Reasoning in knowledge bases with external and internal uncertainties

Phan Dinh Dieu & Tran Dinh Que
Institute of Information Technology

1. Introduction

In the recent years, in A.I. a great deal of attention has been devoted to formalisms dealing with various aspects of reasoning with uncertain information, and a number of theories and methods for handling uncertainty of knowledge has been proposed, notably, the probability theory, the Dempster-Shafer theory, and the possibility theory. Among these theories, the probability one is surely the most developed by its long history and elaborated foundations.

In this paper the uncertainty of a sentence will be given by an interval of possible values for its truth probability. Two types of knowledge with uncertainty will be investigated: external uncertainty and internal uncertainty. The former is given in the form \(< S, I >\), in which \(S\) is a sentence, and \(I = [a, b]\) is a closed subinterval of the unit interval \([0, 1]\); it means that the truth probability of the whole sentence \(S\) lies in the interval \(I\). \(I\) is then called the interval of truth probabilities of \(S\). In the later case, intervals of truth probabilities are given to subsentences of the given sentence \(S\). For example,

\(< S_1, I_1 > \land < S_2, I_2 > \land \ldots \land < S_n, I_n > \leq < S_{n+1}, I_{n+1} >

is a knowledge with internal uncertainty.

Let \(B\) be a knowledge base with these types of uncertainty and \(S\) be a any sentence. A semantics, which underlies a method of deducing the interval of truth probabilities of \(S\) from \(B\), will be given.
The paper is structured as follows: In Section 2 we shall review the probabilistic logic by N.J. Nilsson, and its extension to probabilistic logic with interval values. In Section 3, we shall discuss a method of reasoning from a set of knowledges with internal uncertainty. Section 4 is devoted to a method of reasoning from a knowledge base containing both external and internal uncertainties. Some illustrative examples will be given in Section 5.

2. External Uncertainty

We shall use methods of the interval-valued probabilistic logic for reasoning in knowledge bases with external uncertainty. This logic is based on the semantics given by N.J. Nilsson [4], and is presented in details, e.g., in [6].

Let us give a knowledge base

\[ B = \{< S_i, I_i> | i = 1, \ldots, L \} \]

and let \( S \) be a target sentence. We put \( \Sigma = \{S_1, \ldots, S_L, S\} \), and suppose that \( W_1, \ldots, W_k \) are all \( \Sigma \) - classes of possible worlds. Every class \( W_j \) is characterized by a consistent vector \((u_{i1}, \ldots, u_{ij}, u_j)\) of truth values of sentences \( S_1, \ldots, S_L, S \).

Given a probability distribution \((p_1, \ldots, p_K)\) over the classes \( W_1, \ldots, W_K \), the truth probability of sentence \( S_i \) is defined to be the sum of probabilities of classes of worlds in which \( S_i \) is true, i.e., \( \pi(S_i) = u_{i1}p_1 + \cdots + u_{iK}p_K \). From this semantics it follows that, given the knowledge base \( B \), the derived interval-value \([\alpha, \beta]\) for the truth probability of the sentence \( S \) is defined by

\[\alpha = \min \pi(S), \quad \beta = \max \pi(S),\]

where

\[ \pi(S_i) = u_{i1}p_1 + \cdots + u_{iK}p_K, \]

subject to the constraints

\[ \sum_{j=1}^{K} p_j = 1, \quad p_j \geq 0 (j = 1, \ldots, K). \]

We denote the interval \([\alpha, \beta]\) by \( F(S, B) \), and write \( B \vdash S, F(S, B) \).

Now, let us give a set of sentence \( \Gamma \). We define \( I \) to be the set of all mappings from \( \Gamma \) into \( C[0,1] \) - the set of closed subintervals of \([0,1]\). A such mapping \( I \) assigns to each sentence \( P \in \Gamma \) an interval \( I(P) \in C[0,1] \).

The given knowledge base \( B \) defines an operator \( R_B \) from \( I \) into \( I \) as follows: For every \( I \in I \), we establish a new knowledge base

\[ B^* = B \cup \{< P, I(P) > | P \in \Gamma \}. \]
and then we take for every $P \in \Gamma$ the interval $I'(P) = F(P, B')$. The mapping $I'$ is $R_B(I)$ defined to be the image of $I$ by the operator $R_B : R_B(I) = I'$.

It is easy to see that if $R_B(I) = I'$ then $R_B(I') = I'$; therefore

$$R^n_B(I) = R_B(I') \quad \text{for any } n \geq 1. \quad (*)$$

The calculation of the operator $R_B$ is reduced to the solution of linear programming problems which has to face up a very large computational complexity whenever the sizes of $B$ and $\Gamma$ are large. Some attempts to reducing the size of linear programming problems have been investigated, e.g., a method of reduction is given in [7] for the cases when the core $\{S_1, \ldots, S_L\}$ of $B$ forms a logic program.

