NON-LINEAR VIBRATION OF ECCENTRICALLY STIFFENED LAMINATED COMPOSITE SHELLS

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Abstract. The present paper deals with a non-linear vibration of eccentrically stiffened laminated composite doubly curved shallow shells. The calculations of internal forces and displacements of the shell are based upon the thin shell theory considering the geometrical non-linearity and the Lekhnitsky’s smeared stiffeners technique. From the deformation compatibility equation and the motion equation a system of partial differential equations for stress function and deflection of shell is obtained. The Bubnov-Galerkin’s method and iterative procedure in conjunction with Newmark constant acceleration scheme are used for dynamical analysis of shells to give the frequency-amplitude relation of free non-linear vibration and non-linear transient responses. Numerical results show the influence of boundary conditions and Gauss curvature on the non-linear vibration of shells.

1. INTRODUCTION

Reinforced laminated structures like plates and shallow shells are widely used in air-industry and ship-industry. The stiffening member provides the benefit of added load-carrying static and dynamic capability. When structures subjected to external loads may be appear a large deflection then the geometrical non-linearity of shell must be considered; of course it meets with mathematics difficulty. To solve problem we are concerned with two aspects: to seek an approximated analytical solution which allows to investigate the motion characteristics and to seek solution by numerical methods. The research results for nonlinear vibration of composite plates have been represented in [4, 11] and for cylindrical shells in [2, 5, 7, 8]. Approximated analytical solutions for the vibration problem of doubly curved unstiffened composite shells were given in [1, 6, 9, 10]. In [3] the authors carried out the non-linear dynamic analysis of doubly curved stiffened composite shells by the displacement approach.

The aim of this paper is to search an approximated analytical solution for the dynamic problem of doubly curved eccentrically stiffened laminated shells with negative and positive Gauss curvature and different boundary conditions by using stress function and Bubnov-Galerkin methods to give non-linear vibration equations of shell. Numerical solutions are given by the iterative method and Newmark constant acceleration scheme. The frequency-amplitude relation in the non-linear free vibration and the influence of boundary conditions and Gauss curvature on the solution of dynamic problem of shells are examined.
2. GOVERNING EQUATIONS

Consider a symmetrically laminated composite doubly curved shallow shells of thickness $h$ and in-plane edges $a$ and $b$. The shell is reinforced by eccentrically longitudinal and transversal composite stiffeners and subjected to the transverse load of intensity $q(x_1, x_2, t)$.

Using Kirchoff-Love theory non-linear strain-displacement relations for doubly curved shallow shells are formulated

\[
\begin{align*}
\varepsilon_1^o &= \frac{\partial u}{\partial x_1} - k_1 w + \frac{1}{2} \left( \frac{\partial w}{\partial x_1} \right)^2; \quad \phi_1 = -\frac{\partial^2 w}{\partial x_1^2}; \\
\varepsilon_2^o &= \frac{\partial u}{\partial x_2} - k_2 w + \frac{1}{2} \left( \frac{\partial w}{\partial x_2} \right)^2; \quad \phi_2 = -\frac{\partial^2 w}{\partial x_2^2}; \\
\varepsilon_6^o &= \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} + \frac{\partial w}{\partial x_1} \cdot \frac{\partial w}{\partial x_2}; \quad \phi_6 = -2 \frac{\partial^2 w}{\partial x_1 \partial x_2},
\end{align*}
\]

where $k_1 = \frac{1}{R_1}, k_2 = \frac{1}{R_2}$ are principal curvatures of the shell; $R_1, R_2$ are radii of curvature; $u, v, w$ are displacements of the midle surface point along $x_1, x_2, x_3 \equiv z$ directions respectively; $\varepsilon_i^o$ and $\phi_i (i = 1, 2, 6)$ are strain of the midle surface point and curvature variations satisfying the deformation compatibility equation:

\[
\frac{\partial^2 \varepsilon_1^o}{\partial x_2^2} + \frac{\partial^2 \varepsilon_2^o}{\partial x_1^2} - \frac{\partial^2 \varepsilon_6^o}{\partial x_1 \partial x_2} = \left( \frac{\partial^2 w}{\partial x_1 \partial x_2} \right)^2 - \frac{\partial^2 w}{\partial x_1^2} \cdot \frac{\partial^2 w}{\partial x_2^2} - k_1^2 \frac{\partial^2 w}{\partial x_1^2} - k_2^2 \frac{\partial^2 w}{\partial x_2^2}. \tag{2}
\]

