FORCED OSCILLATION OF THE RECTANGULAR THIN PLATE ON THE ELASTIC FOUNDATION WITH TWO COEFFICIENTS

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§0. INTRODUCTION

Parametric oscillation of the rectangular thin plate on the elastic foundation with two coefficients, when making mention of the creep of material, has been investigated in earlier publications (see for example [2, 3]). However, forced oscillation of the rectangular thin plate, to the author's knowledge, has not been hitherto examined.

This problem is studied here by means of an asymptotic method for high-order systems [1] and boundary value problem [4].

§1. FORMULATION OF THE PROBLEM. THE EQUATION OF MOTION

Now, let's determine forced oscillation of a rectangular thin plate, having thickness \( h \), Young's modulus \( E \), specific mass \( M \) and lengths of edges \( b, c \), which is supported on four edges and lying on the elastic foundation with two coefficients as shown in Fig. 1.

Its motion is loaded by direction force, equally distributed \( q = q(t) \)

The mechanical properties of material, when being straining, has been described by the model of the standard linear body [5] (Fig. 2). Because of the state equation in operator is written by the following form

\[
\sigma = E\varepsilon
\]

\[
E = \frac{E_1 + K \left(1 + \frac{E_2}{E_1}\right) \frac{\partial}{\partial t}}{\left(1 + \frac{K}{E_2}\right) \frac{\partial}{\partial t}}
\]

Using the classical bending equation of plate with regarding the initial strain and the non-linear foundation, superseding the elastic modulus \( E \) by the analogous operator (1.2) into the
expression for the bending hardness

\[ D = \frac{Eh^3}{12(1 - \nu^2)} \]

we get the equation of the problem

\[
\frac{\partial^3 W}{\partial t^3} + \xi \frac{\partial^2 W}{\partial t^2} + \frac{1}{M} \left[ \frac{\partial}{\partial t} \left( D_1 \nabla^4 W - K_2 \nabla^2 W + K_1 W \right) + \xi \left( D_1 \nabla^4 W - K_2 \nabla^2 W + K_1 W \right) \right] = \\
= \frac{1}{M} \left[ \epsilon \left( \frac{\partial f}{\partial t} - \frac{E_2 D_1}{E_1} \frac{\partial}{\partial t} \nabla^4 W \right) + \xi q + \frac{\partial}{\partial t} q \right].
\]

The relevant homogeneous boundary conditions are as follow

\[
W \big|_{x=0,b} = 0, \quad \frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} \big|_{x=0,b} = 0,
\]
\[
W \big|_{y=0,c} = 0, \quad \frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \big|_{y=0,c} = 0.
\]

Here \( W = W(x, y, t) \) - sagging of cross-section, \( \xi = \frac{E_2}{K}, \nu \) - Poisson's ratio, \( K_1, K_2 \) - coefficients of the elastic foundation, \( f \) - elastic reaction, which is a non-linear function of \( \left( W, \frac{\partial W}{\partial x}, \ldots \right) \), \( \nabla^2 \) - Laplace's operator.

\[ D_1 = \frac{E_1 h^3}{12(1 - \nu^2)}, \quad \xi = \frac{E_2}{K} \]

For simplicity, it is supposed that \( M = 1, q = \varepsilon q_0 \sin \gamma t \), when the equation (1.3) is possibly written in the form

\[
\frac{\partial^3 W}{\partial t^3} + \xi \frac{\partial^2 W}{\partial t^2} + \frac{\partial}{\partial t} \left( D_1 \nabla^4 W - K_2 \nabla^2 W + K_1 W \right) + \xi \left( D_1 \nabla^4 W - K_2 \nabla^2 W + K_1 W \right) = \varepsilon F(\theta, x, y, W, \ldots),
\]

where \( d\theta/dt = \gamma \), \( F \) is a periodic function with period \( 2\pi \) relatively \( \theta \).

