

## **FREQUENCY ANALYSIS OF CRACKED FUNCTIONALLY GRADED CANTILEVER BEAM**

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### **ABSTRACT**

In this paper, a functionally graded cantilever beam with an open crack is investigated on the base of Timoshenko beam theory; power law of functionally graded material (FGM) and taking into account actual position of neutral axis instead of the central one. The open and edge crack is modeled by coupled translational and rotational springs stiffness of which is calculated by the formulas conducted accordingly to fracture mechanics. Using the frequency equation obtained in the framework of the theory natural frequencies of the beam are examined along the crack parameters and material properties. This analysis demonstrates that sensitivity of natural frequencies of FGM beam to crack is strongly dependent on the material constants of FGM.

*Keywords:* FGM, Timoshenko beam; cracked beam, modal analysis;

### **1. INTRODUCTION**

Due to advantage properties compared to the laminate composites the functionally graded material (FGM) has been intensively studied recently and got wide application in the high-tech industries. An overview of the problems for manufacturing, modelling and testing FGM was given in [1]. Numerous methods such as the Finite Element Method (FEM) [2]; Spectral Element Method (SEM) [3]; Dynamic Stiffness Method (DSM) [4] or Rayleigh-Ritz method [5] have been developed for analysis of structures made of FGM. Nevertheless, the analytical methods are still the most accurate and efficient for dynamic analysis of functionally graded beam-like structures [6-9]. While the most of the aforementioned studies investigated undamaged beam, the crack problem in FGM has been studied in [10-11]. The most important result of the studies is that a crack in FGM beam can be modeled by an equivalent spring of stiffness calculated from the crack depth. Based on the rotational spring model of crack, Yang and Chen [12] studied free vibration and buckling of Euler-Bernoulli FGM beam with edge cracks. They found that natural frequencies of FGM beam with smaller slenderness and lower ratio of the bottom Young's modulus to the top one are more sensitive to cracks. The transfer matrix method was employed by Wei et al. [13] for obtaining frequency equation of FGM beam

with arbitrary number of cracks in the form of third-order determinant. This simplifies significantly the modal analysis of multiple cracked FGM beam. Aydin [14] has conducted an expression for mode shape of FGM beam with multiple cracks and used it for constructing the frequency equation in the form of an explicit determinant of third-order also. Forced vibration and nonlinear free vibration of cracked FGM beam are investigated in Ref. [15-16]. Based on the exponential law of FGM and rotational spring model of crack, Yu and Chu [17] and Banerjee et al. [18] have applied the FEM and the Frequency Contour Method (FCM) for detecting a crack in Euler-Bernoulli and Timoshenko FGM beams, respectively. Nguyen Tien Khiem and Nguyen Ngoc Huyen obtained a condition for uncoupling of longitudinal and bending vibration in FGM beam and studied uncoupled flexural vibration of the beam [19].

In the present paper, an analytical approach in frequency domain is proposed to study free vibration of functionally graded Timoshenko beam with an open crack modeled by a pair of translational and rotational springs. This is a novelty of present paper in comparison with the previous ones where only rotational spring model of crack was adopted. Using the proposed model of crack, frequency equation of a cracked cantilever is conducted and used for sensitivity analysis of natural frequencies to crack parameters. Numerical results of natural frequencies as functions of crack positions and depths are obtained by MATLAB code.

## 2. GOVERNING EQUATIONS

### 2.1. Model of FGM beam

Consider a beam of length  $L$ , cross-section area  $A = b \times h$  made of FGM with material parameters varying along thickness by the power law

$$\begin{Bmatrix} E(z) \\ G(z) \\ \rho(z) \end{Bmatrix} = \begin{Bmatrix} E_b \\ G_b \\ \rho_b \end{Bmatrix} + \begin{Bmatrix} E_t - E_b \\ G_t - G_b \\ \rho_t - \rho_b \end{Bmatrix} \left( \frac{z}{h} + \frac{1}{2} \right)^n, \quad -h/2 \leq z \leq h/2, \quad (2.1)$$

where  $E$ ,  $G$  and  $\rho$  with indexes  $t$  and  $b$  stand for elasticity, shear modulus and material density at the top and bottom respectively;  $z$  is ordinate from the central axis at high  $h/2$ . Assuming linear theory of shear deformation, the displacement fields in the cross-section at  $x$  are

$$u(x, z, t) = u_0(x, t) - (z - h_0)\theta(x, t); w(x, z, t) = w_0(x, t), \quad (2.2)$$

with  $u_0(x, t)$ ,  $w_0(x, t)$  being the displacements of neutral axis that is located at the high  $h_0$  from the central axis;  $\theta$  is slope of the cross-section. Therefore, constituting equations get the form

$$\varepsilon_x = \partial u_0 / \partial x - (z - h_0) \partial \theta / \partial x; \gamma_{xz} = \partial w_0 / \partial x - \theta \quad (2.3)$$

and

$$\sigma_x = E(z)\varepsilon_x; \tau_{xz} = \kappa G(z)\gamma_{xz}. \quad (2.4)$$

