QUADRATIC AND CUBIC NON-LINEARITIES IN A QUASI-LINEAR FORCED SYSTEM

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Summary. In [3], the difference between the quadratic non-linearity and the cubic one in a quasi-linear parametrically excited system has been analyzed. In the present paper, the same question will be examined for a quasi-linear forced system and analogous results as in [3] will be obtained.

§1. SYSTEM UNDER CONSIDERATION AND DIFFERENT FORMS OF ITS DIFFERENTIAL EQUATION

Let us consider a quasi-linear forced system, described by the differential equation:

\[ \ddot{x} + \nu^2 x = \beta x^2 - \gamma x^3 + \Delta x - h \dot{x} + q \cos \nu t \]  (1.1)

where \( q > 0, \nu > 0 \) are intensity and frequency of forced excitation, respectively; the signification of other symbols has been explained in [3].

Assuming that the order of smallness of \( h \) and \( q \) is \( \varepsilon^2 \), the differential equation (1.1) can be written in the following forms, depending on the orders of smallness of \( \beta, \gamma \) and \( \Delta \):

- if \( \beta, \gamma, \Delta \) are of order \( \varepsilon^2 \), we have:
  \[ \ddot{x} + \nu^2 x = \varepsilon^2 \{ \beta x^2 - \gamma x^3 + \Delta x - h \dot{x} + q \cos \nu t \} \]  (1.2)

- if \( \beta, \gamma, \Delta \) are of order \( \varepsilon \), we have:
  \[ \ddot{x} + \nu^2 x = \varepsilon \{ \beta x^2 - \gamma x^3 + \Delta x \} + \varepsilon^2 \{ - h \dot{x} + q \cos \nu t \} \]  (1.3)

- at last, if \( \beta \) is of order \( \varepsilon \) while \( \gamma \) and \( \Delta \) are of order \( \varepsilon^2 \), we have:
  \[ \ddot{x} + \nu^2 x = \varepsilon \{ \beta x^2 \} + \varepsilon^2 \{ - \gamma x^3 + \Delta x - h \dot{x} + q \cos \nu t \} \]  (1.4)

As in [3], the case in which \( \gamma \) and \( \Delta \) are of different orders is rejected and, for the sake of simplicity, \( \gamma \) is assumed to be positive.

§2. SYSTEM WITH THE NON-LINEARITIES OF ORDER \( \varepsilon^2 \)

First, we shall examine the case described by the differential equation (1.2). As in [3], the asymptotic method is used and we obtain successively
\[-2\nu A_2 = h\nu a + q\sin \theta,\]
\[-2\nu a B_2 = \Delta a - \frac{3}{4}\gamma a^3 + q\cos \theta,\]
and
\[\dot{a} = -\frac{a}{2\nu}\{h\nu + \frac{q}{a}\sin \theta\},\]
\[\dot{\theta} = -\frac{1}{2\nu}\{\Delta - \frac{3}{4}\gamma a^2 + \frac{q}{a}\cos \theta\}.\]

Setting the right-hand sides of (2.2) equal to zero yields:

\[h\nu + \frac{q}{a}\sin \theta = 0,\]
\[\Delta - \frac{3}{4}\gamma a^2 + \frac{q}{a}\cos \theta = 0,\]

and, after eliminating \(\theta\), the relationship between the amplitude \(a\) and the frequency \(\nu\) of the stationary oscillation is obtained:

\[W(a, \nu^2) = a^2 \left\{ \left[ \frac{3}{4}\gamma a^2 - (\nu^2 - 1) \right]^2 + h^2\nu^2 \right\} - q^2 = 0.\]

To study the stability of the stationary oscillation \((a_0, \theta_0)\) the perturbations \(\delta a, \delta \theta\) are introduced, namely:

\[\delta a = a - a_0, \quad \delta \theta = \theta - \theta_0.\]

It is easy to establish the variational system:

\[\left(\delta a\right)' = -\frac{q}{2\nu a_0} \sin \theta_0 \cdot \delta a - \frac{q}{2\nu} \cos \theta_0 \cdot \delta \theta,\]
\[\left(\delta \theta\right)' = -\frac{1}{2\nu a_0} \left\{ -2 \cdot \frac{3}{4}\gamma a_0^2 - \frac{q}{a_0}\cos \theta_0 \right\} \delta a + \frac{q}{2\nu a_0} \sin \theta_0 \cdot \delta \theta,\]

and its characteristic equation:

\[\rho^2 + h\rho + \frac{1}{8\nu^2 a_0} \cdot \frac{\partial W(a_0, \nu^2)}{\partial a_0} = 0.\]