Internal forces of an unstiffened composite shell are calculated by Reddy [10]. Note that in a symmetrically laminated shell the coupling stiffnesses $B_{ij}$ are equal to zero and the extensional $A_{16}, A_{26}$ and bending $D_{16}, D_{26}$ stiffnesses are negligible compared to the other stiffnesses. Using Lekhnitsky’s smeared stiffeners technique we get governing internal force resultants and moments as follows

\[
\begin{align*}
N_1 &= \left( A_{11} + \frac{EA_1}{s_1} \right) \varepsilon_1^o + A_{12} \varepsilon_2^o - \frac{EA_1 z_1}{s_1} \cdot \frac{\partial^2 w}{\partial x_1^2}, \\
N_2 &= A_{12} \varepsilon_1^o + \left( A_{22} + \frac{EA_2}{s_2} \right) \varepsilon_2^o - \frac{EA_2 z_2}{s_2} \cdot \frac{\partial^2 w}{\partial x_2^2}, \tag{3}
N_6 &= A_{66} \varepsilon_6^o, \\
M_1 &= -\left( D_{11} + \frac{EI_1}{s_1} \right) \frac{\partial^2 w}{\partial x_1^2} - D_{12} \frac{\partial^2 w}{\partial x_2^2} + \frac{EA_1 z_1}{s_1} \varepsilon_1^o, \\
M_2 &= -D_{12} \frac{\partial^2 w}{\partial x_1^2} \left( D_{22} + \frac{EI_2}{s_2} \right) \frac{\partial^2 w}{\partial x_2^2} + \frac{EA_2 z_2}{s_2} \varepsilon_2^o, \\
M_6 &= -2D_{66} \frac{\partial^2 w}{\partial x_1 \partial x_2},
\end{align*}
\]

where

\[
(A_{ij}, D_{ij}) = \sum_{k=1}^{n_1} \int_{h_{k-1}}^{h_k} Q_{ij}^{(k)} (1, z^2) \, dz, \quad (i, j = 1, 2, 6)
\]
are stiffnesses of unreinforced composite shells, \(Q^{(k)}_{ij}\) are transformed stiffnesses of \(k^{th}\)-layer, 
\(E\) - elastic modulus of stiffeners. Spacings of the longitudinal and transversal stiffeners are denoted by \(s_1, s_2\). Values \(z_1, z_2\) - eccentricities of the longitudinal and transversal stiffeners with respect to the middle surface of the shell, \(A_1, A_2\) are cross section areas of the stiffeners and \(I_1, I_2\) are inertia moments of stiffener cross sections.

The reverse relations are obtained from equations (3)

\[
\varepsilon_1^0 = A_{22}^* N_1 - A_{12}^* N_2 - A_{12}^* \frac{E A_{22} z_2}{s_2} \frac{\partial^2 w}{\partial x_2^2} + A_{22}^* \frac{E A_{11} z_1}{s_1} \frac{\partial^2 w}{\partial x_1^2},
\]

\[
\varepsilon_2^0 = A_{11}^* N_2 - A_{12}^* N_1 - A_{12}^* \frac{E A_{11} z_1}{s_1} \frac{\partial^2 w}{\partial x_2^2} + A_{11}^* \frac{E A_{22} z_2}{s_2} \frac{\partial^2 w}{\partial x_1^2},
\]

\[
\varepsilon_6^0 = A_{66}^* N_6,
\]

where:

\[
A_{11}^* = \frac{1}{\Delta} \left( A_{11} + \frac{E A_1}{s_1} \right), \quad A_{22}^* = \frac{1}{\Delta} \left( A_{22} + \frac{E A_2}{s_2} \right),
\]

\[
A_{12}^* = \frac{A_{12}}{\Delta}, \quad A_{66}^* = \frac{1}{\Delta}, \quad \Delta = \left( A_{11} + \frac{E A_1}{s_1} \right) \left( A_{22} + \frac{E A_2}{s_2} \right) - A_{12}^2.
\]

Inserting (5) into (4) yields

\[
M_1 = - \left( D_{11}^* \frac{\partial^2 w}{\partial x_1^2} + D_{12}^* \frac{\partial^2 w}{\partial x_2^2} \right) + A_{22}^* \frac{E A_{11} z_1}{s_1} N_1 - A_{12}^* \frac{E A_{11} z_1}{s_1} N_2,
\]

\[
M_2 = - \left( D_{12}^* \frac{\partial^2 w}{\partial x_1^2} + D_{22}^* \frac{\partial^2 w}{\partial x_2^2} \right) + A_{11}^* \frac{E A_{22} z_2}{s_2} N_2 - A_{12}^* \frac{E A_{22} z_2}{s_2} N_1,
\]

\[
M_6 = -2 D_{66}^* \frac{\partial^2 w}{\partial x_1 \partial x_2},
\]

where:

\[
D_{11}^* = D_{11} + \frac{E I_1}{s_1} + A_{22}^* \left( \frac{E A_{11} z_1}{s_1} \right)^2,
\]

\[
D_{22}^* = D_{22} + \frac{E I_2}{s_2} + A_{11}^* \left( \frac{E A_{22} z_2}{s_2} \right)^2,
\]

\[
D_{12}^* = D_{12} + A_{12}^* \frac{E^2 A_1 A_2 z_1 z_2}{s_1 s_2}, \quad D_{66}^* = D_{66},
\]

\[
D_{33}^* = D_{12}^* + 2 D_{66}^* = D_{12} + 2 D_{66} + A_{12}^* \frac{E^2 A_1 A_2 z_1 z_2}{s_1 s_2}
\]