In the latter equation  $\kappa$  is a coefficient introduced to account for the geometry-dependent distribution of shear stress. Hamilton principle allows one to obtain equations of motion in the time domain as

$$I_{11}\ddot{u} - A_{11}u'' - I_{12}\ddot{\theta} = 0;$$

$$\begin{aligned} I_{12}\ddot{u} - I_{22}\ddot{\theta} + A_{22}\theta'' + A_{33}(w' - \theta) &= 0; \\ I_{11}\ddot{w} - A_{33}(w'' - \theta') &= 0, \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} A_{11} &= \frac{2E_0A(R_e + n)}{(R_e + 1)(1+n)}; I_{11} = \frac{2\rho_0A(R_\rho + n)}{(R_\rho + 1)(1+n)}; I_{12} = \frac{2\rho_0Ah}{(R_\rho + 1)} \left[ \frac{2R_\rho + n}{2(2+n)} - \frac{R_\rho + n}{(1+n)} \alpha \right]; \\ A_{22} &= \frac{24E_0I_0}{(R_e + 1)} \left[ \frac{3R_e + n}{3(3+n)} - \frac{2R_e + n}{(2+n)} \alpha + \frac{R_e + n}{(1+n)} \alpha^2 \right]; \\ A_{33} &= \frac{2\kappa G_0A(R_G + n)}{(R_G + 1)(1+n)}; \\ I_{22} &= \frac{24\rho_0I_0}{(R_\rho + 1)} \left[ \frac{3R_\rho + n}{3(3+n)} - \frac{2R_\rho + n}{(2+n)} \alpha + \frac{R_\rho + n}{(1+n)} \alpha^2 \right]; \alpha = 1/2 + h_0/h; \\ h_0 &= \frac{n(R_e - 1)h}{2(n+2)(n+R_e)}, R_e = \frac{E_t}{E_b}, R_\rho = \frac{\rho_t}{\rho_b}; E_0 = \frac{E_b + E_t}{2}; \rho_0 = \frac{\rho_b + \rho_t}{2}; G_0 = \frac{G_b + G_t}{2}; \\ I_0 &= bh^3/12; A = bh. \end{aligned} \quad (2.6)$$

Introducing the displacement amplitudes

$$\{U, \Theta, W\} = \int_{-\infty}^{\infty} \{u_0(x, t), \theta(x, t), w_0(x, t)\} e^{-i\omega t} dt \quad (2.7)$$

Eq. (2.5) get to be

$$\begin{aligned} (\omega^2 I_{11}U + A_{11}U'') - \omega^2 I_{12}\Theta &= 0; \\ (\omega^2 I_{22}\Theta + A_{22}\Theta'') - \omega^2 I_{12}U + A_{33}(W' - \Theta) &= 0; \\ \omega^2 I_{11}W + A_{33}(W'' - \Theta') &= 0. \end{aligned} \quad (2.8)$$

Using the following matrix and vector notations

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}; \mathbf{\Pi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{33} \\ 0 & -A_{33} & 0 \end{bmatrix}; \\ \mathbf{C}(\omega) &= \begin{bmatrix} \omega^2 I_{11} & -\omega^2 I_{12} & 0 \\ -\omega^2 I_{12} & \omega^2 I_{22} - A_{33} & 0 \\ 0 & 0 & \omega^2 I_{11} \end{bmatrix}; \\ \mathbf{z} &= \{U, \Theta, W\}^T, \end{aligned}$$

Eq. (2.8) are rewritten in the form [19]

$$\mathbf{Az}'' + \mathbf{\Pi z}' + \mathbf{Cz} = 0. \quad (2.9)$$

## 2.2. Crack modeling

Assume that the beam has been cracked at the position  $e$  measured from the left end of beam and the crack is modeled by a pair of equivalent springs of stiffness  $T$  for translational spring and  $R$  for rotational one. Therefore, conditions that must be satisfied at the crack are

$$U(e+0) - U(e-0) = N(e)/T; \Theta(e+0) - \Theta(e-0) = M(e)/R; W(e+0) = W(e-0);$$

$$N(e) = N(e+0) = N(e-0); Q(e+0) = Q(e-0); M(e+0) = M(e-0) = M(e), \quad (2.10)$$

where  $N, Q, M$  are respectively internal axial, shear forces and bending moment at section  $x$

$$N = A_{11}U'_x; M = A_{22}\Theta'_x; Q = A_{33}(W'_x - \Theta). \quad (2.11)$$

Substituting (2.11) into (2.10) one can rewrite the latter conditions as

$$U(e+0) = U(e-0) + \gamma_1 U'_x(e); \Theta(e+0) = \Theta(e-0) + \gamma_2 \Theta'_x(e); W(e+0) = W(e-0);$$

$$U'_x(e+0) = U'_x(e-0); \Theta'_x(e+0) = \Theta'_x(e-0); W'_x(e+0) = W'_x(e-0) + \gamma_2 \Theta'_x(e), \quad (2.12)$$

$$\gamma_1 = A_{11}/T; \gamma_2 = A_{22}/R. \quad (2.13)$$

The so-called crack magnitudes  $\gamma_1, \gamma_2$  introduced in (2.13) are function of the material parameters such as elastic modulus and they should be those of homogeneous beam when  $E_t = E_b = E_0$ . On the other hand, using expressions (2.6) the crack magnitudes (2.13) can be rewritten as

$$\gamma_1 = \gamma_a \theta_1(R_E, n); \gamma_2 = \gamma_b \theta_2(R_E, n), \quad (2.14)$$

where

$$\gamma_a = E_0 A / T; \gamma_b = E_0 I_0 / R;$$

$$\theta_1 = \frac{2(R_e + n)}{(R_e + 1)(1 + n)}; \theta_2 = \frac{24}{(R_e + 1)} \left[ \frac{3R_e + n}{3(3 + n)} - \frac{2R_e + n}{(2 + n)} \alpha + \frac{R_e + n}{(1 + n)} \alpha^2 \right]. \quad (2.15)$$

In case of homogeneous beam when  $R_e = 1$  the crack magnitudes must be equal to  $\gamma_{10}, \gamma_{20}$ , that are calculated from crack depth  $a$  for axial [20] and flexural [21] vibrations as

$$\gamma_{10} = E_0 A / T_0 = 2\pi(1 - \nu_0^2) h f_1(z), z = a/h; \quad (2.16)$$

$$f_1(z) = z^2(0.6272 - 0.17248z + 5.92134z^2 - 10.7054z^3 + 31.5685z^4 - 67.47z^5 + 139.123z^6 - 146.682z^7 + 92.3552z^8);$$

$$\gamma_{20} = E_0 I / R_0 = 6\pi(1 - \nu_0^2) h f_2(z); \quad (2.17)$$

$$f_2(z) = z^2(0.6272 - 1.04533z + 4.5948z^2 - 9.9736z^3 + 20.2948z^4 - 33.0351z^5 + 47.1063z^6 - 40.7556z^7 + 19.6z^8).$$

For modal analysis of cracked FGM beam crack magnitudes are proposed herein to be approximately calculated using expressions (2.16-2.17) with  $\gamma_a = \gamma_{10}, \gamma_b = \gamma_{20}$ , i. e.

$$\gamma_1 = F_1(a); \gamma_2 = F_2(a); \quad (2.18)$$

$$F_1(a) = 2\pi(1 - \nu_0^2)h\theta_1 f_1(a); F_2(a) = 6\pi(1 - \nu_0^2)h\sigma_2 f_2(a). \quad (2.19)$$

These functions would be used for calculating the crack magnitudes from given crack depth.

