

A NOTE ON THE MELNIKOV FUNCTION

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Abstract. With one (Poincaré section) parameter and a particular motion law (that associated to certain determined point of homo-heteroclinic orbits), the usual form of the Melnikov function seems to be not convenient for certain problems. Another favourable form can be obtained by using a supplementary parameter - the arbitrary time constant in the general mentioned motion law.

1. INTRODUCTION

As known, for Hamiltonian systems, homo-heteroclinic orbits (when they exist) can be destroyed by weak perturbations to generate - on Poincaré section - stable and unstable invariant manifolds and the distance between these manifolds can be estimated by the so-called Melnikov function ([1], pp. 356-389; [2], pp. 172-181). This distance is defined for manifolds on certain chosen Poincaré section and at certain determined point of the homo-heteroclinic orbits. Therefore, in the usual form of the Melnikov function, besides the parameter specifying the chosen Poincaré section, a particular motion law is used, that associated to the point of interest on the homo-heteroclinic orbits. Varying the Poincaré section parameter, we can only study the variation of the mentioned distance on different Poincaré section at the same point of the homo-heteroclinic orbits. To study the variation of the distance between manifolds on a fixed Poincaré section and along the homo-heteroclinic orbits, we have to "modulate" the aforesaid motion law. This modulation can be realized by using a supplementary parameter as presented below.

2. THE USUAL FORM OF THE MELNIKOV FUNCTION

Let us consider weakly perturbed Hamiltonian systems governed by differential equations of the form:

$$\ddot{x} = f(x) + \varepsilon g(x, t) \quad (2.1)$$

or

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + \varepsilon g_1(x_1, x_2, t); & f_1(x_1, x_2) &= \partial H(x_1, x_2) / \partial x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + \varepsilon g_2(x_1, x_2, t); & f_2(x_1, x_2) &= -\partial H(x_1, x_2) / \partial x_1 \end{aligned} \quad (2.2)$$

where x is a column vector variable with two components (x_1, x_2) ; f and g are column vector functions with two components (f_1, f_2) and (g_1, g_2) , respectively; overdots denote differentiation with respect to time t . $g(x, t)$ represents a small perturbation T -periodic in t ; $H(x_1, x_2)$ is the Hamiltonian (a scalar-valued function) of the unperturbed system ($\varepsilon = 0$); ε is a small parameter.

Assume that the unperturbed Hamiltonian system possesses an orbit C , homoclinic to a saddle 0 . This homoclinic orbit is destroyed by the small perturbation εg and on each Poincaré section S_θ specified by the parameter θ ($0 \leq \theta < T$), two invariant manifolds are generated: a stable manifold C_s and an unstable one C_n ; both these manifolds homoclinic to a new saddle O' , close to O .

The distance between C_s and C_n is defined on certain chosen Poincaré section S_θ and at certain determined point A of C . As known can be estimated by the so-called Melnikov function

$$M(\theta) = \int_{-\infty}^{+\infty} f[x_A(t - \theta)] \wedge g[x_A(t - \theta), t] dt = \int_{-\infty}^{+\infty} f[x_A(\tau)] \wedge g[x_A(\tau), \tau + \theta] d\tau \quad (2.3)$$

where $M(\theta)$ denotes the Melnikov estimation of the distance (along the normal of C) at A between C_s and C_n on the Poincaré section S_θ ; θ is the value of the parameter which specifies the chosen Poincaré section; $x_A(t)$ is the motion law (an unperturbed solution) associated to A i.e. that satisfying the condition $x_A(0) \equiv A$ (thus, the representative point $x_A(t - \theta)$ moves along C and passes through A just at the moment $t = \theta$); the wedge symbol indicates the magnitude of the cross (X) product (i.e. $f \wedge g = f_1 g_2 - f_2 g_1$).