Because the load is perpendicular to the middle surface, then following Volmir [12] the wave propagation in middle surface of the shell can be overpass, it reduces to overlook inertial forces along \(x_1\) and \(x_2\) directions. The motion equations of doubly curved shallow composite shell are of the form

\[
\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = 0,
\]

\[
\frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} = 0.
\]
\[
\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} + N_1 \frac{\partial^2 w}{\partial x_1^4} + 2N_6 \frac{\partial^2 w}{\partial x_1 \partial x_2} + N_2 \frac{\partial^2 w}{\partial x_2^2} \\
+ k_1 N_1 + k_2 N_2 + q = J_0 \frac{\partial^2 w}{\partial t^2} + J_2 \left( \frac{\partial^4 w}{\partial x_1^2 \partial t^2} + \frac{\partial^4 w}{\partial x_2^2 \partial t^2} \right).
\]

(11)

Introduce the stress function

\[
N_1 = h_\sigma_{11} = \frac{\partial^2 \varphi}{\partial x_1^2}, \quad N_2 = h_\sigma_{22} = \frac{\partial^2 \varphi}{\partial x_2^2}, \quad N_{12} = h_\sigma_{12} = -\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}
\]

(12)

to equations (9) and (10) which are satisfied identically.

Substituting (5) into the compatibility equation (2), and (7) into (11), the system of
two motion equations for stress function \( \varphi \) and deflection \( w \) are obtained

\[
A_{11}^* \frac{\partial^4 \varphi}{\partial x_1^4} + \left( A_{66}^* - 2A_{12}^* \right) \frac{\partial^4 \varphi}{\partial x_1^2 \partial x_2^2} + A_{22}^* \frac{\partial^4 \varphi}{\partial x_2^4} - \left[ A_{12}^* \frac{EA_{11} z_1}{s_1} \frac{\partial^4 w}{\partial x_1^4} + A_{22}^* \frac{EA_{22} z_2}{s_2} \frac{\partial^4 w}{\partial x_2^4} \right] \frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \left( \frac{\partial^2 w}{\partial x_1 \partial x_2} \right)^2 \]

(13)

\[
- \frac{\partial^2 w}{\partial x_1^2} \cdot \frac{\partial^2 w}{\partial x_2^2} - k_1 \frac{\partial^2 w}{\partial x_1^2} - k_2 \frac{\partial^2 w}{\partial x_2^2} = 0;
\]

\[
D_{11}^* \frac{\partial^4 w}{\partial x_1^4} + 2D_{33}^* \frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + D_{22}^* \frac{\partial^4 w}{\partial x_2^4} - \left( A_{11}^* \frac{EA_{22} z_1}{s_2} + A_{22}^* \frac{EA_{11} z_2}{s_1} \right) \frac{\partial^4 \varphi}{\partial x_1^2 \partial x_2^2} + A_{11}^* \frac{EA_{11} z_1}{s_1} \frac{\partial^4 \varphi}{\partial x_1^4} + A_{12}^* \frac{EA_{22} z_2}{s_2} \frac{\partial^4 \varphi}{\partial x_2^4} - k_1 \frac{\partial^2 \varphi}{\partial x_1^2} - k_2 \frac{\partial^2 \varphi}{\partial x_2^2} + 2 \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \cdot \frac{\partial^2 w}{\partial x_1 \partial x_2} \]

(14)

\[
- \frac{\partial^2 \varphi}{\partial x_1^2} \cdot \frac{\partial^2 \varphi}{\partial x_2^2} - \frac{\partial^2 \varphi}{\partial x_1^2} \cdot \frac{\partial^2 \varphi}{\partial x_1^2} - \frac{\partial^2 \varphi}{\partial x_1^2} \cdot \frac{\partial^2 \varphi}{\partial x_2^2} - q + J_0 \frac{\partial^2 w}{\partial t^2} - J_2 \left( \frac{\partial^4 w}{\partial x_1^2 \partial t^2} + \frac{\partial^4 w}{\partial x_2^2 \partial t^2} \right) = 0.
\]

3. VIBRATION OF THE SHELL WITH SIMPLY SUPPORTED EDGES

Let consider a simply supported shell at all edges, then the boundary conditions are

\[
w = 0, \quad M_1 = 0, \quad N_1 = 0, \quad N_6 = 0, \quad a; \quad w = 0, \quad M_2 = 0, \quad N_2 = 0, \quad N_6 = 0, \quad b.
\]

(15)

Solutions of stress function and deflection are chosen as

\[
\varphi = g(t) \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b},
\]

\[
w = f(t) \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b}.
\]

(16)

Substitution of (16) into (1), (3), (7) shows that boundary conditions (15) are satisfied.