§2. CONSTRUCTION OF THE ASYMPTOTIC SOLUTION

When \( \varepsilon = 0 \), we have the boundary problem

\[
\frac{\partial^3 W}{\partial t^3} + \xi \frac{\partial^2 W}{\partial t^2} + \frac{\partial}{\partial t} \left( D_1 \nabla^4 W - K_2 \nabla^2 W + K_1 W \right) + \xi \left( D_1 \nabla^4 W - K_2 \nabla^2 W + K_1 W \right) = 0,
\]

with the boundary conditions

\[
W \big|_{x=0,b} = 0, \quad \frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} \big|_{x=0,b} = 0,
\]
\[
W \big|_{y=0,c} = 0, \quad \frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \big|_{y=0,c} = 0.
\]

The solution of this problem can be found in the form

\[ W_0(x, y, t) = Z(x, y)T(t) \]
Substituting (2.3) into (2.1) and (2.2) we obtain
\[
\frac{d^2 T}{dt^2} + \xi \frac{d^2 T}{dt^2} + \beta^2 D_1 \frac{dT}{dt} + \xi \beta^2 D_1 T = 0,
\]
(2.4)
\[
\nabla^4 Z - \frac{K_2}{D_1} \nabla^2 Z + \frac{K_1}{D_1} Z - \beta^2 Z = 0,
\]
(2.5)
\[
Z \bigg|_{x=0, b} = 0, \quad \frac{\partial^2 Z}{\partial x^2} + \nu \frac{\partial^2 Z}{\partial y^2} \bigg|_{x=0, b} = 0,
\]
(2.6)
\[
Z \bigg|_{y=0, c} = 0, \quad \frac{\partial^2 Z}{\partial x^2} + \nu \frac{\partial^2 Z}{\partial y^2} \bigg|_{y=0, c} = 0.
\]
(2.7)
It is easy seen that the solution (2.3) takes form
\[
W_0(x, y, t) = \sum_{r,s=1}^{\infty} A_{rs} Z_{r,s}(x, y) \cos \phi_{rs} + \sum_{r,s=1}^{\infty} D_{rs} Z_{r,s}(x, y) e^{-\xi t},
\]
(2.8)
\[
\Omega^2 = D_1 \beta^2, \quad Z_{rs}(x, y) = \sin \frac{r\pi x}{b} \sin \frac{s\pi y}{c}, \quad \phi_{rs} = (\Omega_{rs} t + \psi_{rs}),
\]
(2.9)
where $A_{rs}, D_{rs}, \psi_{rs}$ are positive constants determined from the initial conditions
\[
\Omega_{rs}^2 = \left\{ D_1 \left[ \left( \frac{r\pi}{b} \right)^2 + \left( \frac{s\pi}{c} \right)^2 \right] + K_1 \right\}.
\]
(2.10)
It is supposed that when $\varepsilon = 0$ there exists a periodic solution with frequency $\Omega_{11}$
\[
\Omega_{11}^2 = \gamma^2 + \varepsilon \delta,
\]
(2.11)
$\delta$ is the detuning coefficient.

With these assumptions, we are going to find the partial solution of the boundary value problem (1.4), (1.5) in the asymptotic form
\[
W(x, y, z) = a Z_{11}(x, y) \cos \phi + \varepsilon U_1(x, y, a, \phi, \delta) + \varepsilon^2 U_2(x, y, a, \phi, \delta) + \ldots
\]
(2.12)
where the functions $U_1, U_2, \ldots$ are periodic with period $2\pi$ relatively $\phi, \theta$, the quantities $a, \psi$ are determined from the equations
\[
\frac{da}{dt} = \varepsilon A_1(a, \psi) + \varepsilon^2 A_2(a, \psi) + \varepsilon^3 \ldots,
\]
(2.13)
\[
\frac{d\psi}{dt} = (\Omega_{11} - \gamma) + \varepsilon B_1(a, \psi) + \varepsilon^2 B_2(a, \psi) + \varepsilon^3 \ldots
\]
By substituting (2.12) into the equation (1.5) and paying attention to (2.13), (1.4), in the better first approximation we get
\[
L_3[U_1] + \xi L_3[U_1] + L_1[D_1 \Delta^2 U_1 + K_1 \nabla^2 U_1 + K_1 U_1] + \xi[D_1 \Delta^2 U_1 - K_2 \nabla^2 U_1 + K_1 U_1] +
\]
\[
+ \left( 2 \Omega_{11}^2 A_1 + 2 \xi \omega_{11} B_1 \right) \cos \phi + \left( 2 \xi \Omega_{11} A_1 - 2 \omega_{11} B_1 \right) \sin \phi \right|_{Z_{11}} = F_1,
\]
(2.14)
with the boundary conditions

