ON A VARIANT OF THE ASYMPTOTIC PROCEDURE (II: WEAKLY NONLINEAR NON-AUTONOMOUS SYSTEMS)

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ABSTRACT. A variant of the asymptotic method is proposed to construct steady solution of weakly nonlinear non-autonomous oscillating systems. The amplitude and the dephase angle of order \( c^0 \) are used as variables, the uniqueness of the asymptotic expansions is assured by stationarity conditions.

1 Introduction

In [3], to determine steady state in weakly nonlinear autonomous oscillating systems, a variant of the asymptotic procedure has been proposed, consisting of two modifications: - the approximate amplitude \( a \) of order \( c^0 \) of the first harmonic is chosen as variable in asymptotic expansions and - the arbitrariness of the latter is removed by initial conditions and by stationarity conditions.

In the present paper, the case of non-autonomous system is considered. Besides the mentioned amplitude of order \( c^0 \), the dephase angle \( \theta \) of same order is used as the second variable and the additional stationarity conditions are used to assure the uniqueness of the asymptotic expansions. It is shown that steady state (stationary oscillation) can be successively determined in each step of approximation and the solution obtained is identical with that given by the Poincaré method.

2 Systems under consideration. The usual asymptotic procedure

Consider a weakly nonlinear non-autonomous oscillating system described by the differential equation:

\[
\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, \omega t),
\]

where \( \omega \) is the exciting frequency; \( f(x, \dot{x}, \omega t) \) is a function of \((x, \dot{x}, \omega t)\), 2\(\pi\)-periodic with respect to \( \omega t \); the significations of other notations have been explained in [3].

For simplicity, \( f(x, \dot{x}, \omega t) \) is assumed to be a finite Fourier series in \( t \) with polynomial in \((x, \dot{x})\) coefficients.

To be able to make a comparison, the usual asymptotic procedure is briefly recalled [1].

First, following asymptotic expansions are used:

\[
x = a \cos \psi + \varepsilon u_1(a, \theta, \psi) + \varepsilon^2 u_2(a, \theta, \psi) + \ldots, \quad \psi = \omega t + \theta,
\]

\[
\dot{a} = \varepsilon A_1(a, \theta) + \varepsilon^2 A_2(a, \theta) + \ldots,
\]

\[
\dot{\theta} = \varepsilon B_1(a, \theta) + \varepsilon^2 B_2(a, \theta) + \ldots,
\]

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where $a$ and $\theta$ are the full amplitude and the full dephase angle of the first harmonic, respectively; $A_i, B_i$ $(i = 1, 2, \ldots)$ are functions of $(a, \theta)$; $u_i$ $(i = 1, 2, \ldots)$ are functions of $(a, \theta, \psi)$, $2\pi$-periodic with respect to $\psi$, do not containing the first harmonics $\sin \psi, \cos \psi$.

Substituting (1.2) into (1.1), using (1.3), (1.4), expanding the right hand side in Taylor series of $\varepsilon$, equating the terms of like powers of $\varepsilon$ yield in the first approximation

$$-2\omega A_1 \sin \psi - 2\omega a B_1 \cos \psi + \omega^2 \left( \frac{\partial^2 u_1}{\partial \psi^2} + u_1 \right) = f^{(1)}(a, \theta, \psi)$$

\[= f(a \cos \psi, -\omega a \sin \psi, \psi - \theta). \tag{1.5} \]

Expanding $f^{(1)}$ in Fourier series, that is:

$$f^{(1)}(a, \theta, \psi) = f^{(0)}_2(a, \theta) + \sum_{n=1}^{N_1} \left[ S_n^{(1)}(a, \theta) \sin n\psi + C_n^{(1)}(a, \theta) \cos n\psi \right], \tag{1.6}$$

then equating the terms of like harmonics, we obtain:

$$A_1(a, \theta) = -\frac{1}{2\omega} S_1^{(1)}(a, \theta), \quad B_1(a, \theta) = -\frac{1}{2\omega} C_1^{(1)}(a, \theta), \tag{1.7}$$

$$\omega^2 \left( \frac{\partial^2 u_1}{\partial \psi^2} + u_1 \right) = f^{(0)}_2(a, \theta) + \sum_{n=2}^{N_1} \left[ S_n^{(1)}(a, \theta) \sin n\psi + C_n^{(1)}(a, \theta) \cos n\psi \right]. \tag{1.8}$$

The expression of $u_1$ is of the form:

$$u_i(a, \theta, \psi) = \frac{1}{\omega^2} \left\{ f^{(0)}_2(a, \theta) - \sum_{n=2}^{N_1} \frac{1}{n^2 - 1} \left[ S_n^{(i)}(a, \theta) \sin n\psi + C_n^{(i)}(a, \theta) \cos n\psi \right] \right\}. \tag{1.9}$$

The same procedure leads to $A_i, B_i, u_i$ in the $i$th approximation $(i = 2, 3, \ldots)$.

