NON - LINEAR VIBRATION OF FUNCTIONALLY GRADED SHALLOW SPHERICAL SHELLS

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Abstract. The present paper deals with the non-linear vibration of functionally graded shallow spherical shells. The properties of shell material are graded in the thickness direction according to the power law distribution in terms of volume fractions of the material constituents. In the derived governing equations geometric non-linearity in all strain-displacement relations of the shell is considered. From the deformation compatibility equation and the motion equation a system of partial differential equations for stress function and deflection of shell is obtained. The Galerkin method and Runge-Kutta method are used for dynamical analysis of shells to give expressions of natural frequencies and non-linear dynamic responses. Numerical results show the essential influence of characteristics of functionally graded materials and dimension ratios on the dynamical behaviors of shells.

1. INTRODUCTION

In recent years, functionally graded materials (FGMs) have gained considerable attention in the engineering applications. FGMs are microscopically inhomogeneous, in which the material properties vary smoothly and continuously from one surface to the other. Studies on the stability and vibration of functionally graded plates, cylindrical and shallow shells have been carried out. About vibration of FGM plates Vel S.S. and Batra R.C. [1] gave three dimensional exact solution for the vibration of FGM rectangular plates; Ferreira A.J. et al. [2] received natural frequencies of FGM plates by meshless method. Natural frequencies and buckling stresses of FGM plates and shallow shells were analyzed by Hiroyuki Matsunaga [3,4] using 2-D higher-order deformation theory. Pradyumna S. and Bandyspadhyay J.N. received natural frequencies of FGM curved panels using high-order finite element formulation [5]. Pradhan S.C et al. [6] and Loy C.T et al. [7] studied vibration of FGM cylindrical shells and the effects of boundary conditions and power law indices on the natural frequencies of shells; Yang Y. and Shen H.S. [8] studied free vibration and parametric response of shear deformable FGM cylindrical panels; Reddy J.N and Cheng Z.Q [9] gave frequency correspondence between membranes and functionally graded spherical shallow shell of polygonal plan form; Dao H.B. and Vu D. L. [10] studied dynamical behaviors of FGM shallow shells with geometrical imperfections and obtained dynamic responses and dynamic critical buckling loads of cylindrical and doubly-curved shallow shells subjected to dynamic loading Research on the stability of FGM cylindrical...
shells under aperiodic axial impulse and dynamic torsional loading can be seen in the works of Sofiyev A.H. et al. [11, 12]. Non-linear buckling analysis of FGM shallow spherical shells under pressure loads was presented by Ganapathi M. [13] by using finite element method, geometric non-linearity is assumed only on the meridional direction in strain-displacement relations. Dao H.B. [14] studied this problem using the approximated analytical method, geometric non-linearity is assumed in all strain-displacement relations.

The present paper deals with the non-linear vibration of functionally graded material shallow spherical shells. Derivations of governing equations of shells considering geometric non-linearity are based on the shell theory according to the von Karman theory for moderately large deflection and small strain with the assumption of power law composition for the constituent materials. The natural frequencies and non-linear dynamic responses of FGM spherical shells subjected to pressure loading are considered. The effects of characteristics of functionally graded materials and dimension ratios of shells on their dynamical behaviors are investigated.

2. GOVERNING EQUATIONS

Consider an axisymmetric functionally graded shallow spherical shell of thickness $h$, base radius $r_0$, shell radius $R$ with the coordinates $\varphi$, $\theta$ and $z$ along the meridional, circumferential and radial-thickness directions respectively as shown in Fig. 1. The shell is made of a mixture of ceramic and metal with continuously varying volume fractions. The materials in outer and inner surfaces of the shell are ceramic and metal respectively. Assume that, the modulus of elasticity $E$ and the mass density $\rho$ of the mixture change in the thickness direction by the power law of distribution; while Poisson’s ratio $\nu$ is assumed to be constant [15]

$$
E(z) = E_m + (E_c - E_m) \left(\frac{2z + h}{2h}\right)^k,
$$

$$
\rho(z) = \rho_m + (\rho_c - \rho_m) \left(\frac{2z + h}{2h}\right)^k,
$$

$$
\nu(z) = \nu = \text{const}
$$

Fig. 1. Geometry of spherical cap
It is convenient to introduce an additional variable $r$ defined by the relation $r = R \sin \varphi$, where $r$ is the radius of the parallel circle. If the rise $H$ of the shell is much smaller than the base radius $r_0$ we can take $\cos \varphi \approx 1$ and $R d\varphi = dr$, such that points of the middle surface may be referred to coordinates $r$ and $\theta$.