\[
\begin{align*}
U_1 \bigg|_{x=0,b} &= 0, \quad \frac{\partial^2 U_1}{\partial x^2} + v \frac{\partial^2 U_1}{\partial y^2} \bigg|_{x=0,b} = 0, \\
U_1 \bigg|_{y=0,c} &= 0, \quad \frac{\partial^2 U_1}{\partial y^2} + v \frac{\partial^2 U_1}{\partial x^2} \bigg|_{y=0,c} = 0.
\end{align*}
\]

(2.15)

The operators

\[
\begin{align*}
L_1[U_1] &= \left( \Omega_{11} \frac{\partial}{\partial \phi} + \gamma \frac{\partial}{\partial \theta} \right) U_1, \quad L_2[U_1] = \left( \Omega_{11} \frac{\partial}{\partial \phi} + \gamma \frac{\partial}{\partial \theta} \right)^2 U_1, \\
L_3[U_1] &= \left( \Omega_{11} \frac{\partial}{\partial \phi} + \gamma \frac{\partial}{\partial \theta} \right)^3, \quad F_1 = F(\theta, x, y, aZ_1, \cos \phi, \ldots).
\end{align*}
\]

To find the function \( U_1 \), we shall expand \( F_1 \) and \( U_1 \) relatively the foundation functions \( \{ Z_{rs}(x, y) \} \)

\[
U_1 = \sum_{rs} U_{1rs}(a, \phi, \theta) Z_{rs}(x, y),
\]

(2.16)

\[
F_1 = \sum_{rs} F_{1rs}(a, \phi, \theta) Z_{rs}(x, y),
\]

(2.17)

where \( F_{1rs} \int_0^b \int_0^c Z_{rs} dx dy )/ \int_0^b \int_0^c Z_{rs}^2 dx dy \) are defined, still \( U_{1rs} \) need be determined. Writing \( U_1 \) in the form (2.16), the boundary conditions (2.15) are selfsatisfied.

Putting (2.16), (2.17) into the equation (2.14) and then comparing the coefficients of the functions \( Z_{rs}(x, y) \), we have the following equations for determining \( U_{1rs}, A_1, B_1 \)

\[
L_3[U_{111}] + \xi L_2[U_{111}] + \Omega_{11}^2 L_1[U_{111}] + \xi \Omega_{11}^2 U_{111} = F_{111} + (2\Omega_{11}^2 A_1 + 2\xi a\Omega_{11} B_1) \cos \phi + (2\xi \Omega_{11} A_1 - 2a\Omega_{11}^2 B_1) \sin \phi,
\]

(2.18)

\[
L_3[U_{1rs}] + \xi L_2[U_{1rs}] + \Omega_{11}^2 L_1[U_{1rs}] + \xi \Omega_{11}^2 U_{1rs} = F_{1rs},
\]

(2.19)

Now, we first expand \( F_{1rs} \) and \( U_{1rs} \) in the Fourier series

\[
F_{1rs} = \sum_{n,m} F_{1nms}(a) e^{i(n\phi + m\theta)},
\]

(2.20)

\[
U_{1rs} = \sum_{n,m} U_{1nms}(a) e^{i(n\phi + m\phi)}.
\]

(2.21)

Here \( F_{1nms} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} F_{1rs} e^{-i(n\phi + m\phi)} d\phi d\theta \) are known, yet \( U_{1nms} \) need be determined.