In the end i.e. in the $n$th approximation, the full amplitude $\alpha_*$ and the full dephase angle $\theta_*$ of stationary oscillation are determined by stationarity conditions which are two following equations:

$$A(a, \theta) = \varepsilon A_1(a, \theta) + \cdots + \varepsilon^n A_n(a, \theta) = 0,$$

$$B(a, \theta) = \varepsilon B_1(a, \theta) + \cdots + \varepsilon^n B_n(a, \theta) = 0. \tag{1.10}$$

The stability conditions take the form:

$$\left( \frac{\partial A}{\partial a} + \frac{\partial B}{\partial \theta} \right)_* < 0, \quad \left( \frac{\partial A}{\partial a} \frac{\partial B}{\partial \theta} - \frac{\partial A}{\partial \theta} \frac{\partial B}{\partial a} \right)_* > 0. \tag{1.11}$$

3 A variant of the asymptotic procedure

In this section, a variant of the asymptotic procedure is applied to discuss the problem of interest.

First, in the expansions (1.2), (1.3), (1.4), the variables $a$ and $\theta$ are now understood as the amplitude and the dephase angle of order $\varepsilon^0$ (not the full amplitude and the full dephase angle) of the first harmonic. Consequently, the requirement on the absence of the first harmonics $\sin \psi, \cos \psi$ in $u_i$ $(i = 1, 2, 3, \ldots)$ must be rejected i.e. $u_i$ may contain the sum $(a_i \cos \psi + b_i \sin \psi)$ where $a_i, b_i$ are constants to be chosen.
The identity (1.5) retains its form; so, from (1.5) we obtain the same expressions (1.7) of \( A_1(a, \theta) \), \( B_1(a, \theta) \) and the same differential equation (1.8) governing \( u_1 \).

However, since \( u_1(a, \theta, \psi) \) may contain the first harmonics \( \sin \psi \) and \( \cos \psi \), the expression (1.9) - is replaced by

\[
\begin{align*}
    u_1(a, \theta, \psi) &= \frac{1}{\omega^2} \left\{ f_0^{(1)}(a, \theta) - \sum_{n=2}^{N_1} \frac{1}{n^2 - 1} \left[ S_n^{(1)}(a, \theta) \sin n\psi \right. \right. \\
    &\left. \left. \quad + C_n^{(1)}(a, \theta) \cos n\psi \right] \right\} + a_1 \cos \psi + b_1 \sin \psi
\end{align*}
\]  

(2.1)

where \( a_1 \), \( b_1 \) are constants to be chosen.

The amplitude and the dephase angle \((a_0, \theta_0)\) of order \( \varepsilon^0 \) of stationary oscillation are immediately determined in the first approximation by the equations

\[
A_1(a, \theta) = \frac{-1}{2\omega} S_1^{(1)}(a, \theta) = 0, \quad B_1(a, \theta) = \frac{-1}{2\omega a} C_1^{(1)}(a, \theta) = 0.
\]  

(2.2)

In the second approximation we have

\[
\begin{align*}
-2\omega A_2 \sin \psi - 2\omega a B_2 \cos \psi + \omega^2 \left( \frac{\partial^2 u_2}{\partial \psi^2} + u_2 \right) &= f_2(a, \theta, \psi) \\
\frac{\partial A_2}{\partial a} - A_2 \frac{\partial A_1}{\partial \theta} - B_2 \frac{\partial B_1}{\partial \theta} &= \cos \psi \left( 2A_1 B_2 + a A_1 \frac{\partial A_1}{\partial a} + a B_1 \frac{\partial B_1}{\partial \theta} \right) \sin \psi \\
-2\omega A_2 \frac{\partial^2 u_1}{\partial \psi \partial a} - 2\omega A_2 \frac{\partial^2 u_1}{\partial \psi \partial \theta} - 2\omega B_2 \frac{\partial^2 u_1}{\partial \psi^2} + u_1 f_x + \left( A_2 \cos \psi - a B_2 \sin \psi + \frac{\omega^2}{2} \frac{\partial u_1}{\partial \psi} \right) f_x \\
&= \left\{ f_0^{(2)}(a, \theta) + a_1 f_0'(a, \theta) + b_1 f_0''(a, \theta) \right\} \\
&+ \sum_{n=1}^{N_2} \left\{ \left[ S_n^{(2)}(a, \theta) + a_1 S_n'(a, \theta) + b_1 S_n''(a, \theta) \right] \sin n\psi \\
&\quad + \left[ C_n^{(2)}(a, \theta) + a_1 C_n'(a, \theta) + b_1 C_n''(a, \theta) \right] \cos n\psi \right\}.
\end{align*}
\]  