Fixing A and varying θ , (2.3) gives - at the same (fixed) point A of C - the variation of the distance between C_s and C_n on different Poincaré sections. To study - on certain chosen (fixed) Poincaré section - the variation of the distance between C_s, C_n along C , we have to fix θ and “modulate” $x_A(t)$ i.e. each point A needs its appropriate motion law.

In particular, to study on S_θ (θ -fixed, given) the distribution of intersection points of C_s and C_n along C , we have to solve the equation

$$\int_{-\infty}^{+\infty} f[x_A(t - \theta)] \wedge g[x_A(t - \theta), t] dt = 0 \quad (2.4)$$

where $x_A(t - \theta)$ is unknown motion law.

This problem becomes simple - as presented below - by introducing a new parameter α in $x_A(t)$.

3. A MORE DETAILED FORM OF THE MELNIKOV FUNCTION

We remark that the unperturbed Hamiltonian system ($\varepsilon = 0$) is autonomous. So, each of its solutions contains an arbitrary time constant i.e. if $x(t)$ is certain solution, $x(t + h)$ with arbitrary constant h is also another solution of same orbit. In other words, corresponding to an unperturbed orbit, we have to deal with-not one-but a family of unperturbed solutions expressed by $x(t + h)$. We can consider $x(t)$ as a particular solution and $x(t + h)$ as the general solution along the aforesaid orbit (not the general solution of the system). Evidently, this remark can be applied to the special orbit - the homoclinic orbit C . Hence, when we find a particular motion law $x(t)$ along C , the general solution along C is $x(t + h)$.

For the Melnikov function, it is convenient to write this general solution as

$$x(t - \theta + \alpha) \quad (3.1)$$

where θ is the Poincaré section parameter; α is the parameter attributed to the point $x(\alpha) = A_\alpha$; $x(t)$ is the motion law associated to the point $A_0 = x(0)$.

Using (3.1), the Melnikov function can be written as:

$$\begin{aligned}
 M(\theta, \alpha) &= \int_{-\infty}^{+\infty} f[x(t - \theta + \alpha)] \wedge g[x(t - \theta + \alpha), t] dt \\
 &= \int_{-\infty}^{+\infty} f[x(\tau)] \wedge g[x(\tau), \tau + \theta - \alpha] d\tau
 \end{aligned}
 \tag{3.2}$$

$M(\theta, \alpha)$ is thus the distance at $A_\alpha = x(\alpha)$ between C_s and C_n on S_θ . Fixing α and varying θ , (3.2) gives - at the same point A_α of C - the distance between C_s and C_n on different Poincaré sections S_θ . If, with θ fixed, we vary α , (3.2) gives the variation of the distance between C_s and C_n along C , on the already chosen (fixed) Poincaré section S_θ . In particular, along C , the distribution of intersection points of C_s and C_n along C on S_θ is determined by the equation:

$$M(\theta, \alpha) = 0 \tag{3.3}$$

where α is unknown and θ is constant.

Since $g(x, t)$ is T -periodic in t , $M(\theta, \alpha)$ is T -periodic in α . Hence, when (3.3) admits a real root α , it also admits an infinite number of roots $\alpha + 2nT$ (n -integer). Correspondingly, we obtain along C , an infinite number of intersection points of C_s and C_n ; these points are located near the points $A_{\alpha+2n\pi} = x(\alpha + 2n\pi)$.

4. AN EXAMPLE

For illustration, we use the example 5.16 presented in ([1], pp. 374-377) i.e. we consider the externally forced Duffing oscillator governed by the differential equation

$$\ddot{x} - x + x^3 = \varepsilon \{ p \cos(\Omega t) - k\dot{x} \} \quad \text{with } \Omega = 1 \tag{4.1}$$

where x is a scalar variable, $p > 0$ is intensity of the external forcing excitation $p \cos t$ of period 2π ; $k > 0$ is the linear damping coefficient.