Instead of the method presented above we can use methods of approximate reasoning, e.g., by means of deductions based on inference rules (see [2]), however, in this case the property $(*)$ may not be satisfied.

3. Internal Uncertainty

In this section a method of reasoning on knowledges with internal uncertainty will be discussed. We limit ourself to consider knowledges given by rules of the form:

$$< S_1, I_1 > \wedge \cdots \wedge < S_n, I_n > \rightarrow < S, I >$$

Let us given a knowledge base $B = \{J_j | j = 1, \ldots, M\}$, where $J_j$ is the rule:

$$J_j = < A_{j1}, I_{j1} > \wedge \cdots \wedge < A_{jm_j}, I_{jm_j} > \rightarrow < A_{cj}, I_{cj} > .$$

Let $\Gamma$ be a set of sentences containing all sentences occuring in rules $J_j(j = 1 \ldots M)$ of $B$. As above we denote by $I$ the set of mappnings $I$ from $\Gamma$ to $C[0,1]$.

For any $I_1, I_2 \in I$, we say that $I_1 \leq I_2$ iff $I_1(P) \subseteq I_2(P)$ for every $P \in \Gamma$.

We say that the rule $J_j$ is satisfied by the mapping $I \in I$, iff $I(A_{jk}) \subseteq I_{kj}$ for every $k = 1, \ldots, m_j$. Note that if $m_j = 0$ then $J_j$ is satisfied by any mapping $I$.

Now we define an operator $t_B$ from $I$ into $I$, which transforms any $I \in I$ into a mapping $t_B(I)$ such that for every $P \in \Gamma$:

$$t_B(I)(P) = I(P) \cap \bigcap_{j \in E} I_{cj},$$

where $E = \{j | A_{cj} = P \text{ and } J_j \text{ is satisfied by } I\}$. Here, for the vacuous case we assume that $\bigcap_{j \in E} I_{cj} = [0,1]$ whenever $E = \emptyset$.

We have the following proposition:
Proposition 1. For any mapping $I \in \mathcal{I}$ there exists always a natural number $n$ such that $t_{\mathcal{B}}^{n+1}(I) = t_{\mathcal{B}}^{n}(I)$. In other words, the process of iteration of the operator $t_{\mathcal{B}}$ on any given $I \in \mathcal{I}$ always halts after a finite number of steps.

Proof. Suppose that $E_0, E_1, E_2, \ldots$ are the set of indexes of rules which are satisfied by $I, t_{\mathcal{B}}(I), t_{\mathcal{B}}^2(I), \ldots$, respectively.

Let $h_i = |E_i|$ ($i = 1, 2, \ldots$) be the number of elements of the set $E_i$ ($h_i/i = 1, 2, \ldots$) is then a sequence of integers such that

$$0 \leq h_0 \leq h_1 \leq \cdots \leq h_n \leq \cdots \leq M.$$ 

Consider two cases

(i) There exists a number $n$ such that $h_{n-1} = M$, i.e., $J_j$ is satisfied by the mapping $t_{\mathcal{B}}^{n-1}(I)$ for every $J_j = 1, \ldots, M$. In this case we have $t_{\mathcal{B}}^{n}(I) = t_{\mathcal{B}}^{n+1}(I)$.

(ii) There exists a number $n$ such that $h_{n-1} = h_n < M$. In this case we have $E_{n-1} = E_n$, and it is easy to see that

$$t_{\mathcal{B}}^{n}(I) = t_{\mathcal{B}}^{n+1}(I).$$

The proposition has been proved.

From this proposition we can define an operator $T_{\mathcal{B}}$ as follows: For any $I \in \mathcal{I}$, $T_{\mathcal{B}}(I) = t_{\mathcal{B}}^{n}(I)$, where $n$ is the least number such that $t_{\mathcal{B}}^{n}(I) = t_{\mathcal{B}}^{n+1}(I)$.

Let $S$ be a sentence. We denote by $\Gamma$ the set consisting of $S$ and of all sentences occurring in rules $J_j$ of $\mathcal{B}$. Let $I$ be the mapping which assigns the interval $[0,1]$ for every sentence in $\Gamma$. Then $T_{\mathcal{B}}(I)(S)$ can be considered as the interval-value for the truth probability of the sentence $S$ derived from the knowledge base $\mathcal{B}$.