Suppose that \( U_{111} \) does not contain \( \cos \phi, \sin \phi \), putting the expressions (2.20) and (2.21) into the equation (2.18) and (2.19) and then comparing the coefficients of the functions \( e^{i(n\phi + m\phi)} \), we receive

\[
U_{1nms}(a) = \frac{\int_0^{2\pi} \int_0^{2\pi} F_{1rs}(a, \phi, \theta) e^{-i(n\phi + m\phi)} d\phi d\theta}{4\pi^2 [\xi + i(m\Omega_{11} + n\gamma)] [\Omega_{11}^2 + (m\Omega_{11} + n\gamma)^2] \int_0^{2\pi} \int_0^{2\pi} Z_{rs}^2 dx dy}.
\]

(2.22)
Consequently the function $U_1$ is determined by the following expression

$$
U_1 = \sum_{n,m,r,s=1}^{\infty} \int_0^b \int_0 c F_1 Z_{rr} e^{-i(n\phi + m\psi)} \, dx \, dy \, d\phi \, d\psi \, Z_{ss} e^{i(n\phi + m\psi)}
$$

$$
= \frac{1}{4\pi^2 \left[ \Omega^2_m - (m\Omega_1 + n\gamma)^2 \right]} \int_0^b \int_0 c Z_{ss} \, dx \, dy
$$

$$
= \frac{1}{4\pi^2 \left[ \Omega^2_m - (m\Omega_1 + n\gamma)^2 \right]} \neq 0.
$$

By comparing the coefficients of the function $\cos \phi$, $\sin \phi$ in (2.18), we get the following equational system to determine the quantities $A_1$, $B_1$

$$
2\Omega_1^2 A_1 + 2\xi a \Omega_1 B_1 = -\frac{1}{4\pi^2} \sum_{\sigma} e^{i\sigma \phi} \int_0^b \int_0 c F_{111} e^{-i(\phi - \sigma)} \cos \phi \, d\phi \, d\sigma = -G(a, \psi),
$$

$$
2\xi \Omega_1 A_1 - 2a \Omega_1^2 B_1 = -\frac{1}{4\pi^2} \sum_{\sigma} e^{i\sigma \phi} \int_0^b \int_0 c F_{111} e^{-i(\phi - \sigma)} \sin \phi \, d\phi \, d\sigma = -H(a, \phi).
$$

From here, we have

$$
A_1 = -\frac{(\Omega_1 G + \xi H)}{\Omega_1 (\Omega_1^2 + \xi^2)}, \quad B_1 = -\frac{(\xi G - \Omega_1 H)}{a \Omega_1 (\Omega_1^2 + \xi^2)}.
$$

Thus, in the better first approximation the solution of the given boundary problem (2.12) is determined.

§3. CONCRETE CASE

Suppose that the right hand side of the equation (1.5) is of the form

$$
F = \xi \left\{ -K_1 W^3 - K_2 \left[ \frac{2}{2} \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right] \right\} +
$$

$$
+ \frac{\partial}{\partial t} \left\{ -K_1 W^3 - K_2 \left[ \left( \frac{\partial W}{\partial x} \right)^2 + \left( \frac{\partial W}{\partial y} \right)^2 \right] \right\} +
$$

$$
+ \xi\gamma_0 \sin \gamma t + \gamma_0 \gamma \cos \gamma t - \frac{E}{E_1} D_1 \frac{\partial}{\partial t} (\nabla^4 W).
$$

(3.1)

Using the above expressions, in the first approximation we have

$$
W = a \sin \frac{\pi x}{b} \sin \frac{\pi y}{c} \cos(\gamma t + \psi).
$$

(3.2)

Here $a$, $\psi$ are determined from the following equational system

$$
2\gamma \frac{da}{dt} = -h \xi \gamma a - P_0 \cos \psi,
$$

$$
2\gamma \frac{d\psi}{dt} = a(\Omega_1^2 - \gamma^2) - Q a^3 + h \Omega_1^2 a + P_0 \sin \psi,
$$

(3.3)

where

$$
h = \frac{D_1 E_2}{E_1 (\Omega_1^2 + \xi^2)} \varepsilon, \quad P_0 = \frac{16\gamma_0}{\pi^2} \varepsilon,
$$

$$
Q = \frac{6}{64} \left[ -3K_1 + \frac{K_2}{2} \left( \frac{\pi^4}{b^4} + \frac{\pi^4}{c^4} \right) \right] \varepsilon.
$$