The non-linear strain-displacement relationships based upon the von Karman theory for moderately large deflection and small strain are

\[
\varepsilon_0^r = \frac{\partial v}{\partial r} - \frac{w}{R} + \frac{1}{2} \left( \frac{\partial w}{\partial r} \right)^2,
\]

\[
\varepsilon_0^\theta = \frac{1}{r} \left( \frac{\partial u}{\partial \theta} + v \right) - \frac{w}{R} + \frac{1}{2} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} \right)^2,
\]

\[
\gamma_0^r = r \frac{\partial}{\partial r} \left( \frac{u}{r} \right) + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{1}{r} \frac{\partial w}{\partial \theta} \frac{\partial w}{\partial r},
\]

\[
\chi_r = \frac{\partial^2 w}{\partial r^2},
\]

\[
\chi_\theta = \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial w}{\partial r},
\]

\[
\chi_{r\theta} = \frac{1}{r^2} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2},
\]

where $u, v, w$ - displacements of the middle surface points along circumferential, meridional and radial directions respectively;

$\varepsilon_0^r, \varepsilon_0^\theta, \gamma_0^{r\theta}$ - strains in the middle surface;

$\chi_r, \chi_\theta, \chi_{r\theta}$ - changes of curvatures and twist.

They must be relative in the deformation compatibility equation

\[
1 \frac{\partial^2 \varepsilon_0^r}{\partial \theta^2} + \frac{1}{r} \frac{\partial \varepsilon_0^r}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \gamma_0^r}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 \gamma_0^{r\theta}}{\partial r \partial \theta} (r \gamma_0^{r\theta}) = -\frac{\Delta w}{R} + \chi_{r\theta}^2 - \chi_r \chi_\theta,
\]

where $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$.

Integrating the stress-strain equations and their moments through the thickness of the shell ($-\frac{h}{2} \leq z \leq \frac{h}{2}$), taking into account the Young modulus $E(z)$ and the mass density $\rho(z)$ being power functions of $z$ described by Eq.(1), we obtain the expressions of internal forces and moments resultants

\[
N_r = \frac{E_1}{1-\nu^2} (\varepsilon_0^r + \nu \varepsilon_0^\theta) - \frac{E_2}{1-\nu^2} (\chi_r + \nu \chi_\theta),
\]

\[
N_\theta = \frac{E_1}{1-\nu^2} (\varepsilon_0^\theta + \nu \varepsilon_0^r) - \frac{E_2}{1-\nu^2} (\chi_\theta + \nu \chi_r),
\]

\[
N_{r\theta} = \frac{E_1}{2(1+\nu)} \gamma_0^{r\theta} - \frac{E_2}{1+\nu} \chi_{r\theta}
\]
According to Love’s theory the equations of motion are

\[ \varepsilon^0_r = \frac{1}{E_1}(N_r - \nu N_\theta) + \frac{E_2}{E_1} \chi_r, \]
\[ \varepsilon^0_\theta = \frac{1}{E_1}(N_\theta - \nu N_r) + \frac{E_2}{E_1} \chi_\theta, \]
\[ \gamma^0_{r\theta} = \frac{2(1+\nu)}{E_1} N_{r\theta} + \frac{2E_2}{E_1} \chi_{r\theta} \]