Using expressions (16) to equations (13), (14) and Bubnov-Galerkin method leads to

\[
\left[ A_{11} \left( \frac{m\pi}{a} \right)^4 + \left( A_{66} - 2A_{12} \right) \left( \frac{mn\pi^2}{ab} \right)^2 + A_{22} \left( \frac{n\pi}{b} \right)^4 \right] \frac{ab}{4} g(t) = \left[ B_{11} \left( \frac{m\pi}{a} \right)^4 \right]
\]

\[
- B_{12} \left( \frac{mn\pi^2}{ab} \right)^2 + B_{22} \left( \frac{n\pi}{b} \right)^4 + k_1 \left( \frac{n\pi}{b} \right)^2 + k_2 \left( \frac{m\pi}{a} \right)^2 \right] \frac{ab}{4} f(t) - \frac{4mn\pi^2}{3ab} f^2(t)
\]

(17)
and
\[
\begin{align*}
&D_{11}^* \left( \frac{m\pi}{a} \right)^4 + 2D_{33}^* \left( \frac{mn\pi^2}{ab} \right)^2 + D_{22}^* \left( \frac{n\pi}{b} \right)^4 \frac{ab}{4} f(t) \\
&+ B_{11}^* \left( \frac{m\pi}{a} \right)^4 - B_{12}^* \left( \frac{mn\pi^2}{ab} \right)^2 + B_{22}^* \left( \frac{n\pi}{b} \right)^4 + k_1 \left( \frac{n\pi}{b} \right)^2 + k_2 \left( \frac{m\pi}{a} \right)^2 \frac{ab}{4} g(t) \\
&- \frac{8mn\pi^2}{3ab} f(t) g(t) - q \frac{4ab}{mn\pi^2} + \left[ J_0 + J_2 \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \right] \frac{ab}{4} \frac{d^2 f}{dt^2} = 0,
\end{align*}
\]
where denote the coefficients
\[
B_{11}^* = A_{12}^* \frac{EA_{12}z_1}{s_1}, \quad B_{22}^* = A_{12}^* \frac{EA_{22}z_1}{s_2}, \quad B_{12}^* = A_{11}^* \frac{EA_{22}z_2}{s_2} + A_{22}^* \frac{EA_{12}z_1}{s_1},
\]
with \(m, n\) are odd numbers.

**Remark:** When a shell is reinforced by centrically stiffeners then \(B_{ij}^* = 0\); a plate is reinforced by eccentrically stiffeners then \(k_1 = k_2 = 0, B_{ij}^* \neq 0\); a plate is reinforced plate by centrically stiffeners or unrefinced plate then \(k_1 = k_2 = 0, B_{ij}^* = 0\).

The equations (17) and (18) can be rewritten as following
\[
g(t) = A f(t) - B f^2(t), \quad H_1 f(t) + H_2 g(t) - H_3 f(t) g(t) - H_4(t) + H_5 \ddot{f} = 0
\]
with the coefficient
\[
A = \frac{B_{11}^* \left( \frac{m\pi}{a} \right)^4 - B_{12}^* \left( \frac{mn\pi^2}{ab} \right)^2 + B_{22}^* \left( \frac{n\pi}{b} \right)^4 + k_1 \left( \frac{n\pi}{b} \right)^2 + k_2 \left( \frac{m\pi}{a} \right)^2}{A_{11}^* \left( \frac{m\pi}{a} \right)^4 + (A_{66}^* - 2A_{12}^*) \left( \frac{mn\pi^2}{ab} \right)^2 + A_{22}^* \left( \frac{n\pi}{b} \right)^4},
\]
\[
B = \frac{16mn\pi^2}{3a^2b^2 \left[ A_{11}^* \left( \frac{m\pi}{a} \right)^4 + (A_{66}^* - 2A_{12}^*) \left( \frac{mn\pi^2}{ab} \right)^2 + A_{22}^* \left( \frac{n\pi}{b} \right)^4 \right]},
\]
\[
H_1 = \left[ D_{11}^* \left( \frac{m\pi}{a} \right)^4 + 2D_{33}^* \left( \frac{mn\pi^2}{ab} \right)^2 + D_{22}^* \left( \frac{n\pi}{b} \right)^4 \right] \frac{ab}{4},
\]
\[
H_2 = \left[ B_{11}^* \left( \frac{m\pi}{a} \right)^4 - B_{12}^* \left( \frac{mn\pi^2}{ab} \right)^2 + B_{22}^* \left( \frac{n\pi}{b} \right)^4 + k_1 \left( \frac{n\pi}{b} \right)^2 + k_2 \left( \frac{m\pi}{a} \right)^2 \right] \frac{ab}{4},
\]
\[
H_3 = \frac{8mn\pi^2}{3ab},
\]
\[
H_4(t) = q(t) \frac{4ab}{mn\pi^2},
\]
\[
H_5 = \left[ J_0 + J_2 \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \right] \frac{ab}{4}.
\]
Inserting (19) into (20) yields
\[
H_5 \ddot{f}(t) + (H_1 + H_2 A) f(t) - (H_2 B + H_3 A) f^2(t) + H_3 B f^3(t) = H_4(t).
\]
The vibration equation of a shell is of the form
\[
\ddot{f}(t) + m_1 f(t) - m_2 f^2(t) + m_3 f^3(t) = \ddot{q}_{mn}(t),
\]
where denote