Since \(h > 0\), the sufficient condition for stability is:

\[\frac{\partial W(a_0, \nu^2)}{\partial a_0} > 0.\]

Obviously, in the case considered, the system is nearly identical to the classical forced one [1, 2]: the hardness of the system is determined by the cubic non-linearity and, in the first approximation, the quadratic non-linearity does not affect the amplitude, the phase as the stability of the stationary oscillation.

§3. SYSTEM WITH THE NON-LINEARITIES OF ORDER \(\epsilon\)

For the second case, described by the differential equation (1.3), the unknown functions in the asymptotic expansion are:
\[ A_1 = 0, \quad B_1 = -\frac{1}{2\nu} \left( \Delta - \frac{3}{4} \gamma a^2 \right), \]
\[ u_1 = \frac{\beta a^2}{2\nu^2} - \frac{\beta a^2}{6\nu^2} \cos 2\psi + \frac{2a^2}{32\nu^2} \cos 3\psi, \]
\[ -2\nu A_2 = \nu a + q \sin \theta, \]
\[ -2\nu a B_2 = \frac{5\gamma^2}{6\nu^2} - \frac{3\gamma^2}{12\nu^2} a^2 + \frac{\alpha}{4\nu^2} \left[ \Delta - \frac{3}{4} \gamma a^2 \right]^2 + q \cos \theta, \]

and the differential equations for \( a \) and \( \theta \), in the second approximation, are of the form:
\[ \dot{a} = -a \left( \frac{\gamma}{a} \sin \theta + \frac{\sigma}{a} \right), \]
\[ \dot{\theta} = -\frac{1}{2\nu} \left( \frac{5}{24\nu^2} \left( \frac{3}{4} \gamma a^2 \right)^2 - \frac{3\nu^2 - 1 - 2\sigma}{2\nu^2} \left( \frac{3}{4} \gamma a^2 \right) + \left( \nu^2 - 1 \right) + \frac{\left( \nu^2 - 1 \right)^2}{4\nu^2} + \frac{q}{a} \cos \theta \right), \]

where \( \sigma = \frac{5\gamma^2}{6} : 372. \)

The amplitude \( a \) and the phase \( \theta \) of the stationary oscillation satisfy the equations:
\[ h\nu + \frac{\gamma}{a} \sin \theta = 0, \]
\[ \frac{5}{24\nu^2} \left( \frac{3}{4} \gamma a^2 \right)^2 - \frac{3\nu^2 - 1 - 2\sigma}{2\nu^2} \left( \frac{3}{4} \gamma a^2 \right) + \left( \nu^2 - 1 \right) + \frac{\left( \nu^2 - 1 \right)^2}{4\nu^2} + \frac{q}{a} \cos \theta = 0, \]

and the relationship between \( a \) and \( \nu \) is:
\[ W(a, \nu^2) = a^2 \left\{ \left[ \frac{5}{24\nu^2} \left( \frac{3}{4} \gamma a^2 \right)^2 - \frac{3\nu^2 - 1 - 2\sigma}{2\nu^2} \left( \frac{3}{4} \gamma a^2 \right) + \left( \nu^2 - 1 \right) + \frac{\left( \nu^2 - 1 \right)^2}{4\nu^2} \right]^2 + k^2 \nu^2 \right\} - q^2 = 0 \]

It is noted that, for each given value \( \nu^2 \) in the neighbourhood of unity, the algebraic equation (3.4) of unknown \( a^2 \) has no more three acceptable solutions. Indeed, (3.4) can be rewritten in the form:
\[ \left[ \frac{5}{24\nu^2} X^2 - \frac{3\nu^2 - 1 - 2\sigma}{2\nu^2} X + \left( \nu^2 - 1 \right) + \frac{\left( \nu^2 - 1 \right)^2}{4\nu^2} \right]^2 = \frac{Q^2}{X} - k^2 \nu^2, \]

where:
\[ X = \frac{3}{4} \gamma a^2, \quad Q^2 = \frac{3}{4} \gamma q^2 \]