and

\[ M_r = \frac{E_2}{1-\nu^2}(\varepsilon^0_r + \nu \varepsilon^0_\theta) - \frac{E_3}{1-\nu^2}(\chi_r + \nu \chi_\theta) = \frac{E_2}{E_1} N_r - \frac{E_1 E_3 - E_2^2}{E_1(1-\nu^2)}(\chi_r + \nu \chi_\theta), \]
\[ M_\theta = \frac{E_2}{1-\nu^2}(\varepsilon^0_\theta + \nu \varepsilon^0_r) - \frac{E_3}{1-\nu^2}(\chi_\theta + \nu \chi_r) = \frac{E_2}{E_1} N_\theta - \frac{E_1 E_3 - E_2^2}{E_1(1-\nu^2)}(\chi_\theta + \nu \chi_r), \]
\[ M_{r\theta} = \frac{E_2}{2(1+\nu)} \gamma^0_{r\theta} - \frac{E_3}{1+\nu} \chi_{r\theta} = \frac{E_2}{E_1} N_{r\theta} - \frac{E_1 E_3 - E_2^2}{E_1(1+\nu)} \chi_{r\theta}, \]

where

\[ E_1 = \int_{-h/2}^{h/2} E(z)dz = E_m h + \frac{(E_c - E_m)h}{k + 1} = E^*_1 h, \]
\[ E_2 = \int_{-h/2}^{h/2} E(z)zdz = \frac{(E_c - E_m)kh^2}{2(k + 1)(k + 2)} = E^*_2 h^2, \]
\[ E_3 = \int_{-h/2}^{h/2} E(z)z^2 dz = \frac{E_m h^3}{12} + (E_c - E_m)h^3 \left( \frac{1}{k + 3} - \frac{1}{k + 2} + \frac{1}{4k + 4} \right) = E^*_3 h^3, \]

with

\[ E^*_1 = E_m + \frac{E_c - E_m}{k + 1}, \]
\[ E^*_2 = \frac{(E_c - E_m)k}{2(k + 1)(k + 2)}, \]
\[ E^*_3 = \frac{E_m}{12} + (E_c - E_m) \left( \frac{1}{k + 3} - \frac{1}{k + 2} + \frac{1}{4k + 4} \right). \]

Suppose that the shallow spherical shell is acted on by external uniform pressure \( q \).

According to Love’s theory the equations of motion are

\[ \frac{1}{r} \frac{\partial}{\partial r} (rN_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (N_{r\theta}) - \frac{N_\theta}{r} = \rho_1 \frac{\partial^2 v}{\partial t^2}, \]
\[ \frac{1}{r} \frac{\partial}{\partial r} (rN_\theta) + \frac{1}{r} \frac{\partial N_r}{\partial \theta} + \frac{N_{r\theta}}{r} = \rho \frac{\partial^2 u}{\partial t^2}, \]
\[ \frac{1}{r} \left[ \frac{\partial^2}{\partial r^2} (rM_r) + 2 \left( \frac{\partial^2 M_{r\theta}}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial M_r}{\partial \theta} \right) \right] + \frac{1}{r} \frac{\partial^2 M_\theta}{\partial \theta^2} - \frac{\partial M_\theta}{\partial r} + \frac{1}{r} (N_r + N_\theta) + \]
\[ + \frac{1}{r} \frac{\partial}{\partial r} \left( rN_r \frac{\partial w}{\partial r} + N_{r\theta} \frac{\partial w}{\partial \theta} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( N_{r\theta} \frac{\partial w}{\partial r} + N_\theta \frac{\partial w}{\partial \theta} \right) + q = \rho_1 \frac{\partial^2 w}{\partial t^2}, \]
where
\[ \rho_1 = \frac{h/2}{\int_{-h/2}^{h/2} \rho(z)dz = \rho_m h + \frac{(\rho_c - \rho_m)h}{k + 1} = \rho^* h,} \] (13)
with \( \rho^* = \rho_m + \frac{\rho_c - \rho_m}{k + 1} \).