### 2.3. Characteristic equation

Continuous solution of Eq. (2.9) sought in the form  $\mathbf{z}_0 = \mathbf{d}e^{\lambda x}$  yields the equation

$$[\lambda^2 \mathbf{A} + \lambda \mathbf{\Pi} + \mathbf{C}]\mathbf{d} = 0. \quad (2.20)$$

The latter equation would have nontrivial solution with respect to constant vector  $\mathbf{d}$  under the condition

$$\det[\lambda^2 \mathbf{A} + \lambda \mathbf{\Pi} + \mathbf{C}] = 0,$$

that can be in turn expressed in the form

$$(\lambda^2 A_{11} + \omega^2 I_{11})[(\lambda^2 A_{33} + \omega^2 I_{11})(\lambda^2 A_{22} + \omega^2 I_{22}) - \omega^2 I_{11} A_{33}] - (\lambda^2 A_{33} + \omega^2 I_{11})\omega^4 I_{12}^2 = 0.$$

This is in fact a cubic equation with respect to  $\eta = \lambda^2$  that could be elementarily solved and results in three roots  $\eta_1, \eta_2, \eta_3$ . Introducing the notations

$$\lambda_{1,4} = \pm k_1 = \pm \sqrt{\eta_1}; \lambda_{2,5} = \pm k_2 = \pm \sqrt{\eta_2}; \lambda_{3,6} = \pm k_3 = \pm \sqrt{\eta_3}, \quad (2.21)$$

general continuous solution of Eq. (2.9) is represented as

$$\mathbf{z}_0(x, \omega) = \mathbf{G}(x, \omega)\mathbf{C}, \quad (2.22)$$

with  $\mathbf{C} = (C_1, \dots, C_6)^T = (d_{11}, \dots, d_{16})^T$  and

$$\mathbf{G}(x, \omega) = [\mathbf{G}_1(x, \omega) \quad \mathbf{G}_2(x, \omega)]; \quad (2.23)$$

$$\mathbf{G}_1(x, \omega) = \begin{bmatrix} \alpha_1 e^{k_1 x} & \alpha_2 e^{k_2 x} & \alpha_3 e^{k_3 x} \\ e^{k_1 x} & e^{k_2 x} & e^{k_3 x} \\ \beta_1 e^{k_1 x} & \beta_2 e^{k_2 x} & \beta_3 e^{k_3 x} \end{bmatrix}; \mathbf{G}_2(x, \omega) = \begin{bmatrix} \alpha_1 e^{-k_1 x} & \alpha_2 e^{-k_2 x} & \alpha_3 e^{-k_3 x} \\ e^{-k_1 x} & e^{-k_2 x} & e^{-k_3 x} \\ -\beta_1 e^{-k_1 x} & -\beta_2 e^{-k_2 x} & -\beta_3 e^{-k_3 x} \end{bmatrix},$$

$$\alpha_j = \frac{\omega^2 I_{12}}{\omega^2 I_{11} + \lambda_j^2 A_{11}}; \beta_j = \frac{\lambda_j A_{33}}{(\omega^2 I_{11} + \lambda_j^2 A_{33})}; j = 1, 2, 3.$$

Using (2.22), it can be found that solution of Eq. (2.9) denoted by  $\mathbf{S}(x)$  satisfying the conditions

$$\mathbf{S}(0) = (S_1^0, S_2^0, 0)^T; \mathbf{S}'(0) = (0, 0, S_3^0)^T. \quad (2.24)$$

is represented as

$$\mathbf{S}(x) = [\mathbf{\Phi}(x)]\{\mathbf{S}^0\}, \quad (2.25)$$

where  $\mathbf{S}^0 = \{S_1^0, S_2^0, S_3^0\}^T$  and matrix

$$[\Phi(x)] = \begin{bmatrix} \alpha_1 \cosh k_1 x & \alpha_2 \cosh k_2 x & \alpha_3 \cosh k_3 x \\ \cosh k_1 x & \cosh k_2 x & \cosh k_3 x \\ \beta_1 \sinh k_1 x & \beta_2 \sinh k_2 x & \beta_3 \sinh k_3 x \end{bmatrix} \cdot \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix}; \quad (2.26)$$

$$\Delta_1 = (\delta_{11}S_1^0 + \delta_{12}S_2^0 + \delta_{13}S_3^0) / \Delta; \Delta_2 = (\delta_{21}S_1^0 + \delta_{22}S_2^0 + \delta_{23}S_3^0) / \Delta;$$

$$\Delta_3 = (\delta_{31}S_1^0 + \delta_{32}S_2^0 + \delta_{33}S_3^0) / \Delta; \Delta = k_1\beta_1(\alpha_2 - \alpha_3) + k_2\beta_2(\alpha_3 - \alpha_1) + k_3\beta_3(\alpha_1 - \alpha_2);$$