(2.3)

Equating the terms of like harmonics, we get:

\[
\begin{align*}
A_2(a, \theta) &= \frac{-1}{\omega} \left\{ S_1^{(2)}(a, \theta) + a_1 S_1'(a, \theta) + b_1 S_1''(a, \theta) \right\}, \\
B_2(a, \theta) &= \frac{-1}{\omega a} \left\{ C_1^{(2)}(a, \theta) + a_1 C_1'(a, \theta) + b_1 C_1''(a, \theta) \right\},
\end{align*}
\]  

(2.4)

\[
\begin{align*}
\omega^2 \left( \frac{\partial^2 u_2}{\partial \psi^2} + u_2 \right) &= \left\{ f_0^{(2)}(a, \theta) + a_1 f_0'(a, \theta) + b_1 f_0''(a, \theta) \right\} \\
&+ \sum_{n=2}^{N_2} \left\{ \left[ S_n^{(2)}(a, \theta) + a_1 S_n'(a, \theta) + b_1 S_n''(a, \theta) \right] \sin n\psi \\
&\quad + \left[ C_n^{(2)}(a, \theta) + a_1 C_n'(a, \theta) + b_1 C_n''(a, \theta) \right] \cos n\psi \right\}.
\end{align*}
\]  

(2.5)
Note that $A_2$, $B_2$ as $u_2$ remain indeterminate because of the presence of $a_1$, $b_1$. A question arises: how to choose $a_1$ and $b_1$? It is natural to impose on $A_2$, $B_2$ the conditions

$$A_2(a_0, \theta_0) = 0, \quad B_2(a_0, \theta_0) = 0$$

which are called stationarity conditions (not $\epsilon A_1 + \epsilon^2 A_2 = 0$, $\epsilon B_1 + \epsilon^2 B_2 = 0$ and not $a$, $b$ but $a = a_0$, $b = b_0$). In detailed form:

$$a_1 S'_1(a_0, \theta_0) + b_1 S''_1(a_0, \theta_0) + S^{(2)}_1(a_0, \theta_0) = 0,$$

$$a_1 C'_1(a_0, \theta_0) + b_1 C''_1(a_0, \theta_0) + C^{(2)}_1(a_0, \theta_0) = 0.$$  

Assuming that the determinant

$$D = (S'_1 C''_1 - S''_1 C'_1)_0 \neq 0,$$  

we obtain:

$$a_1 = a_{10} = D_1/D, \quad b_1 = b_{10} = D_2/D,$$  

where

$$D_1 = (S^{(2)}_1 C'_1 - S'_1 C''_1)_0, \quad D_2 = (S^{(2)}_1 C''_1 - S''_1 C'_1)_0.$$  

Now, with $a_1$, $b_1$ given by (2.8), the expressions of $A_2$, $B_2$ and that of $u_1$ are fully determined.

The expression of $u_2(a, \theta, \psi)$ is:

$$u_2(a, \theta, \psi) = \frac{1}{\omega^2} \left\{ \int_0^{N_2} (a, \theta) + a_{10} f'_0(a, \theta) + b_{10} f''_0(a, \theta) \right\} + a_2 \cos \psi + b_2 \sin \psi$$

$$- \frac{1}{\omega^2} \sum_{n=1}^{N_2} \frac{1}{n^2 - 1} \left\{ \left[ S'^{(2)}_n(a, \theta) + a_{10} S'_n(a, \theta) + b_{10} S''_n(a, \theta) \right] \sin n\psi + \left[ C'^{(2)}_n(a, \theta) + a_{10} C'_n(a, \theta) + b_{10} C''_n(a, \theta) \right] \cos n\psi \right\},$$

where $a_2$, $b_2$ are two constants to be chosen.