\[ m_1 = \frac{H_1 + H_2 A}{H_5}, \quad m_2 = \frac{H_2 B + H_3 A}{H_5}, \quad m_3 = \frac{H_3 B}{H_5}, \quad q_{mn}(t) = \frac{H_4(t)}{H_5}. \]  

For linear free vibration the equation (22) gets form

\[ \ddot{f}(t) + m_1 f(t) = 0, \]

one can determine the fundamental frequency of vibration

\[ \omega_0 = \sqrt{m_1} = \sqrt{\frac{H_1 + H_2 A}{H_5}}. \]

4. VIBRATION OF THE SHELL WITH SIMPLY SUPPORTED AND CLAMPED EDGES

Suppose that a shallow shell is simply supported on edges \( x_1 = 0, \ x_1 = a \) and clamped on edges \( x_2 = 0, \ x_2 = b \). Then in edges the following conditions are taken

\[ w = 0, \quad M_1 = 0, \quad N_1 = 0, \quad N_6 = 0, \text{ in } x_1 = 0, \ a; \]

\[ w = 0, \quad \frac{\partial w}{\partial x_2} = 0, \quad N_2 = 0, \quad N_6 = 0, \text{ in } x_2 = 0, \ b, \]  

(25)

The boundary conditions (25) are satisfied when the stress function and deflection are chosen as follows

\[ \varphi = g(t) \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b}, \]

\[ w = f(t) \sin \frac{m\pi x_1}{a} \left( 1 - \cos \frac{2n\pi x_2}{b} \right). \]

(26)

Substituting expressions (26) into equations (13), (14) and using Bubnov-Galerkin method we get

\[ \begin{bmatrix} A_{11}^* \frac{m^4 \pi^4 b}{4a^3} + (A_{66}^* - 2A_{12}^*) \frac{m^2 n^2 \pi^4}{4ab} + A_{22}^* \frac{n^4 \pi^4 a}{4b^3} \end{bmatrix} g(t) = \begin{bmatrix} B_{11}^* \frac{4m^4 \pi^4 b}{3na^3} \\ -B_{12}^* \frac{4m^2 n^3 \pi^3}{3ab} + B_{22}^* \frac{16n^3 \pi^3 a}{3b^3} + k_1 \frac{4n \pi a}{3b} + k_2 \frac{4m^2 \pi b}{3na} \end{bmatrix} f(t) - \frac{256mn \pi^2}{45ab} f^2(t) \]  

and

\[ \begin{bmatrix} D_{11}^* \frac{3m^4 \pi^4 b}{4a^3} + 2D_{33}^* \frac{2m^2 n^2 \pi^4}{a^2 b^2} + D_{22}^* \frac{4n^4 \pi^4 a}{b^3} \end{bmatrix} f(t) + \]

\[ + \begin{bmatrix} B_{11}^* \frac{4m^4 \pi^4 b}{3na^3} - B_{12}^* \frac{4m^2 n^3 \pi^3 a}{3ab} + B_{22}^* \frac{4n^3 \pi^3 a}{3b^3} + k_3 \frac{4n \pi a}{3b} + k_2 \frac{4m^2 \pi b}{3na} \end{bmatrix} g(t) \]

\[- \frac{512mn \pi^2}{45ab} f(t) g(t) - \frac{2ab}{mn\pi} + \left[ \frac{3ab}{4} + \frac{J_0}{4a} + J_2 \right] \pi^2 \left( \frac{3m^2 b}{4a} + \frac{n^2 a}{b} \right) \frac{d^2 f}{dt^2} = 0, \]

where denote the coefficients

\[ B_{11}^* = A_{12}^* \frac{E A_{11} z_1}{s_1}, \quad B_{22}^* = A_{12}^* \frac{E A_{22} z_2}{s_2}, \quad B_{12}^* = A_{11}^* \frac{E A_{22} z_2}{s_2} + A_{22}^* \frac{E A_{11} z_1}{s_1}. \]
and \( m, n \) are odd numbers.
Similarly we get the vibration equation of the shell in the form (21), where the terms \( m_i, (i = 1, 2, 3) \) can be calculated by (22) but with coefficients