By taking the inertia forces \( \rho \frac{\partial^2 u}{\partial t^2} \rightarrow 0 \) and \( \rho \frac{\partial^2 v}{\partial t^2} \rightarrow 0 \) into consideration because of \( u \ll w, v \ll w \) [16] equations (10) and (11) are satisfied by introducing the stress function \( F \)
\[ N_r = \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2}, \quad N_\theta = \frac{\partial^2 F}{\partial r^2}, \quad N_{r\theta} = \frac{1}{r^2} \frac{\partial F}{\partial \theta} - \frac{1}{r} \frac{\partial^2 F}{\partial r \partial \theta} \] (14)
and the equation (12) can be rewritten as
\[ \frac{1}{r} \left[ \frac{\partial^2}{\partial r^2} (r M_r) + 2 \left( \frac{\partial^2 M_{r\theta}}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial M_\theta}{\partial \theta} \right) + \frac{1}{r} \frac{\partial^2 M_\theta}{\partial \theta^2} - \frac{\partial M_{r\theta}}{\partial r} \right] + \frac{1}{r} (N_r + N_\theta) + N_r \frac{\partial^2 w}{\partial r^2} + 2 N_{r\theta} \frac{\partial^2 w}{\partial r \partial \theta} + N_\theta \frac{\partial w}{\partial r} - \frac{2}{r^2} N_{r\theta} \frac{\partial \theta}{\partial \theta} + q = \rho_1 \frac{\partial^2 w}{\partial t^2}. \] (15)

The substitution of eqs. (6) into the compatibility equation (4) and eqs. (7) into equation (15), taking into account relations (2), (3) and (14) yields a system of equations in terms of the stress function \( F \) and the deflection \( w \)
\[ \frac{1}{E_1} \Delta \Delta F = -\frac{\Delta w}{R} + \left( \frac{1}{r} \frac{\partial \omega}{\partial \theta} - \frac{1}{r^2} \frac{\partial \omega}{\partial \theta} \right)^2 - \frac{\partial^2 w}{\partial t^2} \left( \frac{1}{r} \frac{\partial \omega}{\partial \theta} + \frac{1}{r^2} \frac{\partial \omega}{\partial \theta} \right), \] (16)
\[ \rho_1 \frac{\partial^2 w}{\partial t^2} + \frac{E_1 E_3 - E_2^2}{E_1 (1 - \nu^2)} \Delta \Delta w \quad \frac{1}{E_1} \Delta \Delta F = -\frac{1}{R} \frac{\partial F}{\partial r} + \frac{1}{r} \frac{\partial^2 F}{\partial \theta^2} \left( \frac{1}{r^2} \frac{\partial \omega}{\partial \theta} - \frac{1}{r^2} \frac{\partial \omega}{\partial \theta} \right) + \frac{2}{r^2} \left( \frac{1}{r^2} \frac{\partial \omega}{\partial \theta} - \frac{1}{r^2} \frac{\partial \omega}{\partial \theta} \right), \] (17)

3. NON-LINEAR DYNAMICAL ANALYSIS

The mentioned equations (16) and (17) combining with boundary conditions and initial conditions can be used in non-linear dynamical analysis of functionally graded spherical shells.

Suppose that the FGM spherical shell is clamped at its base edge \( r = r_0 \), the boundary conditions are
\[ w = 0, \frac{\partial w}{\partial r} = 0 \quad \text{and} \quad N_r = \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial r \partial \theta} = 0 \quad \text{at} \quad r = r_0. \]

The boundary conditions can be satisfied, if the deflection \( w \) and the stress function \( F \) are represented by
\[ w(r, \theta, t) = f(t) \frac{r^2 (r_0 - r)^2}{r_0^2} \sin n \theta \]
\[ F(r, \theta, t) = \eta(t) \frac{r^2 (r_0 - r)^2}{r_0^2} \sin n \theta \] (18)
Applying Galerkin method to Eqs. (16) and (17) in the range \(0 \leq \theta \leq 2\pi\), \(0 \leq r \leq r_0\) yields a set of two equations with respect to \(f(t)\) and \(\eta(t)\)

\[
\eta = \frac{2E_1}{38 - 20n^2 + 12n^4} \left[ \frac{f r_0^2 (3 + 2n^2)}{7R} \right],
\]

\[
\rho_1 \frac{d^2 f}{dt^2} + \frac{E_1 E_3 - E_2^2}{E_1(1 - \nu^2)} \cdot \frac{7(38 - 20n^2 + 12n^4)}{4r_0^4} \frac{(3 + 2n^2)}{2Rr_0^2} \eta - \frac{2nf \eta}{\pi r_0^3} = \frac{14q}{\pi n},
\]