In the third approximation, we determine $a_2$, $b_2$ using the stationarity conditions

$$A_3(a_0, \theta_0) = 0, \quad B_3(a_0, \theta_0) = 0$$

(not $\epsilon A_1 + \epsilon^2 A_2 + \epsilon^3 A_3 = 0$, $\epsilon B_1 + \epsilon^2 B_2 + \epsilon^3 B_3 = 0$ and not $a_1\theta$ but $a_0$, $\theta_0$).

Continuing this procedure, in the $n$th approximation, we determine $a_{n-1}$, $b_{n-1}$ using the stationary conditions

$$A_n(a_0, \theta_0) = 0, \quad B_n(a_0, \theta_0) = 0.$$  

It is necessary to insist again that stationarity conditions are not expressed by two equations as in the usual asymptotic procedure but by a set of equations

$$A_1(a_0, \theta_0) = 0, \quad B_1(a_0, \theta_0) = 0, \quad A_2(a_0, \theta_0) = 0, \quad B_2(a_0, \theta_0) = 0,$$

$$\ldots \quad \ldots$$

$$A_n(a_0, \theta_0) = 0, \quad B_n(a_0, \theta_0) = 0.$$  

Also note that the expressions of $A_i(a, \theta)$, $B_i(a, \theta)$ ($i = 2, 3, \ldots, n$) differ from the corresponding ones in the usual asymptotic procedure.

The stability conditions take the same formal form (1.11) given on Section 1, $(\alpha_*, \theta_*)$ must be replaced by $(a_0, \theta_0)$.  

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4 Comparison with the Poincaré method

As in [3], let us compare the variant presented in Section 2 with the Poincaré method. Following the latter, the dimensionless time \( \tau = \omega t \) is introduced and the differential equation (1.1) is written as:

\[
\omega^2 (x'' + x) = \varepsilon f(x, \omega x', \tau),
\]

where primes denote derivation with respect to \( \tau \).

Expanding \( x(\tau) \) in powers of \( \varepsilon \), that is:

\[
x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \ldots,
\]

then equating the terms of like powers of \( \varepsilon \) in both sides of (3.1) yield:

\[
\begin{align*}
\omega^2 (x_0'' + x_0) &= 0, \\
\omega^2 (x_1'' + x_1) &= f(x_0, \omega x_0', \tau), \\
\omega^2 (x_2'' + x - 2) &= x_1 f_{x_2}(x_0, \omega x_0', \tau) + \omega x_1' f_{x_2}(x_0, \omega x_0', \tau),
\end{align*}
\]

The general solution of (3.3) takes the form

\[
x_0 = a_0 \cos(\tau + \theta_0),
\]

where \( a_0, \theta_0 \) are two constants to be determined.

With regard to (3.6), the differential equations (3.4), (3.5) become:

\[
\begin{align*}
\omega^2 (x_1'' + x_1) &= f(a_0 \cos(\tau + \theta_0), -\omega a_0 \sin(\tau + \theta - 0), \tau), \\
\omega^2 (x_2'' + x_2) &= x_1 f_{x_2}(a_0 \cos(\tau + \theta_0), -\omega a_0 \sin(\tau + \theta_0), \tau) \\
&+ \omega x_1' f_{x_2}(a_0 \cos(\tau + \theta_0), -\omega a_0 \sin(\tau + \theta_0), \tau),
\end{align*}
\]

Except the absence of \( A_1, B_1, A_2, B_2 \), the difference between (3.7), (3.8) and (1.5), (2.2) is of formal character and consists only in the difference between the notations \( \tau = \omega t + \theta_0 \rightarrow \psi, \tau \rightarrow \psi - \theta_0, x_1 \rightarrow u_1, x_2 \rightarrow u_2, (\cdot') = \frac{\partial}{\partial \tau} \rightarrow \frac{\partial}{\partial \psi} \). However, for stationary oscillation \( A_1 = B_1 = A_2 = B_2 = 0 \). So, stationary oscillation obtained in Section 2 by the variant of the asymptotic procedure coincides with that determined by (3.7) by the Poincaré method.