Putting $x = x_1$, $\dot{x} = x_2$, the equation (4.1) can be written as

$$\begin{aligned}
 \dot{x}_1 &= x_2, \\
 \dot{x}_2 &= x_1 - x_1^3 + \varepsilon(p \cos t - kx_2).
 \end{aligned}
 \tag{4.2}$$

Hence:

$$f_1 = x_2, \quad f_2 = x_1 - x_1^3, \quad g_1 = 0, \quad g_2 = p \cos t - kx_2, \tag{4.3}$$

$$H(x, \dot{x}) = H(x_1, x_2) = \frac{x_1^2}{2} + \frac{x_1^4}{4} - \frac{x_1^2}{2}. \tag{4.4}$$

The unperturbed Hamiltonian system ($\varepsilon = 0$) possesses two homoclinic orbits - the right C and the left C' - homoclinic to the saddle (origin) O , respectively encircling two centers $I(1, 0)$ and $I'(-1, 0)$, intersecting the abscissa axis $Ox \equiv Ox_1$ at $A(\sqrt{2}, 0)$ and $A'(-\sqrt{2}, 0)$.

The equation of these orbits is

$$H(x, \dot{x}) = H(x_1, x_2) = 0 \quad \text{or} \quad \dot{x} = x_2 = \pm \sqrt{x_1^2 - \frac{x_1^4}{2}} \tag{4.5}$$

where $+$ ($-$) corresponds to upper (lower)-half homoclinic orbits, respectively.

The perturbation $\varepsilon(p \cos t - kx_2)$ destroys C and C' to generate four invariant manifolds: the right and left unstable manifolds C_n and C'_n and the right and left stable one C_s and C'_s .

Below, our study is devoted to the right homoclinic orbit C and its corresponding stable and unstable invariant manifolds C_s and C_n . A particular (unperturbed) motion law along C is

$$x(t) = x_1(t) = \sqrt{2}\operatorname{sech}(t), \quad \dot{x}(t) = x_2(t) = -\sqrt{2}\operatorname{sech}(t)\tanh(t). \tag{4.6}$$

For $t = 0$, we have

$$x_1(0) = \sqrt{2}\operatorname{sech}(0) = \sqrt{2}, \quad x_2(0) = -\sqrt{2}\operatorname{sech}(0)\tanh(0) = 0. \tag{4.7}$$

Thus, the motion (4.6) is that associated to the point $A(\sqrt{2}, 0)$. The usual form of the Melnikov function is:

$$M(\theta) = - \int_{-\infty}^{+\infty} \sqrt{2}\operatorname{sech}(t - \theta)\tanh(t - \theta) \{p \cos t + k\sqrt{2}\operatorname{sech}(t - \theta)\tanh(t - \theta)\} dt. \tag{4.8}$$

It gives the estimation of the distance between C_s and C_n on the Poincaré section S_θ at the point $A(\sqrt{2}, 0)$. As in [1], if we vary θ , we can study only the variation of the distance between C_s and C_n on different Poincaré sections S_θ at the same point A . The “general” motion law along C is:

$$x(t + \alpha) = \sqrt{2}\operatorname{sech}(t + \alpha), \quad \dot{x}(t + \alpha) = -\sqrt{2}\operatorname{sech}(t + \alpha)\tanh(t + \alpha) \tag{4.9}$$

and the Melnikov function with two parameters is

$$M(\alpha, \theta) = - \int_{-\infty}^{+\infty} \sqrt{2}\operatorname{sech}(t + \alpha - \theta)\tanh(t + \alpha - \theta) \times \\ \times \{p \cos t + k\sqrt{2}\operatorname{sech}(t + \alpha - \theta)\tanh(t + \alpha - \theta)\} dt \tag{4.10}$$

or

$$M(\alpha, \theta) = - \int_{-\infty}^{+\infty} \sqrt{2}\operatorname{sech}(\tau)\tanh(\tau) \{p \cos(\tau + \theta - \alpha) + k\sqrt{2}\operatorname{sech}(\tau)\tanh(\tau)\} d\tau. \tag{4.11}$$