\[
A = \frac{B_{11}^*}{3n^3} 4m^4 \pi^3 b + B_{12}^* \frac{4m^2}{3ab} + B_{22}^* \frac{16n^3 \pi^3 a}{3b^3} + k_1 \frac{4n \pi a}{3b} + k_2 \frac{4m^2 \pi b}{3na} + \frac{A_{11}^*}{4a^3} m^4 n^4 b + (A_{66}^* - 2A_{12}^*) \frac{m^2 n^2 \pi^4}{4ab} + A_{22}^* \frac{n^4 \pi^4 a}{4b^3}
\]

\[
B = \frac{256mn^2}{45ab} \left[ \frac{A_{11}^*}{4a^3} m^4 n^4 b + (A_{66}^* - 2A_{12}^*) \frac{m^2 n^2 \pi^4}{4ab} + A_{22}^* \frac{n^4 \pi^4 a}{4b^3} \right],
\]

\[
H_1 = D_{11}^* \frac{3m^4 \pi^4 b}{4a^3} + 2D_{33}^* \frac{2m^2 n^2 \pi^4}{a^2 b^2} + D_{22}^* \frac{4n^4 \pi^4 a}{b^3},
\]

\[
H_2 = B_{11}^* \frac{4m^4 \pi^3 b}{3n^3} - B_{12}^* \frac{4m^2 n^3 \pi^3}{3ab} + B_{22}^* \frac{4n^3 \pi^3 a}{3b^3} + k_1 \frac{4n \pi a}{3b} + k_2 \frac{4m^2 \pi b}{3na},
\]

\[
H_3 = \frac{512mn^2}{45ab}, \quad H_4(t) = q(t) \frac{2ab}{m^2}, \quad H_5 = J_0 3ab^4 + J_2 \pi^2 \left( \frac{3m^2 b^2}{4a} + \frac{n^2 a}{b} \right).
\]

5. NON-LINEAR DYNAMICAL ANALYSIS OF DOUBLY CURVED SHALLOW COMPOSITE SHELLS

5.1. Frequency-amplitude relation of non-linear free vibration

The equation of non-linear free vibration can be obtained from (21)

\[
\ddot{f}(t) + m_1 f(t) - m_2 f^2(t) + m_3 f^3(t) = 0.
\]

(29)

Using harmonic equilibrium method with seeking solution as \( f(t) = A \cos(\omega t) \), after some transformations we get the frequency-amplitude relation of non-linear free vibration

\[
\nu^2 = \frac{\omega^2}{\omega_0^2} = 1 - \frac{8\Omega_0}{3\pi} A + \frac{3K}{4} A^2,
\]

(30)

where \( \Omega_0 = m_2/m_1, K = m_3/m_1 \), terms \( m_1, m_2, m_3 \) are determined from (22), \( \nu \) is ratio of non-linear vibration frequency and fundamental frequency, \( A \) - amplitude of non-linear vibration.

In particular case for an unreinforced plate or a centrically reinforced plate we have \( \Omega_0 = \frac{m_2}{m_1} = 0 \), the expression (30) gets form

\[
\nu^2 = 1 + \frac{3K}{4} A^2.
\]

5.2. Dynamic analysis of composite shell in non-linear vibration

Let's consider non-linear vibration of a shallow composite shell subjected to excited load \( q \equiv q(x_1, x_2, t) = q_0 \sin(\Omega t) \). Now the equation (21) can be represented in the form of Duffing equation

\[
\ddot{f}(t) + \left[ m_1 - m_2 f(t) + m_3 f^2(t) \right] f(t) = \bar{q}_{mn}(t),
\]

(31)

where:
a) For a shell with simply supported edges
\[
\bar{q}_{mn}(t) = \frac{16q_0 \sin(\Omega t)}{mn\pi^2 \left[ J_0 + J_2 \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \right]}
\]

b) For a shell with simply supported edges and clamped edges
\[
\bar{q}_{mn}(t) = \frac{8q_0 \sin(\Omega t)}{m\pi \left[ 3J_0 + J_2 \pi^2 \left( \frac{3m^2}{a^2} + \frac{4n^2}{b^2} \right) \right]}
\]