Assuming furthermore that  $S_1^0 = \gamma_1 U'_x(e), S_2^0 = S_3^0 = \gamma_2 \Theta'_x(e)$  or  $\mathbf{S}^0 = [\Sigma]\{\mathbf{z}'_0(e)\}$  with

$$\Sigma = \begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & \gamma_2 & 0 \end{bmatrix}, \quad (2.27)$$

a particular solution  $\mathbf{z}_c(x)$  of Eq. (2.9) that satisfies initial conditions

$$\mathbf{z}_c(0) = \{\gamma_1 U'_x(e), \gamma_2 \Theta'_x(e), 0\}^T; \mathbf{z}'_c(0) = (0, 0, \gamma_2 \Theta'_x(e))^T, \quad (2.28)$$

is

$$\mathbf{z}_c(x) = [\Phi(x)][\Sigma]\{\mathbf{z}'_0(e)\} = [\mathbf{G}_c(x)]\{\mathbf{z}'_0(e)\}. \quad (2.29)$$

Using the matrix-function notation

$$\mathbf{K}(x) = \begin{cases} \mathbf{G}_c(x) & : x > 0; \\ 0 & : x \leq 0; \end{cases}; \mathbf{K}'(x) = \begin{cases} \mathbf{G}'_c(x) & : x > 0; \\ 0 & : x \leq 0; \end{cases} \quad (2.30)$$

one is able to prove that the function

$$\mathbf{z}(x) = \mathbf{z}_0(x) + \mathbf{K}(x-e)\mathbf{z}'_0(e) \quad (2.31)$$

is general solution of Eq. (2.9) satisfying conditions (2.12) at the cracked section.

It can be easily to verify that boundary conditions for cantilever beam are

$$u(0, t) = w(0, t) = \theta(0, t) = 0; N(L, t) = M(L, t) = Q(L, t) = 0. \quad (2.32a); (2.32b)$$

Applying conditions (2.32a) for solution (2.31) leads to

$$\mathbf{B}_{01}\mathbf{C}_1 + \mathbf{B}_{02}\mathbf{C}_2 = 0, \quad (2.33)$$

$$\mathbf{C}_1 = \{C_1, C_2, C_3\}^T; \mathbf{C}_2 = \{C_4, C_5, C_6\}^T;$$

$$\mathbf{B}_{01} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 1 & 1 & 1 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix}; \mathbf{B}_{02} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 1 & 1 & 1 \\ -\beta_1 & -\beta_2 & -\beta_3 \end{bmatrix}.$$

Obviously, the above equation allows the vectors  $\mathbf{C}_1, \mathbf{C}_2$  to be expressed as

$$\mathbf{C}_1 = [\mathbf{B}_{01}]^{-1}\mathbf{D}, \mathbf{C}_2 = -[\mathbf{B}_{02}]^{-1}\mathbf{D}$$

with an arbitrary constant vector  $\mathbf{D}$ , so that solution  $\mathbf{z}_0(x)$  can be rewritten in the form

$$\mathbf{z}_0(x, \omega) = \mathbf{G}_0(x, \omega)\mathbf{D}, \quad (2.34)$$

where

$$\mathbf{G}_0(x, \omega) = \mathbf{G}_1(x, \omega)\mathbf{B}_{01}^{-1} - \mathbf{G}_2(x, \omega)\mathbf{B}_{02}^{-1}.$$

Consequently, one obtains

$$\mathbf{z}(x) = [\mathbf{G}_0(x, \omega) + \mathbf{K}(x-e)\mathbf{G}'_0(e, \omega)]\{\mathbf{D}\} = [\mathbf{G}_L(x, \omega)]\{\mathbf{D}\}. \quad (2.35)$$

Applying boundary condition (3.32b) for solution (2.34) one gets

$$[\mathbf{B}_{LL}(\omega)]\{\mathbf{D}\} = 0, \quad (2.36)$$

$$\mathbf{B}_{LL}(\omega) = \mathbf{B}_L \left\{ \mathbf{G}_L(x, \omega) \right\}_{x=L}; \mathbf{B}_L = \begin{bmatrix} A_{11}\partial_x & 0 & 0 \\ 0 & A_{22}\partial_x & 0 \\ 0 & -A_{33} & A_{33}\partial_x \end{bmatrix}.$$

So that characteristic or frequency equation of the cracked FGM beam is obtained as

$$\Lambda(\omega) = \det[\mathbf{B}_{LL}(\omega)] = 0. \quad (2.37)$$

Positive root  $\omega_j$  of this equation provide desired natural frequency of the beam. In the case of intact beam the frequency equation (2.37) is reduced to

$$\Lambda_0(\omega) = \det[\mathbf{B}_{L0}(\omega)] = 0. \quad (2.38)$$

$$\mathbf{B}_{L0}(\omega) = \mathbf{B}_L \left\{ \mathbf{G}_0(x, \omega) \right\}_{x=L}.$$

Thus, forward problem is to calculate natural frequencies of cracked or uncracked FGM beam by solving Eq. (2.37) or (2.38).

### 3. NUMERICAL RESULTS AND DISCUSSION

#### 3.1. Comparative study

To investigate effect of actual position of neutral axis on natural frequencies of Timoshenko FGM cantilevered beam, it is examined an undamaged beam studied in [4] that is composed from steel:  $E_b = 210GPa$ ,  $\rho_b = 7800kg/m^3$ ,  $\mu_1 = 0.3$  at the bottom and Aluminum Oxide ( $Al_2O_3$ ):  $E_t = 390GPa$ ,  $\rho_t = 3960kg/m^3$ ,  $\mu_t = 0.25$  at the top surface.

Tables 1 shows first five natural frequencies computed in the present paper for various slenderness ratio  $L/h$ , and power law index  $n$ . Comparison with those obtained in [4] where neutral axis is assumed coincident with the middle one shows that effect of actual position of neutral axis on the lower natural frequencies is clearly observed in the case of small slenderness,  $L/h=5$ , and  $n=2$ . In this case natural frequencies calculated with actual position of neutral axis are lower than those computed by the centroid axis theory. However, higher natural frequencies of FGM beam with greater slenderness and power law index are not very much changed by the correcting position of neutral axis.