In the plane \((XY)\) let us draw the graphs \( Y_1 \) and \( Y_2 \) of the left and right-hand sides of (3.5), respectively (Fig. 1). The graph \( Y_1 \) has two minima on the abscissa axis: \( M_1 \) near \( X = 0 \) and \( M_2 \) near \( X = 24/5 \). The graph \( Y_2 \) is a hyperbola quite near its asymptotes: the abscissa line \( Y = -h^2 \nu^2 \). For \( h = 0 \) the mentioned graphs intersect themselves at two points \( P \) and \( Q \) located in the neighbourhood and on both sides of \( M_2 \). When \( h \) increases, the graph \( Y_2 \) is shifted downward, \( P \) and \( Q \) approach \( M_2 \) then coincide in it and finally disappear. Obviously, the abscissae of \( P \) and \( Q \) are two solutions of the equation (3.5). However, these solutions must be rejected since the corresponding values \( a^2 = 4X/h^2 \) are too large (for standard variable). Thus, the equation (3.5) (i.e., the equation (3.4)) admits no more three acceptable solutions near \( X = 0 \).

To study the stability of the stationary oscillation \((a_0, \theta_0)\), we use:
- the variational system:

\[ (\delta a) = \frac{\nu}{2\nu a_0} \sin \theta_0 \cdot \delta a - \frac{\sigma}{2\nu} a_0 \cdot \delta \theta, \]
\[ (\delta \theta) = -\frac{1}{2\nu a_0} \left\{ 4 \cdot \frac{5}{24\nu^2} \left( \frac{3}{4} \gamma a_0^2 \right)^2 - 2 \cdot \frac{3\nu^2 - 1 - 2\sigma}{2\nu^2} \left( \frac{3}{4} \gamma a_0^2 \right) - \frac{q}{a} \cos \theta_0 \right\} \delta a + \frac{\nu}{2\nu a_0} \sin \theta_0 \cdot \delta \theta, \]
and its characteristic equation:
\[
\rho^2 + k_0 + \frac{1}{8\nu \alpha_0} \frac{\partial W(\alpha_0, \nu^2)}{\partial \alpha_0} = 0.
\]  
From (3.7) the following sufficient stability condition is deduced:
\[
\frac{\partial W(\alpha_0, \nu^2)}{\partial \alpha_0} > 0.
\]  
\[\text{Fig. 1}\]

The structure of (3.4) shows that the character (soft-hardness) of the system is determined by the cubic non-linearity \(\gamma \) (\(\gamma > 0\), the system belongs to hard kind) and the quadratic one \(\beta \) (\(\beta \) - of order \(\varepsilon\)) plays only a supplementary role (it makes the system less hard). Figure 2 shows the resonance curves for the typical case:
\[
h^2 = 0.0002; \quad q^2 = 0.00025; \quad \frac{3}{4} \gamma = 0.24; \quad \sigma = 0 \text{(a) and } \sigma = 0.05 \text{(b)}
\]

§4. SYSTEM WITH THE NON-LINEARITIES OF DIFFERENT ORDERS

In the third case, described by the differential equation (1.4), the quadratic non-linearity is of order \(\varepsilon\) while the cubic one is of order \(\varepsilon^2\). The asymptotic method gives us successively:
\[
A_1 = 0, \quad B_1 = 0, \quad u_1 = \frac{\beta \alpha^2}{2\nu^2} - \frac{\beta \alpha^2}{6\nu^2} \cos 2\phi,
\]
\[- 2\nu A_2 = h\nu a + q \sin \delta,
\]
\[- 2\nu B_2 = \left(\frac{5\beta^2}{6\nu^2} - \frac{3}{4}\right) \alpha^2 + \Delta a + q \cos \theta,
\]
and
\[
\dot{a} = - \frac{a}{2\nu} \left\{ h\nu + \frac{q}{a} \sin \delta \right\},
\]
\[
\dot{\theta} = - \frac{1}{2\nu} \left\{ \left( \frac{1}{\nu^2} - \frac{1}{\nu^2} \frac{5\beta^2}{6} \alpha^2 + (\nu^2 - 1) + \frac{q}{a} \cos \theta \right) \right\}.
\]
where
\[
\frac{1}{\nu_1^2} = \frac{3}{4\gamma} + \frac{5\beta^2}{6}.
\]

