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Short Communication

DUAL APPROACH TO AVERAGED VALUES OF FUNCTIONS:

ADVANCED FORMULAS

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Abstract. Averaged values play major roles in the study of dynamic processes. The use of those values allows transforming varying processes to some constant characteristics that are much easier to be investigated. In order to extend the use of averaged values one may apply the dual approach which suggests a consideration of two different aspects of a problem in question. This short communication proposes new advanced formulas for averaged values of functions based on the dual conception.

Key words: advanced formula, averaged value, global-local averaged, dual approach, Duffing oscillator.

1. INTRODUCTION

Averaged values play major roles in the study of dynamic processes. The use of those values allows transforming varying processes to some constant characteristics that are much easier to be investigated. In order to extend the use of averaged values one may apply the dual approach recently proposed and developed in [1]-[2]. One of significant advantages of the dual conception is its consideration of two different aspects of a problem in question which allows investigation to be more appropriate. This advantage has been used in [3] to propose some new global and local averaged values of functions. The main objective of the short communication is to present a further extension of global-local averaged values of function. New advanced formulas for averaged values of functions are proposed based on the dual conception.

2. ADVANCED FORMULAS FOR AVERAGED VALUES OF DETERMINISTIC FUNCTIONS

Let $y(x)$ be an integrable deterministic function of $x \in [a, b]$. The following advanced formulas for averaged value of $y(x)$ can be considered.

Classical formula. Define

$$I^0(y(x), a, b) \;=\; y(x) \tag{1}$$

Classical formula for averaged value of $y(x)$ is well-known as follows

$$< y(x) > \;=\; < I^0(y(x), a, b) > = \;\frac{1}{b-a}\int_a^b I^0(y(x), a, b)dx = \frac{1}{b-a}\int_a^b y(x)dx. \tag{2}$$

Formula I.1. Define a function of $x \in [a, b]$ by

$$I^1(y(x), a, b) \;=\; \frac{1}{2}\Big(\frac{1}{x-a}\int\limits_a^x y(\sigma)d\sigma + \frac{1}{b-x}\int\limits_x^b y(\sigma)d\sigma\Big). \tag{3}$$

Formula I.1 for averaged value of $y(x)$ is defined by

$$< y(x) >_1 \;=< I^1(y(x), a, b) > = \;\frac{1}{b-a}\int_a^b I^1(y(x), a, b)dx. \tag{4}$$

Formula I.2. Define

$$\begin{aligned} I^2(y(x), a, b) \;&=\; I^1(I^1(y(x), a, b), a, b) = \\ &= \frac{1}{2}\Big(\frac{1}{x-a}\int\limits_a^x I^1(y(\sigma), a, b)\, d\sigma + \frac{1}{b-x}\int\limits_x^b I^1(y(\sigma), a, b)\, d\sigma\Big). \end{aligned} \tag{5}$$

Formula I.2 for averaged value of $y(x)$ is defined by

$$< y(x) >_2 \;=< I^2(y(x), a, b)) > = \;\frac{1}{b-a}\int_a^b I^2(y(x), a, b))dx. \tag{6}$$

Analogously, one can define

$$I^n(y(x), a, b) \;=\; I^1(I^{n-1}(y(x), a, b), a, b) \tag{7}$$

and Formula I.n for averaged value of $y(x)$ is defined by

$$< y(x) >_n \;=< I^n(y(x), a, b) > = \;\frac{1}{b-a}\int_a^b I^n(y(x), a, b)dx. \tag{8}$$

Further, according to (3) define

$$I^1(y(r), a, x) \;=\; \frac{1}{2}\Big(\frac{1}{r-a}\int\limits_a^r y(\sigma)d\sigma + \frac{1}{x-r}\int\limits_r^x y(\sigma)d\sigma\Big),$$

$$I^1(y(s), x, b) = \frac{1}{2}(\frac{1}{s-x}\int\limits_x^s y(\sigma)d\sigma + \frac{1}{b-s}\int\limits_s^b y(\sigma)d\sigma), \tag{9}$$

and

$$A^1(y(x), a, b) = \frac{1}{2}(\frac{1}{x-a}\int\limits_a^x I^1(y(r), a, x)dr + \frac{1}{b-x}\int\limits_x^b I^1(y(s), x, b)ds). \tag{10}$$

It is seen that $I^1(y(r), a, x)$ is a function of $r \in [a, x]$, $I^1(y(s), x, b)$ is a function of $s \in [x, b]$, and $A^1(y(x), a, b)$ is a function of $x \in [a, b]$. Formula A.1 for averaged value of function $y(x)$ is defined by

$$< y(x) >^1 \; =< A^1(y(x), a, b) > = \frac{1}{b-a}\int_a^b A^1(y(x), a, b)dx, \tag{11}$$