Table 1. Comparison of frequency parameters,  $\lambda = (\omega L^2 / h) \sqrt{\rho_b / E_b}$ , for undamaged FGM cantilever beam: Present – actual and Ref. [4] – centroid position of neutral axis.

$L/h$		5		10		20		30	
$n$	Fr. No.	Present	Ref.[4]	Present	Ref.[4]	Present	Ref.[4]	Present	Ref.[4]
0.1	1	1.7377	1.7574	1.7854	1.7966	1.8020	1.8070	1.8060	1.8089
	2	9.3254	9.0511	10.6630	10.782	11.1116	11.196	11.2359	11.278
	3	14.1039	14.095	28.0582	28.190	30.4454	30.800	31.1476	31.325
	4	22.3755	22.682	28.3600	28.404	56.3576	56.379	60.2240	60.681
	5	37.5464	37.747	51.8239	51.618	58.0334	58.897	84.5711	84.569
0.2	1	1.6294	1.6638	1.6804	1.7010	1.7011	1.7107	1.7061	1.7126
	2	8.6806	8.9969	9.9804	10.208	10.4534	10.600	10.5981	10.678
	3	13.4167	13.390	26.3172	26.781	28.5635	29.161	29.3406	29.657
	4	20.9419	21.482	26.9510	26.895	53.4530	53.562	56.6568	57.449
	5	35.5698	35.754	48.9926	48.878	54.4140	55.762	80.3494	80.343
0.5	1	1.4308	1.4911	1.4852	1.5244	1.5118	1.5332	1.5183	1.5348
	2	7.5158	8.0609	8.7058	9.1477	9.2390	9.4992	9.4075	9.5691
	3	12.0814	12.012	22.8654	24.024	25.1327	26.130	25.9870	26.576
	4	18.3974	19.243	24.5762	24.098	47.3355	48.048	50.0731	51.475
	5	32.0833	32.022	43.5626	43.787	48.3416	49.962	72.0863	72.072
1.0	1	1.2809	1.3557	1.3345	1.3864	1.3636	1.3945	1.3705	1.3960
	2	6.6597	7.3164	7.7397	8.3146	8.3071	8.6383	8.4791	8.7027
	3	10.9037	10.811	20.3079	21.623	22.5403	23.755	23.3925	24.165
	4	16.4188	17.441	22.1864	21.886	42.3703	43.246	45.0165	46.795
	5	28.9477	28.989	38.7308	39.732	43.5214	45.402	64.8906	64.870
2.0	1	1.1757	1.2471	1.2252	1.2762	1.2519	1.2839	1.2583	1.2853
	2	6.1047	6.7053	7.1063	7.6440	7.6240	7.9501	7.7835	8.0112
	3	9.8238	9.7403	18.6028	19.481	20.6777	21.851	21.4688	22.239
	4	15.0301	15.937	20.0358	20.088	38.6088	38.961	41.3021	43.049
	5	26.4566	26.428	35.5854	36.403	39.4505	41.733	58.4651	58.442
5.0	1	1.1030	1.1446	1.1405	1.1722	1.1604	1.1795	1.1651	1.1809
	2	5.8414	6.1274	6.6706	7.0111	7.0955	7.3014	7.2209	7.3594
	3	8.8103	8.7633	17.3374	17.527	19.2998	20.057	19.9462	20.425
	4	14.3168	14.516	17.7729	18.391	35.0124	35.053	38.4225	39.525
	5	24.9321	24.009	32.4530	33.2625	36.5838	38.278	52.5932	52.580
10	1	1.0629	1.0867	1.0962	1.1130	1.1105	1.1199	1.1138	1.1212
	2	5.6345	5.8159	6.4753	6.6562	6.8192	6.9324	6.9167	6.9876
	3	8.3591	8.3430	16.6488	16.686	18.6108	19.394	19.1378	19.394
	4	13.5189	13.776	17.0828	17.459	33.3597	33.372	36.9246	37.532
	5	22.8182	22.783	31.2177	31.575	35.3365	36.345	50.0635	50.058