(3.4)

(3.5)
Vanishing the right part of the equation (3.3), we obtain the stationary solution \( a_0, \psi_0 \) related to the frequency \( \gamma \) and amplitude \( \eta_0 \) of the force \( \eta \)

\[
(f(a_0, \eta^2) = \left[a_0(\Omega_{11}^2 - \gamma^2) - Q_0a_0 + h\Omega_{11}^2a_0\right]^2 - P_0^2 + h^2 \xi^2 \eta_0^2 a_0^2 = 0, \quad (3.6)
\]

\[
\gamma^2 = -Q_0a_0^2 + (1 + h)\Omega_{11}^2 \pm \sqrt{P_0^2 - h^2 \xi^2 \eta_0^2 a_0^2}. \quad (3.7)
\]

The relation (3.7) is plotted in Fig. 3 for the case

\( Q = -1; \quad \Omega_{11}^2 = 1; \quad h^2 = 0.09; \quad P_0^2 = 0.045; \quad \xi^2 = 2, 3. \)

In Fig. 4 for the case

\( Q = +1; \quad \Omega_{11}^2 = 1; \quad h^2 = 0.09; \quad P_0^2 = 0.045; \quad \xi^2 = 2, 3. \)

In Fig. 5 for the case

\( Q = 0; \quad \Omega_{11}^2 = 1; \quad h^2 = 0.09; \quad P_0^2 = 0.045; \quad \xi^2 = 2, 3. \)

To study the stability of stationary oscillations, we set into the equation (3.3) for \( a = a_0 + \delta a \) and \( \psi = \psi_0 + \delta \psi \), where \( \delta a, \delta \psi \) are small perturbations. Neglecting the small quantities of higher order, we receive following variational equations for \( a_0 \neq 0 \)

\[
2\gamma \frac{d}{dt}(\delta a) = -h \xi \gamma \delta a + \frac{P_0}{\xi^2} \sin \psi_0 \delta \psi, \quad (3.8)
\]

\[
2a_0 \gamma \frac{d}{dt}(\delta \psi) = \left[\left(\Omega_{11}^2 - \gamma^2\right) - 3QA_0^2 + h\Omega_{11}^2\right] \delta a + \frac{P_0}{\xi^2} \cos \psi_0 \delta \psi.
\]

The characteristic equation of (3.8) is

\[
4\gamma^2 \lambda^2 + \left[\left(\Omega_{11}^2 - \gamma^2\right) - Q_0^2 + h\Omega_{11}^2\right]\left[\left(\Omega_{11}^2 - \gamma^2\right) - 3QA_0^2 + h\Omega_{11}^2\right] + h^2 \xi^2 \gamma^2 = 0 \quad (3.9)
\]

From here it is easy to see that the condition for stability stationary oscillation is

\[
\left[\left(\Omega_{11}^2 - \gamma^2\right) - Q_0^2 + h\Omega_{11}^2\right]\left[\left(\Omega_{11}^2 - \gamma^2\right) - 3QA_0^2 + h\Omega_{11}^2\right] + h^2 \xi^2 \gamma^2 > 0. \quad (3.10)
\]
The inequality (3.10) will be satisfied, when
\[
\frac{\partial f(a_0, r^2)}{\partial a_0} > 0. \tag{3.11}
\]
In Fig. (3, 4), the solid curves correspond to stable states of vibration where the stability condition (3.11) is being valid.

CONCLUSION

1. The equation of motion for a rectangular thin plate on an elastic foundation with two coefficients was set up. Its solution has been found by means of an asymptotic method for high-order systems, further the stability condition of the stationary oscillation has been investigated.

2. Taking into account two coefficients of the elastic foundation, the partial frequency increases. Thus, the problem is changed on the peculiarity.

3. From the presented Fig. 3, 4, 5 we can see the parameters \( b, c, K_1, K_2 \) may be chosen so that the system observed has hard or soft character.

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