With (4.11), it is easy to estimate the distance between C_s and C_n on any Poincaré section S_θ and at any point of the homoclinic orbit C . For instance, at $A(\sqrt{2}, 0)$ and on $S_0, S_{\pi/2}$

we obtain respectively

$$M(\alpha = 0, \theta = 0) = -\sqrt{2} \int_{-\infty}^{+\infty} \operatorname{sech}(\tau) \tanh(\tau) \{p \cos \tau + k\sqrt{2} \operatorname{sech}(\tau) \tanh(\tau)\} d\tau, \quad (4.12)$$

$$M(\alpha = 0, \theta = \frac{\pi}{2}) = -\sqrt{2} \int_{-\infty}^{+\infty} \operatorname{sech}(\tau) \tanh(\tau) \{-p \sin \tau + k\sqrt{2} \operatorname{sech}(\tau) \tanh(\tau)\} d\tau. \quad (4.13)$$

Let $B(1, \sqrt{2}/2)$ be the maximum point of C . The value α_* corresponding to B satisfies the equations:

$$\sqrt{2} \operatorname{sech}(\alpha_*) = 1, \quad -\sqrt{2} \operatorname{sech}(\alpha_*) \tanh(\alpha_*) = \frac{\sqrt{2}}{2}. \quad (4.14)$$

Solving (4.11), we find $\alpha_* = \ln(\sqrt{2}-1) \approx -0.88137$. Therefore the distances at B between C_s and C_n on $S_0, S_{\pi/2}$ are estimated respectively by the integrals:

$$M(\alpha_*, 0) = -\sqrt{2} \int_{-\infty}^{+\infty} \operatorname{sech}(\tau) \tanh(\tau) \{p \cos(\tau - \alpha_*) + k\sqrt{2} \operatorname{sech}(\tau) \tanh(\tau)\} d\tau, \quad (4.15)$$

$$M(\alpha_*, \frac{\pi}{2}) = -\sqrt{2} \int_{-\infty}^{+\infty} \operatorname{sech}(\tau) \tanh(\tau) \{-p \sin(\tau - \alpha_*) + k\sqrt{2} \operatorname{sech}(\tau) \tanh(\tau)\} d\tau. \quad (4.16)$$

Following the procedure given in [1], we expand (4.11) as:

$$M(\alpha, \theta) = -kI_1 - p\sqrt{2} \cos(\theta - \alpha)I_2 + p\sqrt{2} \sin(\theta - \alpha)I_3 \quad (4.17)$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} \dot{x}^2(\tau) d\tau, & I_2 &= \int_{-\infty}^{+\infty} \operatorname{sech}(\tau) \tanh(\tau) \cos \tau d\tau, \\ I_3 &= \int_{-\infty}^{+\infty} \operatorname{sech}(\tau) \tanh(\tau) \sin \tau d\tau. \end{aligned} \quad (4.18)$$

Using (4.5), it is easy to calculate the first integral

$$I_1 = \int_0^{\sqrt{2}} \sqrt{x^2 - x^4/2} dx - \int_{\sqrt{2}}^0 \sqrt{x^2 - x^4/2} dx = 2 \int_0^{\sqrt{2}} \sqrt{x^2 - x^4/2} dx = \frac{4}{3}. \quad (4.19)$$

The second integral $I_2 = 0$ because its integral is odd.

Using $I_2 = 0$ and the transformations $u = e^\tau, v = e^{-\tau}$, the third integral can be put in the forms:

$$I_3 = \frac{1}{i} \int_{-\infty}^{+\infty} \operatorname{sech}(\tau) \tanh(\tau) e^{i\tau} d\tau = \frac{2}{i} \int_0^\infty u_i \frac{u^2 - 1}{(u^2 + 1)^2} du, \tag{4.20}$$

$$I_3 = \frac{2}{i} \int_{-\infty}^{+\infty} (-1)^i v^i \frac{v^2 - 1}{(v^2 + 1)^2} dv = \frac{2}{i} e^{-\pi} \int_{-\infty}^{+\infty} v_i \frac{v^2 - 1}{(v^2 + 1)^2} dv. \tag{4.21}$$