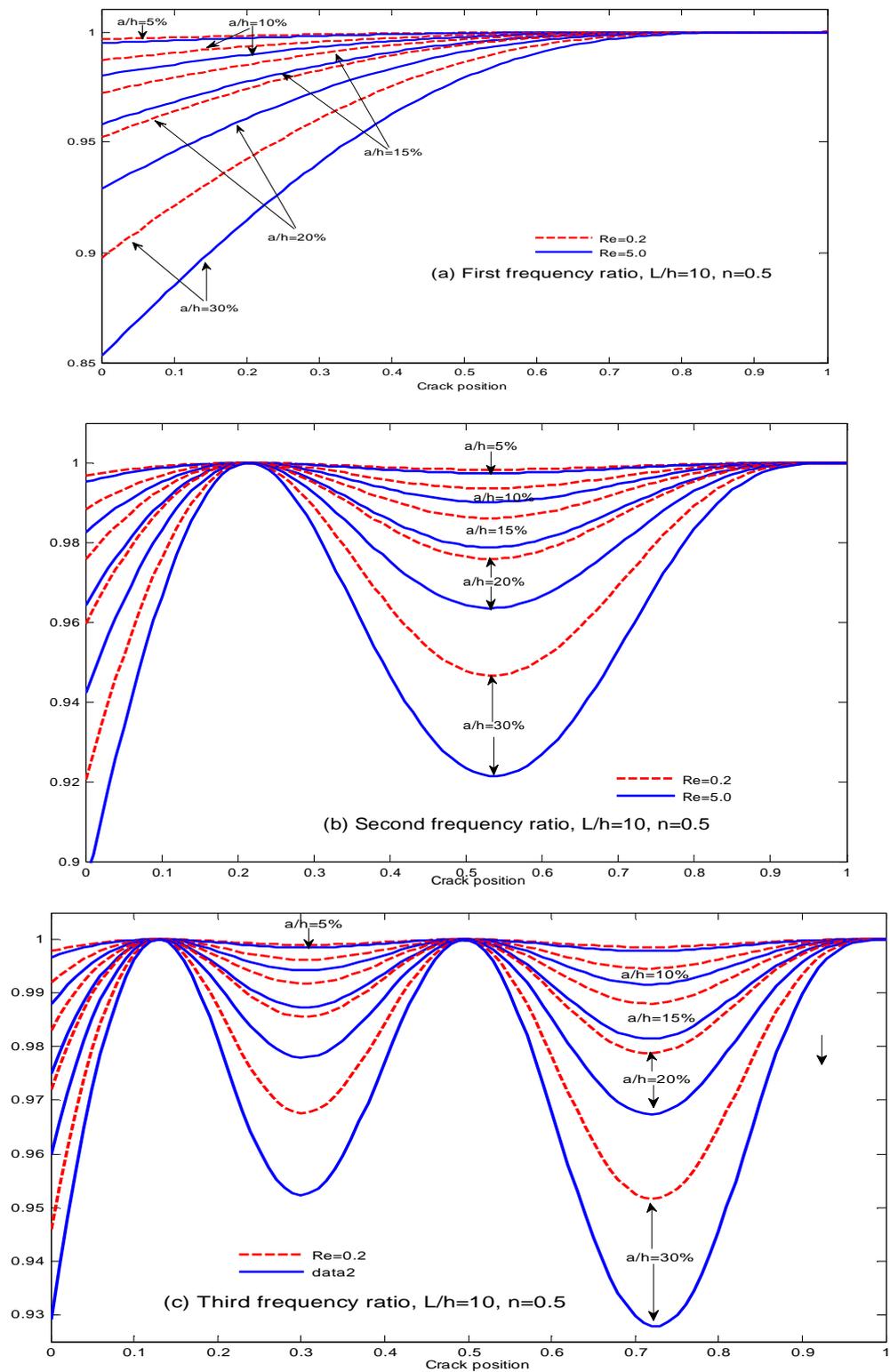


Figure 1. Sensitivity of natural frequencies (a- first, b- second, c- third) in dependence on crack depth (5 % - 30 %) and elasticity modulus ratio  $R_e = 0.2$  &  $5.0$  with  $L/h = 10$ ,  $n = 5$ .

### 3.2. Sensitivity of natural frequencies to crack

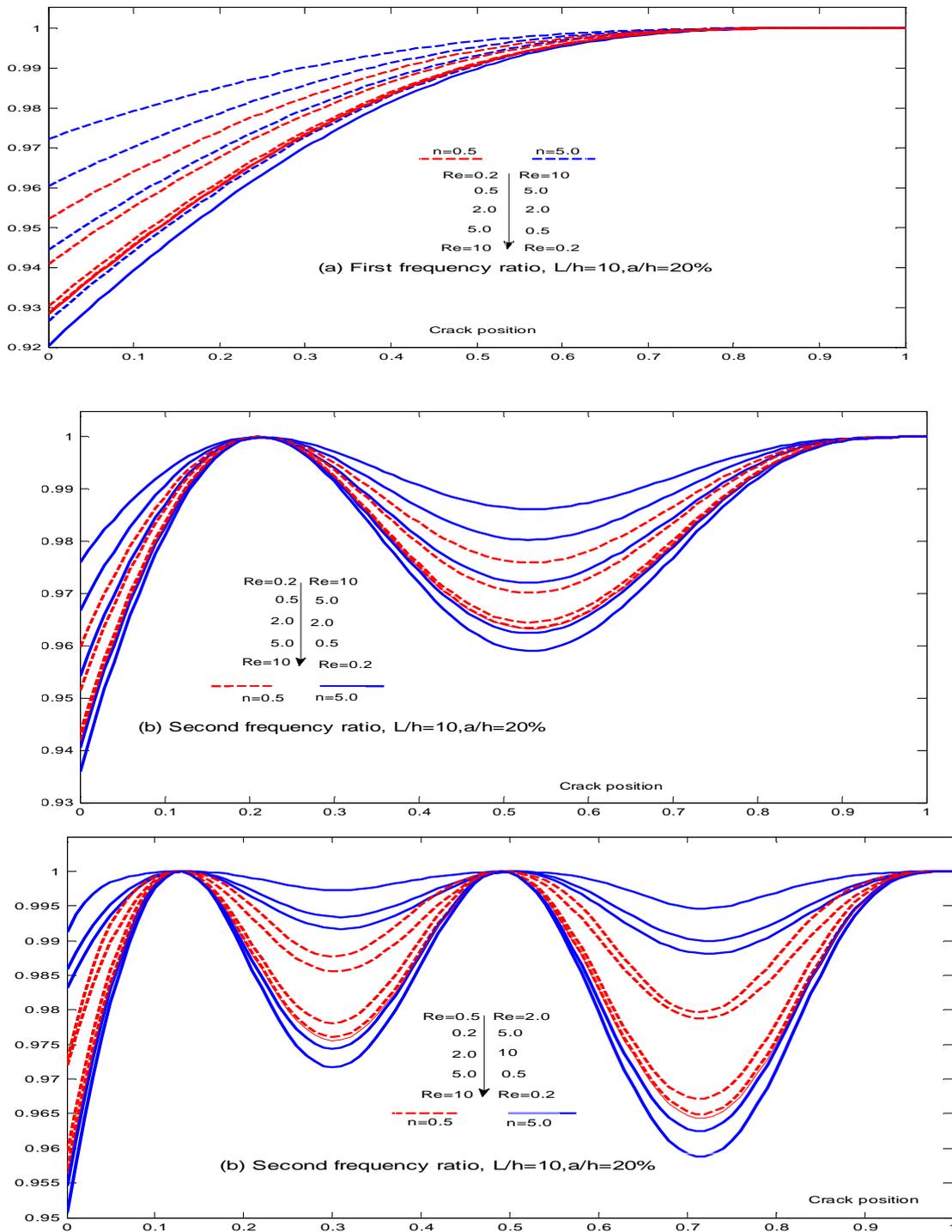


Figure 2. Sensitivity of natural frequencies (a- first, b- second, c - third) in dependence on the elasticity modulus ratio  $R_e$  (0.1 – 10) and  $n=0.5;5.0$  with  $L/h=10, a/h=20\%$ .

