUNG BÌNH NGẪU NHIÊN B Aç CAO
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Phương pháp trung bình ngẫu nhiên bậc cao đã được áp dụng rộng rãi đối với các hệ dao động phi tuyến một bậc tự do chịu kích động ngẫu nhiên đang ở trạng và ổn định. Trong bài báo này, phương pháp tiếp tục được trình bày đối với các hệ phi tuyến yếu hai bậc tự do chịu kích động ngẫu nhiên đang ở trạng. Sau đó phương pháp được áp dụng để xác định nghiệm xấp xỉ bậc hai của phương trình Fokker-Planck đối với hệ có cần phải tuyến bậc ba.