The equations for \(a\) and \(\vartheta\) in the stationary oscillation are:
\[
\begin{align*}
\hbar \nu + \frac{q}{a} \sin \vartheta &= 0, \\
\left( \frac{1}{\nu^2} - \frac{1}{\nu_1^2} \right) \frac{5\beta^2}{6} a^2 + (\nu^2 - 1) \frac{q}{a} \cos \vartheta &= 0,
\end{align*}
\]
and the relationship between \(a\) and \(\nu\) is:
\[
W(a, \nu^2) = a^2 \left\{ \left[ \left( \frac{1}{\nu^2} - \frac{1}{\nu_1^2} \right) \frac{5\beta^2}{6} a^2 + (\nu^2 - 1) \frac{q}{a} \cos \vartheta \right] \right\} - \frac{q^2}{a} = 0.
\]

To study the stability of the stationary oscillation \((a_0, \vartheta_0)\) we use:
- the variational system:
\[
(\delta a)^{\prime} = -\frac{q}{2\nu a_0} \sin \vartheta_0 \cdot \delta a - \frac{q}{2\nu} \cos \vartheta_0 \cdot \delta \vartheta,
\]
\[
(\delta \vartheta)^{\prime} = -\frac{1}{2\nu a_0} \left\{ 2 \left( \frac{1}{\nu^2} - \frac{1}{\nu_1^2} \right) \frac{5\beta^2}{6} a_0^2 - \frac{q}{a_0} \cos \vartheta_0 \right\} \delta a + \frac{q}{2\nu a_0} \sin \vartheta_0 \cdot \delta \vartheta,
\]
- and its characteristic equation:
\[
\rho^2 + h^2 + \frac{1}{8\nu^2 a_0} \frac{\partial W(a_0, \nu^2)}{\partial a_0} = 0
\]
The stability condition is as previously:
\[
\frac{\partial W(a_0, \nu^2)}{\partial a_0} > 0
\]

Analogous to the corresponding case in [3], the interesting phenomenon in the last case is that the character (soft-hardness) of the system under consideration depends on \(\nu_1^2\) i.e. on the ratio of \(\beta^2\) and \(\gamma:\)
- if \(\nu_1^2\) is enough greater than 1, the system belongs to soft kind,
- if \(\nu_1^2\) is enough less than 1, the system becomes hard one,
- if \(\nu_1^2\) is close to 1, the system is neutralized, it becomes a linear one.

Figure 3 shows the resonance curves for the typical case: \(h^2 = 0.0003;\ \frac{q^2}{a^2} = 0.0003;\ \frac{5\beta^2}{6} = 0.04\) and \(\frac{3\gamma}{4} = 0\ (a)\ (the\ resonance\ curve\ leans\ to\ the\ left),\ \frac{3\gamma}{4} = 0.04\ (b),\ \frac{3\gamma}{4} = 0.08\ (c)\ (the\ resonance\ curve\ leans\ to\ the\ right).\)

CONCLUSION

The results obtained show the difference between the quadratic non-linearity and the cubic one. If the two non-linearities are of the same order of smallness, the cubic non-linearity is the dominant factor, on the contrary, if they are of different orders, the character (soft-hardness) of the system depends on both them, in equal degree.

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REFERENCES


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PHI TUYỂN BẠC HAI VÀ BẠC BA TRONG MỘT HỆ Á TUYỂN CUONGL BỨC

Tiếp tục van đề đạt ra trong [3], bài báo này xét tror các số hàng dần hồi phi tuyến bậc hai và bậc ba trong một hệ dao động á tuyến cường bức. Kết quả thu được cho thấy:
- Nếu hai số hàng phi tuyến nối trong cấp (e hoặc e²), số hàng bậc ba quyết định tính cùng mem của hệ, số hàng bậc hai chỉ có ảnh hưởng bổ sung;
- Nếu số hàng bậc hai & cấp e và số hàng bậc ba & cấp e², tính cùng mem của hệ phụ thuộc cả vào hai số hàng đó và vi vậy, tùy "ty so" giữa chúng, hệ có thể thuộc loại cùng hoặc mem hoặc bi "trung hoa".