The change in natural frequencies caused by a crack is usually called sensitivity of the natural frequencies to crack. The natural frequency sensitivity is represented in this paper by a ratio of the damaged to undamaged frequencies as function of crack location along the beam length. Such indicator for the natural frequency sensitivity is investigated herein in dependence on the material and geometry parameters of a FGM cantilever beam. Results are shown in Figs. 1-5 for combinations of various crack depth  $a/h$ , slenderness ratio  $L/h$ , power law index  $n$  and elasticity modulus ratio  $R_e$ .

First, it is observed in the Figures that, likely to the homogeneous beam, a natural frequency could be unchanged if crack occurred at some positions on beam. Such positions are called critical points (or frequency node) for vibration mode with the unchanged frequency. For instance, the free end of homogeneous cantilever beam is a consistent critical point for all modes including either axial or flexural vibration. Approximate critical points for first three vibration modes with undamaged natural frequency  $\omega_k^0$  of an FGM beam are given in Table 2.

*Table 2.* Possible critical points for FGM Timoshenko cantilever beam.

Mode	First frequency node	Second frequency node	Third frequency node
1	1.0		
2	0.22	1.0	no
3	0.13	0.49	1.0

Observation of the graphs given in Figs. 1-4 provide that the sensitivity of natural frequencies is monotonically reducing with growing crack depth and it is dependent also on the material and geometry of the beam. Namely, the sensitivity is increasing with elasticity modulus ratio  $R_E = E_t / E_b$  for  $n < 1$  and decreasing when  $n > 1$ . The latter implies that increase of elasticity modulus from bottom to top of Timoshenko beam makes the natural frequencies more or less sensitive to crack dependently on that  $n < 1$  or  $n > 1$ . Similarly, it is observed from Fig. 3 that natural frequency sensitivity is increasing with  $n$  for a fixed  $R_E < 1$  and would be decreasing if  $R_E > 1$ . Fig. 4 shows that natural frequencies of flexural vibration modes become less sensitive to crack for increasing slenderness ratio and it is independent on whatever material the beam is made of but the axial mode frequencies show to be most sensitive to crack when  $L/h = 10$ .