Hence, we can write:

$$I_3 = \frac{2}{i(1 + e^{-\pi})} \int_{-\infty}^{+\infty} u^2 \frac{u^2 - 1}{(u^2 + 1)^2} du. \tag{4.22}$$

The integral (4.22) has a two folds singularity at $u = i$. Following the method of residues, we obtain

$$I_3 = \frac{2}{i(1 + e^{-\pi})} \cdot 2\pi i \cdot \lim_{u \rightarrow i} \frac{d}{du} \left\{ (u - i)^2 \frac{u^2(u^2 - 1)}{(u^2 + 1)^2} \right\} = \pi \operatorname{sech} \frac{\pi}{2}. \tag{4.23}$$

Finally:

$$M(\alpha, \theta) = -\frac{4}{3}k + p\pi\sqrt{2} \sin(\theta - \alpha) \operatorname{sech} \frac{\pi}{2}. \tag{4.24}$$

Thus:

$$\begin{aligned} M(0, 0) &= -\frac{4}{3}k; & M\left(0, \frac{\pi}{2}\right) &= -\frac{4}{3}k + p\pi\sqrt{2} \operatorname{sech} \frac{\pi}{2} \\ M(\alpha_*, 0) &= -\frac{4}{3}k + p\pi\sqrt{2} \sin \alpha_* \operatorname{sech} \frac{\pi}{2} \\ M(\alpha_*, \frac{\pi}{2}) &= -\frac{4}{3}k + p\pi\sqrt{2} \cos \alpha_* \operatorname{sech} \frac{\pi}{2} \end{aligned} \tag{4.25}$$

Using (4.24), we can easily study the distribution of intersection points of C_s and C_n along C and - for instance - on S_0 . Indeed, by solving the equation

$$M(\alpha, \theta = 0) = -\frac{4}{3}k - p\pi\sqrt{2} \sin \alpha \operatorname{sech} \frac{\pi}{2} = 0 \tag{4.26}$$

we obtain

$$\sin \alpha = -\frac{4k}{3p\pi\sqrt{2} \operatorname{sech} \frac{\pi}{2}}. \tag{4.27}$$

For $k \approx 0.5990222$, $\sin \alpha \approx -\sqrt{2}/2$, $\alpha_1 = -\frac{\pi}{4} + 2k\pi$, $\alpha_2 = -\frac{3\pi}{4} + 2k\pi$ corresponding to α_1 , we obtain:

- for $k = 0, -1, -2, \dots$ $x_0 = 1.0676459, x_{-1} = 0.002408, x_{-2} = 0.000004$
- for $k = 1, 2, 3, \dots$ $x_1 = 0.0115845, x_2 = 0.00000216, x_3 = 0.0000004$.

These are respectively the abscissae of the points of the upper and lower half-homoclinic orbit C close intersection points of C_s and C_n .

Similarly, from α_2 , we can imagine another infinite number of intersection points of C_s and C_n , distributed along and close to C and accumulated in the neighbourhood of O .

5. CONCLUSION

From the above presented note, we see that, by introducing the time constant in the motion law i.e. by using the general motion law along homo-heteroclinic orbits, the Melnikov function can be put in a form containing two parameters and shows to be more convenient.

REFERENCES

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MỘT GHI CHÚ VỀ HÀM MELNIKOV

Dạng quen biết của hàm Melnikov trong lý thuyết chuyển động hỗn loạn chỉ chứa một tham số xác định thiết diện Poincaré được lựa chọn. Nếu đưa thêm một tham số gán cho các điểm trên đường đồng dị clinic, có thể viết hàm Melnikov chứa hai tham số. Dạng sau tỏ ra thuận tiện hơn khi nghiên cứu sự phân bố giao điểm các đa tạp ổn định và không ổn định.