Analogously, one can define

$$A^n(y(x), a, b) = A^1(A^{n-1}(y(x), a, b), a, b) \tag{12}$$

and Formula A.n for averaged value of $y(x)$ is defined by

$$< y(x) >^n \; =< A^n(y(x), a, b) > = \frac{1}{b-a}\int_a^b A^n(y(x), a, b)dx. \tag{13}$$

One can also define mixed products as

$$A^n(I^m(y(x), a, b), a, b) = A^1(A^{n-1}(I^m(y(x), a, b), a, b), a, b) \tag{14}$$

and Formula A.nI.m for averaged value of $y(x)$ as

$$< y(x) >^n_m \; =< A^n(I^m(y(x), a, b), a, b) > = \frac{1}{b-a}\int_a^b A^n(I^m(y(x), a, b), a, b)dx. \tag{15}$$

3. APPLICATION TO NONLINEAR EQUATIONS

For illustration of possible uses of the proposed advanced formulas consider the following equation

$$\ddot{z} + \gamma z^3 = 0, \; z(0) = a, \; \dot{z}(0) = 0. \tag{16}$$

Let x is a solution of the linear equation

$$\ddot{x} + kx = 0, \; x(0) = a, \; \dot{x}(0) = 0. \tag{17}$$

The equation error is to be

$$e(x) = \gamma x^3 - kx$$

The value of k can be determined from a minimum condition, for example,

$$L(e^2(x)) \ \to \ \min_k \tag{18}$$

where L is a linear averaging operator. Thus, from (18) one has

$$\omega^2 = k = \gamma \frac{L\left(x^4\right)}{L\left(x^2\right)} \tag{19}$$

- Taking

$$L(\cdot) = \frac{1}{a}\int_0^a (\cdot)dx \tag{20}$$

one gets

$$\omega_0^2 = k_0 = \gamma \frac{\frac{1}{a}\int_0^a x^4 dx}{\frac{1}{a}\int_0^a x^2 dx} = \gamma \frac{a^4}{5}\frac{3}{a^2} = \frac{3}{5}\gamma a^2, \tag{21}$$

or,

$$\omega_0 = \sqrt{\frac{3}{5}\gamma a^2} = 0.77460\sqrt{\gamma}a. \tag{22}$$

- Taking

$$L(\cdot) = < \cdot >_1 = \frac{1}{a}\int_0^a I^1(\ \cdot\ , 0, a)dx, \tag{23}$$

one gets

$$\omega_1^2 = k_1 = \gamma \frac{< x^4 >_1}{< x^2 >_1} = \gamma \frac{\frac{1}{a}\int_0^a \frac{1}{10}(a^4 + a^3x + a^2x^2 + ax^3 + 2x^4)dx}{\frac{1}{a}\int_0^a \frac{1}{6}(a^2 + ax + 2x^2)dx} =$$

$$= \gamma \frac{149a^4}{600}\frac{36}{13a^2} = \frac{447}{650}\gamma a^2,$$

or,

$$\omega_1 = \sqrt{\frac{447}{650}\gamma a^2} = 0.82927\sqrt{\gamma}a. \tag{24}$$

- Taking

$$L(\cdot) = < \cdot >_2 = \frac{1}{a}\int_0^a I^2(\ \cdot\ , 0, a)dx, \tag{25}$$

one gets

$$\omega_2^2 = k_2 = \gamma \frac{< x^4 >_2}{< x^2 >_2} = \gamma \frac{\frac{1}{a}\int_0^a \frac{1}{1200}(209a^4 + 119a^3x + 79a^2x^2 + 54ax^3 + 48x^4)dx}{\frac{1}{a}\int_0^a \frac{1}{72}(19a^2 + 10ax + 8x^2)dx} =$$
$$= 0.26494\frac{27}{10}\gamma a^2 = 0.71535\gamma a^2, \quad (26)$$

or,

$$\omega_2 = \sqrt{0.71535\gamma a^2} = 0.84578\sqrt{\gamma}a. \quad (27)$$

Comparing the solution (27) with the exact solution ω_e and the one ω_l obtained by the classical criterion of equivalent linearization, respectively,

$$\omega_e = 0.847\sqrt{\gamma}a, \ \omega_l = 0.866\sqrt{\gamma}a, \quad (28)$$

one shows a significant accuracy of (27).

4. CONCLUSION

In this short communication the main idea of the dual conception is further extended to suggest new advanced formulas for averaged values of functions. These advanced formulas contain the classical formula of averaged value as a particular case. In the example of Duffing oscillator it is shown that advanced formulas can give a series of approximate solutions that are more accurate than the conventional one obtained by the classical criterion of equivalent linearization.

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