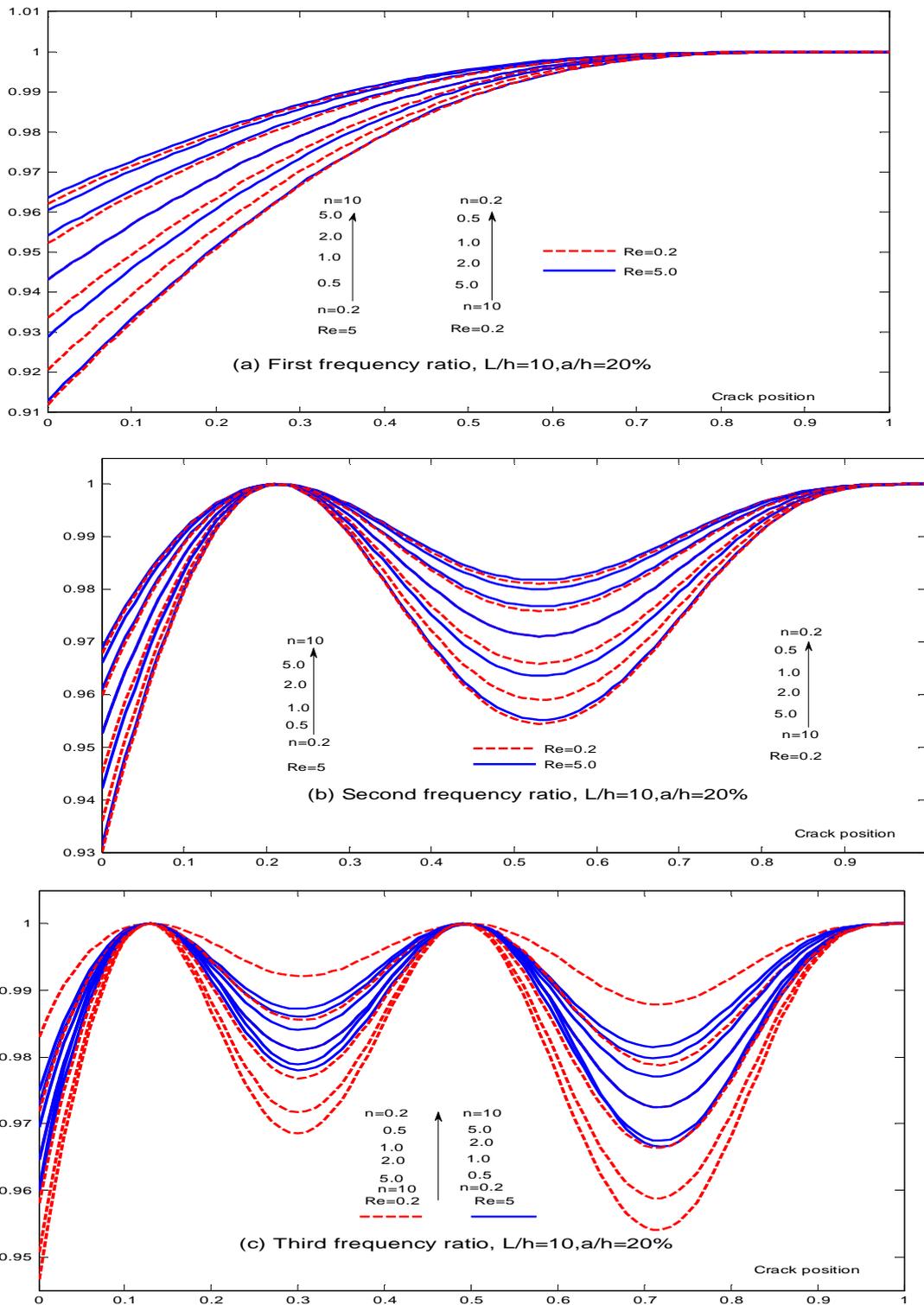


Figure 3. Sensitivity of natural frequencies (a- first, b- second, c-third) in dependence on the power law index  $n = 0.2 - 10$ ; the slenderness ratio  $L/h=10$  with elasticity modulus ratio  $R_e = 0.2$  &  $5.0$  and crack depth  $a/h = 20\%$ .

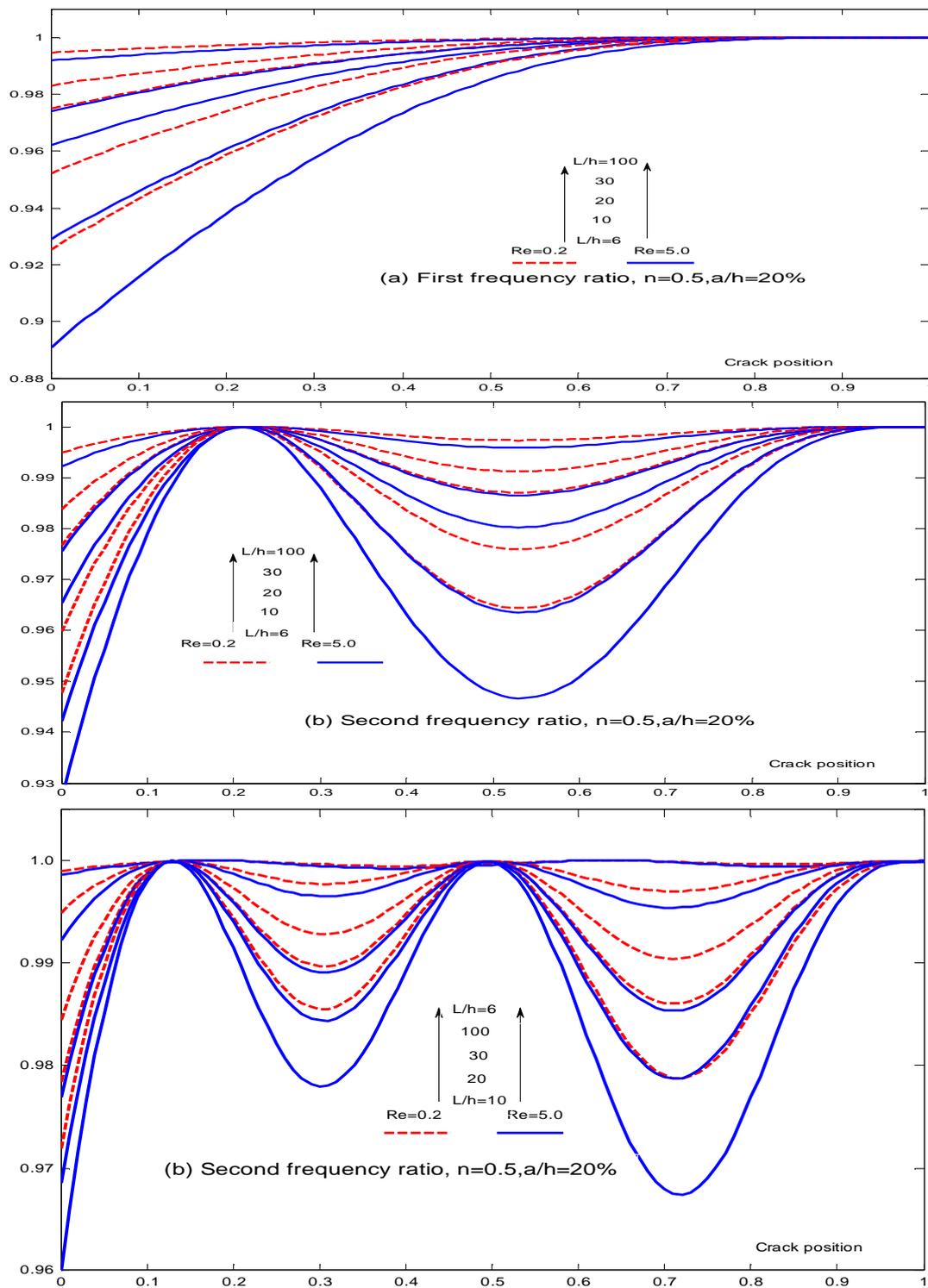


Figure 4. Sensitivity of natural frequencies (a- first, b- second, c-third) in dependence on the slenderness ratio  $L/h = 5 - 50$  with elasticity modulus ratio  $R_e = 0.2$  &  $5.0$ ,  $n = 0.5$  &  $5.0$  and crack depth  $a/h = 20\%$ .

#### 4. CONCLUSION

Major results obtained in the present paper are as follows:

A consistent theory of vibration beam has been formulated in the frequency domain for functionally graded Timoshenko beam that can be used for analysis of either free or forced vibrations in the beam.

Frequency equation for functionally graded Timoshenko beam with single crack modeled by coupled translation and rotation springs was constructed in a form that is applicable straightforward to frequency analysis of the beam. Application of the equation for natural frequency analysis of FGM beam demonstrates that natural frequencies of flexural vibration modes are more sensitive to crack than those of axial vibration modes and the natural frequency sensitivity is strongly dependent on both material and geometry parameters of functionally graded Timoshenko beam.

The theory proposed in the present work can be further developed for analysis and identification of FGM beam with multiple cracks.

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