Put \( K(f) = m_1 - m_2 f(t) + m_3 f^2(t) \), \( \bar{q}_{mn}(t) = F(t) \), equation (31) can be rewritten in the form
\[
\ddot{f}(t) + K(f) f(t) = F(t).
\]  
Equation (32) can be solved by iterative method with Newmark scheme. Dividing calculated process by time-step \( \Delta t \), corresponding \((n + 1)^{th}\) step we have \( t_{n+1} = (n + 1)\Delta t \). Using Newmark scheme we can rewrite (32) as follows
\[
K^*(f)_{n+1} \cdot f_{n+1} = F_{n+1}^*,
\]  
where
\[
K^*(f)_{n+1} = K(f)_{n+1} + \frac{4}{\Delta t^2},
\]
\[
F_{n+1}^* = F_{n+1} + \left( \frac{4}{\Delta t^2} \ddot{f}_n + \frac{4}{\Delta t} \dot{f}_n + f_n \right).
\]  
The solution \( f_{n+1} \) is determined at the time \( t_{n+1} = (n + 1)\Delta t \), then velocity and acceleration can be calculated respectively by formulas
\[
\ddot{f}_{n+1} = \frac{4}{\Delta t^2} (f_{n+1} - f_n) - \frac{4}{\Delta t} \dot{f}_n - \ddot{f}_n,
\]
\[
\dot{f}_{n+1} = \dot{f}_n + \frac{\Delta t}{2} (\ddot{f}_n + \ddot{f}_{n+1}).
\]  
Since the coefficient \( K^*(f)_{n+1} \) in (33) is non-linear, we use an iterative method to solve the equation. Using here the direct iteration technique the equation (33) can be expressed as
\[
K^*(f)_{n+1}^{(k)} \cdot f_{n+1}^{(k+1)} = F_{n+1}^*,
\]  
where \( k \) is the iteration number. At any fixed time for the \((k + 1)^{th}\) iteration, the stiffness \( K^*(f)_{n+1}^{(k)} \) is calculated from the \( k^{th} \) iteration. Calculating process stops when the convergence criterion is satisfied
\[
\left| \frac{f_{n+1}^{(k)} - f_{n+1}^{(k+1)}}{f_{n+1}^{(k)}} \right| \leq \varepsilon,
\]  
where \( \varepsilon > 0 \) is a given small value.
6. NUMERICAL EXAMPLES

The shallow shell considered here is a panel with in-plane edges $a = b = 2$ m, curvatures $k_1 = 1/R_1$, $k_2 = 1/R_2$. The shell is simply supported at all its edges or simply supported-clamped. The skin of the shell had 4 plies $[45/−45/−45/45]$, each ply being 1.5 mm. The material of the shell is AS4/3501 graphite/epoxy with parameters $E_1 = 144.8$ GPa, $E_2 = 9.67$ GPa

$$G_{12} = G_{13} = 4, 14 \text{ GPa}, \quad G_{23} = 3.45 \text{ GPa}, \quad \nu_{12} = 0.3, \quad \rho = 1389.23 \text{ kg/m}^3,$$

where $E_1$, $E_2$ are the elastic modulus in directions $x_1$ and $x_2$ respectively, $\nu_{12}$ - the Poisson coefficient, $G_{12}, G_{13}$ and $G_{23}$ are the shear modulii in 12, 13 and 23 planes. Material of reinforced stiffener has elastic modulus $E = 600$ GPa. The height of the stiffener is equal to 12 mm, while their width 4 mm. The spacings of longitudinal stiffener and transversal stiffeners $s_1 = 50$ mm, and $s_2 = 50$ mm respectively.

The time-step $\Delta t$ is taken as $T/300$ where period $T = 2\pi/\Omega$, $\Omega$ is a frequency of excited load and $t_n = n.\Delta t$ are used for numerically solving non-linear differential equation by Newmark method. The applied harmonic uniform load is of the form $q(x_1, x_2, t) = p\sin \Omega t$, where the magnitude $p$ may be taken as 7,500 N/m$^2$, 15,000 N/m$^2$, 30,000 N/m$^2$. Two cases of Gauss curvature are taken into consideration: $k = k_1.k_2 > 0$ when $R_1 = R_2 = 5$m and $k = k_1.k_2 < 0$ when $R_1 = 3$ m, $R_2 = -10$ m, we take respectively $\Omega = 1400$ and $\Omega = 800 (s^{-1})$, these frequency values of excited load are near to ones of fundamental frequency values when $k > 0$: $\omega_0 = 1416$ (in case SS-SS), $\omega_0 = 1390$ (in case SS-CC); when $k < 0$: $\omega_0 = 780$ (in case SS-SS), $\omega_0 = 749$ (in case SS-CC) respectively (calculated data according to (3.10)).

The Figs. 1, 2, 3 show graphs of maximum deflection $w_{max} = f$ by 25 period, that means the non-linear transient responses, for cases of boundary condition with simply supported edges (SS-SS) and simply supported-clamped edges (SS-CC) when Gauss curvature $k = k_1.k_2 > 0$ and magnitude $p$ of excited load as 7,500 N/m$^2$, 15,000 N/m$^2$, 30,000 N/m$^2$ respectively. The Fig. 4 shows relation of maximum deflection and velocity of maximum deflection when Gauss curvature $k = k_1.k_2 > 0$, $p = 30,000$ N/m$^2$.