Eliminating \(\eta\) from these two equations leads to a non-linear second-order ordinary differential equation for \(f\)

\[
\rho_1 \frac{d^2 f}{dt^2} + \left[ \frac{E_1 E_3 - E_2^2}{E_1(1 - \nu^2)} \cdot \frac{7(38 - 20n^2 + 12n^4)}{4r_0^4} \frac{(3 + 2n^2)}{2Rr_0^2} \eta - \frac{2nf \eta}{\pi r_0^3} \right] \frac{f}{\pi n} = \frac{14q(t)}{\pi n}.
\]

where \(n\) - odd number.

By use of notations (8), (9) and (13) Eq. (19) can be rewritten in the form

\[
\frac{d^2 f}{dt^2} + \frac{E_1^*}{\rho^* R^2} \left[ \frac{7(E_1^* E_3^* - E_2^*^2)}{4\alpha^4 E_1^*^2(1 - \nu^2)} \left( \frac{h}{R} \right)^2 \frac{(3 + 2n^2)}{7(38 - 20n^2 + 12n^4)} \right] \frac{f}{\pi n} - \frac{14q(t)}{\pi n \rho^* h}
\]

where putting \(r_0 = \alpha R\), \(\alpha\) is the ratio of the base radius and the shell radius.

The obtained equation (20) is a governing equation for dynamical analysis of functionally graded shallow spherical shells. Based on this equation the non-linear vibration of FGM spherical shells can be investigated and the post-buckling analysis of shells can be performed.

4. NON-LINEAR VIBRATION OF FGM SHALLOW SPHERICAL SHELLS

Consider a FGM shallow spherical shell acted on by an uniformly distributed excited pressure load \(q(t) = Q \sin \Omega t\), the equation of motion has of the form

\[
\frac{d^2 f}{dt^2} + \frac{E_1^*}{\rho^* R^2} \left[ \frac{7(E_1^* E_3^* - E_2^*^2)}{4\alpha^4 E_1^*^2(1 - \nu^2)} \left( \frac{h}{R} \right)^2 \frac{(3 + 2n^2)}{7(38 - 20n^2 + 12n^4)} \right] \frac{f}{\pi n} - \frac{14Q \sin \Omega t}{\pi n \rho^* h}
\]

From Eq. (21) the fundamental frequencies of natural vibration of the shell \(\omega_n\) can be determined by the relation

\[
\omega_n^2 = \frac{E_1^*}{\rho^* R^2} \left[ \frac{7(E_1^* E_3^* - E_2^*^2)}{4\alpha^4 E_1^*^2(1 - \nu^2)} \left( \frac{h}{R} \right)^2 \frac{(3 + 2n^2)}{7(38 - 20n^2 + 12n^4)} \right]
\]
The FGM shallow shell considered here is a spherical cap of radius \( R = 5 \text{m} \), base radius \( r_0 = 0.6R \), i.e. \( \alpha = 0.6 \), thickness \( h = 0.01 \text{m} \). The materials mixture consists of Silicon nitride \( \text{Si}_3\text{N}_4 \) \((E_c = 348.43 \times 10^9 \text{N/m}^2, \rho_c = 2370 \text{kg/m}^3)\) and Stainless steel SUS 304 \((E_m = 201.04 \times 10^9 \text{N/m}^2, \rho_m = 8166 \text{kg/m}^3)\). The Poisson ratio is chosen to be 0.3 for simplicity.