5 Example

As an illustration, let us consider the system:

\[
\ddot{x} + x = \varepsilon \{h(1 - x^2)\dot{x} + e \cos t\}, \quad h = 0.04, \ e = 0.03.
\]

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In the first approximation, we have:

\[-2A_1 \sin \psi - 2aB_1 \cos \psi + \left( \frac{\partial^2 u_1}{\partial \psi^2} + u_1 \right) = f^{(1)}(a, \theta, \psi) = \left\{ \epsilon \sin \theta - ha \left(1 - \frac{a^2}{4}\right) \right\} \sin \psi + \epsilon \cos \theta \cos \psi + \frac{1}{4} ha^3 \sin 3\psi \]  

(4.2)

from which we obtain:

\[A_1(a, \theta) = -\frac{1}{2} \left( \epsilon \sin \theta - ha \left(1 - \frac{a^2}{4}\right) \right), \quad B_1(a, \theta) = -\frac{\epsilon}{2a} \cos \theta, \]  

(4.3)

\[\frac{\partial^2 u_1}{\partial \psi^2} + u_1 = \frac{1}{4} ha^3 \sin 3\psi, \quad u_1(a, \theta, \psi) = -\frac{ha^3}{32} \sin 3\psi + a_1 \cos \psi + b_1 \sin \psi. \]  

(4.4)

By solving the equations

\[A_1(a, \theta) = 0, \quad B_1(a, \theta) = 0, \]  

(4.5)

three stationary oscillations are determined

\[\theta_{01} = -\frac{\pi}{2}, \quad a_{01} \approx 2.3028; \]
\[\theta_{02} = \frac{\pi}{2}, \quad a_{02} = 1; \quad \theta_{03} = \frac{\pi}{2}, \quad a_{03} \approx 1.3028. \]  

(4.6)

(4.7)

In the second approximation, we have to examine the identity (2.3); for the example considered, it has the form:

\[\ldots + \left( \frac{\partial^2 u_2}{\partial \psi^2} + u_2 \right) = \ldots + u_1 f_2 + \frac{\partial u_1}{\partial \psi} f_2 = \]

\[= \left( -\frac{ha^3}{32} \sin 3\psi + a_1 \cos \psi + b_1 \sin \psi \right) \left(2ha^2 \cos \psi \sin \psi \right) \]

\[+ \left( -\frac{3ha^3}{32} \cos 3\psi - a_1 \sin \psi + b_1 \cos \psi \right) h \left(1 - a^2 \cos^2 \psi \right) \]

\[= \left[ \frac{h^2 a^5}{128} + \frac{b_1 h}{2} \left(1 - \frac{a^2}{2}\right) \right] \cos \psi + a_1 \frac{h^2}{2} \left( \frac{3a^2}{2} - 1 \right) \sin \psi + \text{(higher harmonics)}, \]  

(4.8)

where non-written terms do not contain \(A_1, B_1, A_2, B_2\).

By substituting \(a = a_0, \theta = \theta_0\) then by vanishing the coefficient of the first harmonics \(\cos \psi\) and \(\sin \psi\) in (4.8) we obtain:

\[a_{10} = 0, \quad b_{19} = -ha_0^5/64 \left(1 - \frac{a_0^2}{2}\right). \]  

(4.9)

The stability conditions are

\[\epsilon \left( \frac{\partial A_1}{\partial a} + \frac{\partial B_1}{\partial \theta} \right)_0 + \epsilon^2 = \epsilon \left( \frac{h}{2} \left(1 - \frac{3a_0^2}{4}\right) + \frac{e}{2a_0} \sin \theta_0 \right) + \ldots < 0, \]

\[\epsilon^2 \left( \frac{\partial A_1}{\partial a} \frac{\partial B_1}{\partial \theta} - \frac{\partial A_1}{\partial \theta} \frac{\partial B_1}{\partial a} + \epsilon^3 \ldots \right)_0 = \epsilon^2 \left( \frac{he}{4a_0} \left(1 - \frac{3a_0^2}{4}\right) \sin \theta_0 + \frac{e^2}{4a_0^2} \cos^2 \theta_0 \right) + \epsilon^3 \ldots > 0. \]  

(4.10)

It is easy to verify that the stationary oscillation (3.14) is stable and other two stationary oscillations (3.15) are unstable.
6 Conclusion

A variant of the asymptotic procedure is proposed for weakly non-linear non autonomous systems. It differs from the usual one by two modifications: 1) the use of the amplitude and the dephase angle order \( \varepsilon^0 \) as variables in asymptotic expansion, 2) the use of the stationarity condition in each step of approximation. The formulas obtained are identical with those given the well-known Poincaré method.

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