![Fig. 1. Non-linear transient responses with $p=7,500$ N/m$^2$](image1.png)

![Fig. 2. Non-linear transient responses with $p=15,000$ N/m$^2$](image2.png)
Fig. 3. Non-linear transient responses with $p=30,000 \text{ N/m}^2$

Fig. 4. Deflection-velocity relation

Fig. 5 shows the graph of maximum deflection $w_{\text{max}} = f$ by 25 period (non-linear transient responses) for cases of boundary condition with simply supported edges (SS-SS) and simply supported-clamped edges (SS-CC) when Gauss curvature $k = k_1, k_2 < 0$. Relation for maximum deflection and velocity of maximum deflection when $p=7,500 \text{ N/m}^2$ is expressed in Fig. 6.

Fig. 5. Non-linear transient responses

Fig. 6. Deflection-velocity relation

The Figs. 7, 8, 9 show non-linear transient responses and graphs $f - \dot{f}$ by 25 period respectively for cases with Gauss curvature $k < 0$, when $p=1,5000 \text{ N/m}^2$, and Figs. 10, 11, 12 for cases with Gauss curvature $k < 0$, $p=30,000 \text{ N/m}^2$.

Figs. 13, 14 show the non-linear transient responses for unstiffened and stiffened shell with simply supported at all edges, simply supported-clamped edges. The skin of the shell had 4 plies, each ply being 1 mm. The spacings $s_1, s_2$ of stiffeners vary differently, consequently the effect of stiffeners on the transient responses of shell is represented apparently.
Fig. 7. Non-linear transient responses when $p=1,500$ N/m$^2$

Fig. 8. Deflection-velocity relation (SS-SS)

Fig. 9. Deflection-velocity relation (SS-CC)

Fig. 10. Non-linear transient responses

Fig. 11. Relation of deflection-velocity (SS-SS)

Fig. 12. Relation of deflection-velocity (SS-CC)

Discussion
- Although the excited load is harmonic, non-linear responses of shell are not harmonic by timing, however having a specific cycle for the shell of positive Gauss curvature $k > 0$. For non-linear vibration of a shell when fundamental frequency is equal to frequency of excited load may occur the phenomenon of resonance, but in this considered example the
frequency of excited load is near to the fundamental frequency we observe the phenomenon like harmonic beat of a linear vibration and the amplitude of external load is smaller, then the cycle of non-linear transient responses is larger.

- For the shell with negative Gauss curvature $k < 0$, the phenomenon of cycle is not clear and its turn is rather complicated and needed to be examined more carefully.
- When $k > 0$ the amplitude of non-linear vibration for the shell with SS-SS edges is greater than with SS-CC edges, when $k < 0$ the above rule is not changed completely.
- Non-linear responses and relation of deflection $f$-velocity $\dot{f}$ for a shell with $k > 0$ is stable, a sudden phenomenon does not happen but for a shell with $k < 0$ this rule is not true and needed to be examined more carefully.

7. CONCLUSION

The governing equations for dynamic problem of eccentrically stiffened laminated composite shallow shell are given and analysed by use of the thin shell theory considering geometrical non-linearity and the Lekhnitsky's smeared stiffeners technique. Using stress function and Bubnov-Galerkin methods the equation of non-linear vibration for shallow shell in the form of Duffing equation is obtained. Numerical solution is carried out by iterative method using Newmark calculated scheme. Obtained results show the influence of boundary conditions and Gauss curvature on the non-linear vibration of eccentrically stiffened laminated shallow shell.

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**DAO ĐỘNG PHI TUYẾN CỦA VỎ COMPOSITE LỚP CÓ GÂN GIA CƯỜNG**

Bài báo trình bày dao động phi tuyến của vỏ thoát composite lớp hai độ cong có gân gia cường lệch tâm. Tính toán nội lực và chuyển vị dựa trên lý thuyết với mock có tính đến yếu tố phi tuyến hình học và kỹ thuật tính gần gia cường theo Lekhnitoky. Dựa trên phương trình tương thích biên dạng và phương trình chuyển động của vỏ để thiết lập hệ phương trình dao hàm riêng theo hàm ứng suất và độ vồng của vỏ vò. Sử dụng phương pháp Bubnov-Gaberkin và phương pháp bước lập có dùng số độ Newmark để phân tích động lực của vỏ. Đảm nhận được liên hệ tận số-bién độ của dao động phi tuyến tự do và đáp ứng tức thời phi tuyến của vỏ. Các kết quả tính toán số cho thấy ảnh hưởng rõ rệt của điều kiện biên và độ cong Gauss đến dao động phi tuyến của vỏ.