Five first natural frequencies of FGM spherical shells are shown in the Table 1.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \omega_1 )</th>
<th>( \omega_2 )</th>
<th>( \omega_3 )</th>
<th>( \omega_4 )</th>
<th>( \omega_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 (Ceramic)</td>
<td>670.8294</td>
<td>685.9349</td>
<td>734.8552</td>
<td>825.8576</td>
<td>837.2293</td>
</tr>
<tr>
<td>1</td>
<td>398.6771</td>
<td>408.2874</td>
<td>435.6813</td>
<td>491.8383</td>
<td>498.6416</td>
</tr>
<tr>
<td>2</td>
<td>350.8065</td>
<td>358.4667</td>
<td>384.6829</td>
<td>431.4895</td>
<td>437.4192</td>
</tr>
<tr>
<td>( \infty ) (Metal)</td>
<td>274.5143</td>
<td>280.6957</td>
<td>300.7148</td>
<td>337.9545</td>
<td>342.6079</td>
</tr>
</tbody>
</table>

Obviously the natural frequencies of FGM shallow spherical shells are observed to be dependent on the constituent volume fraction, they decrease when increasing the power law index \( k \). This is attributed due to the stiffness reduction because of the increase in the metallic volumetric fraction. When \( k = 0 \), representing a full ceramic shell the natural frequencies are considerably greater than frequencies of other FGM shells, when \( k = \infty \) representing a full metal shell the frequencies are smaller than those of others. The reason is the higher value of assumed modulus of elasticity of the ceramic constituent and the lower value of elasticity modulus of the metal constituent.

The non-linear dynamic responses of FGM spherical shells acted on by the harmonic uniformly external pressure load \( q(t) = Q \sin \Omega t \) are obtained by solving Eq. (21) combined with the initial conditions and by use of the Runge-Kutta method. Fig. 2 shows non-linear responses of FGM spherical shells with various power law indices subjected to the excited load of magnitude \( Q = 3000 \text{N/m}^2 \) and frequency \( \Omega = 300(\text{s}^{-1}) \) different far from the natural frequencies of FGM shells with \( k = 0, 1, 2 \).

From the obtained results we can see that amplitudes of non-linear vibration of FGM shells increase when increasing the power law index \( k \) but frequencies decrease when increasing \( k \). The non-linear dynamic responses perform the phenomenon of periodic cycles.

The pictures of non-linear responses of FGM shells are quite different when the frequencies of excited load are near to the natural frequencies of shells. Indeed, in the following examples we consider the non-linear dynamic responses of functionally graded spherical caps with various power law indices under external pressure loads \( Q \sin \Omega t \), where the external frequencies \( \Omega \) are chosen near to the natural frequencies and the external amplitudes are taken in two cases: (a) \( Q = 1500 \text{N/m}^2 \) and (b) \( Q = 500 \text{N/m}^2 \) respectively. Fig. 3a and Fig. 3b show the non-linear responses of a full ceramic shell (with \( k=0 \)) , the natural frequency of which is \( \omega = 837, 2293(\text{s}^{-1}) \) and the frequency of external load is taken as \( \Omega = 812, 0760(\text{s}^{-1}) \).
Fig. 2. Non-linear dynamic responses of FGM spherical shells with various $k$

Fig. 3a. Non-linear responses of a full ceramic shell with $k = 0$ in case (a)

Fig. 3b. Non-linear responses in case (b)

Fig. 4a and Fig. 4b show the non-linear responses of a FGM shell with power law index $k=1$, the natural frequency of the shell $\omega = 498.6416 s^{-1}$ and external frequency $\Omega = 466.2267 s^{-1}$

The non-linear responses of the FGM shell with power law index $k=2$ are indicated in the Fig. 5a and Fig. 5b, where the natural frequency of the shell $\omega = 437.4192 s^{-1}$ and external frequency $\Omega = 402.4303 s^{-1}$

And the non-linear responses of a full metal shell ($k = \infty$) are presented in the Fig. 6a and Fig. 6b with $\omega = 342.6079 s^{-1}$ and $\Omega = 303.2078 s^{-1}$

From obtained results we observe the interesting phenomenon like harmonic beat of a linear vibration, where the number of the beats is greater when increasing the power law index $k$, i.e. the period of beat of non-linear responses is shorter when increasing the index $k$. Furthermore frequencies of non-linear vibration of FGM shells also decrease when increasing the index $k$. Decreasing the amplitude of external load leads to decreasing the
vibration amplitude, of course, but the period of beat, the frequency of non-linear vibration of FGM shells are unchanged.

Next, the detailed investigation for dynamical characteristics of functionally graded spherical caps is carried out for different geometrical parameters. According to the relation
calculated values of natural frequencies of the FGM spherical caps with $k=1$ corresponding to different dimension ratios $R/h$ and $H/r_0$, where $H = R \left(1 - \sqrt{1 - \left(\frac{r_0}{R}\right)^2}\right)$ is the rise of the shell, are given in the Table 2 and Table 3 respectively.

Table 2. Natural frequencies of FGM spherical cap with $k = 1, \alpha = 0, 6$ versus ratios $R/h$

<table>
<thead>
<tr>
<th>$R/h$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>$\omega_4$</th>
<th>$\omega_5$</th>
</tr>
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<tr>
<td>600</td>
<td>323.1338</td>
<td>337.6230</td>
<td>341.7290</td>
<td>409.4741</td>
<td>415.4575</td>
</tr>
<tr>
<td>500</td>
<td>398.6771</td>
<td>408.2874</td>
<td>435.6813</td>
<td>491.8383</td>
<td>498.6416</td>
</tr>
<tr>
<td>400</td>
<td>522.5359</td>
<td>517.5065</td>
<td>599.0702</td>
<td>615.8767</td>
<td>623.5149</td>
</tr>
<tr>
<td>300</td>
<td>761.7481</td>
<td>710.1611</td>
<td>936.6334</td>
<td>824.2681</td>
<td>831.9664</td>
</tr>
<tr>
<td>200</td>
<td>1.3840e+003</td>
<td>1.1472e+003</td>
<td>1.8740e+003</td>
<td>1.2496e+003</td>
<td>1.2506e+003</td>
</tr>
</tbody>
</table>

Table 3. Natural frequencies of FGM spherical cap with $k = 1$ versus ratios $H/r_0$

<table>
<thead>
<tr>
<th>$H/r_0$</th>
<th>$\omega_1$</th>
<th>$\omega_2$</th>
<th>$\omega_3$</th>
<th>$\omega_4$</th>
<th>$\omega_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.268</td>
<td>1.2550e+003</td>
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Fig. 7. Effect of dimension ratio $R/h$ on non-linear dynamic responses

Fig. 7 shows the effect of dimension ratios $R/h$ on the non-linear responses of FGM spherical cap with $k = 1$ under external load with $\Omega = 350s^{-1}$ and $Q = 1500N/m^2$.

Effect of dimension ratio $H/r_0$ on non-linear dynamic responses of FGM spherical cap under external load with $\Omega = 350s^{-1}$ and $Q = 3000N/m^2$ is presented in the Fig. 8.
It can be noted that the rate of decrease in the natural frequencies is high with the increase in dimension ratio $R/h$; but with the decrease in dimension ratios $H/r_0$. What about amplitudes of non-linear dynamic responses we can see opposite phenomenon, i.e. they increase when increasing ratio $R/h$ or decreasing ratio $H/r_0$.

5. CONCLUSIONS

The governing equations for non-linear dynamical analysis of functionally graded shallow spherical shells, including geometric non-linearity, are derived. Derivations are based on the classical shell theory and with the assumption of power law composition for the constituent materials.

Non-linear vibration of functionally graded shallow spherical shells is studied. Frequencies of non-linear vibration decrease, while amplitudes increase when increasing the power law index. The non-linear dynamic responses perform the phenomenon like periodic cycles when the frequency of external load is far from natural frequency of the shell and the phenomenon like harmonic beat of a linear vibration when the frequency of external load is near to natural frequency of the shell.

From the obtained results it is observed that the dynamical characteristics of FGM shallow spherical shells significantly depend on the value of material power-law index, apart from geometric shell parameters.

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