We consider now the knowledge bases consisting both types of knowledges with external and internal uncertainty. Let $\mathcal{B}$ be such a knowledge base, we can write $\mathcal{B} = \mathcal{B}^E \cup \mathcal{B}^I$, where $\mathcal{B}^E$ consists of knowledges with external uncertainty. Suppose that

$$\mathcal{B}^E = \{< S_i, I_i > | i = 1, \ldots, L \},$$

$$\mathcal{B}^I = \{J_j | j = 1, \ldots, M \},$$

where

$$J_j = < A_{j_1}, I_{j_1} > \wedge \cdots \wedge < A_{j_m}, I_{j_m} > \rightarrow < A_{e_j}, I_{e_j} > .$$

Let $S$ be any (target) sentence. Our problem is to deduce from the knowledge base $\mathcal{B}$ the interval value for the truth probability of the sentence $S$. 
For this purpose we put \( \Gamma \) to be the set consisting of the sentence \( S \) and all distinct sentences occurring in \( B_\beta \) and \( B_\ell \). Denote by \( I \) the set of mappings from \( \Gamma \) to \( C[0,1] \). Let \( I_0 \) be the mapping defined by

\[
I_0(P) = \begin{cases} 
I_i & \text{if } P = S_i \text{ for some } i = 1, \ldots, L \\
[0,1] & \text{otherwise.}
\end{cases}
\]

\( I_0 \) is called the initial assignment (of interval-value to sentences in \( \Gamma \)).

Now we define a sequence of assignments \( I_n \ (n = 0,1,\ldots) \) initiated by \( I_0 \) and given recursively as follows:

\[
I_n = \begin{cases} 
\mathcal{R}(I_{n-1}) & \text{if } n \text{ is odd} \\
\mathcal{T}(I_{n-1}) & \text{if } n \text{ is positive even.}
\end{cases}
\]

Here \( \mathcal{R} \) stands for \( R_{\beta\ell} \), and \( \mathcal{T} \) stands for \( T_{\beta\ell} \).

Proposition 2. Let \( B \) be a knowledge base, and \( S \) be any given sentence. There exists a natural number \( n \) such that \( I_{n+2} = I_{n+1} = I_n \).

Proof. From the definition of the sequence \( \{I_n\} \ (i = 0,1,\ldots) \) we can write

\[
I_0 \mathcal{R} I_1 \mathcal{T} I_2 \mathcal{R} \ldots \mathcal{R} I_{n-2} \mathcal{T} I_{n-1} \mathcal{R} I_n \mathcal{T} I_{n+1} \mathcal{R} \ldots
\]

Let \( h_i \) be a number of rules \( J_i \) satisfied by \( I_i (i = 1,3,5,\ldots) \). Then, \( \{h_i\} \) is a sequence of integers such that

\[
0 \leq h_1 \leq h_3 \leq \cdots \leq h_{n-2} \leq h_n \leq \cdots \leq M.
\]

Consider two cases

(i) There exists a number \( n \) such that \( h_{n-2} = M \), i.e, \( J_J \) is satisfied by the mapping \( I_{n-2} \), for every \( j = 1,\ldots,M \). Then, any rule \( J_j (j = 1,\ldots,M) \) is also satisfied by the mappings

\[
I_{n-1} = \mathcal{T}(I_{n-1})
\]

\[
I_n = \mathcal{R}(I_n)
\]

Thus

\[
I_{n+1} = \mathcal{T}(I_n) = I_n
\]

\[
I_{n+2} = \mathcal{R}(I_{n+1}) = \mathcal{R}(I_n) = I_n
\]

or \( I_n = I_{n+1} = I_{n+2} \).

(ii) There exists a number \( n \) such that \( h_n = h_{n-2} = M \). Then the sets of rules satisfied by \( I_n \) and by \( I_{n-2} \) are the same. Therefore,

\[
I_{n+1} = \mathcal{T}(I_n) = I_n
\]

\[
I_{n+2} = \mathcal{R}(I_{n+1}) = I_{n+1}
\]

and we have again \( I_n = I_{n+1} = I_{n+2} \).

This completes our proof.
Let $n$ be the least number having the property stated in the proposition 2. We denote this $n$ by $I^*$, and call it to be the resulting assignment deduced from $B$ to sentences in $\Gamma$.

The interval $I^*(S)$ is defined to be the interval value for the truth probability of the sentence $S$ derived from the knowledge base $B$. We write also:

$$B \vdash < S, I^*(S) >$$

5. Examples

This section presents two examples illustrating the method of reasoning in a knowledge base consisting of both types of knowledge with internal and external uncertainty.

Example 1.

Given a knowledge base $B = B^E \cup B^I$ where $B^E$ is the set of sentences

\begin{align*}
B & \rightarrow A : [1,1] \\
A & \rightarrow C : [1,1] \\
B & : [2,6] \\
C & : [6,7]
\end{align*}

and $B^I$ is the set of rules

\begin{align*}
J_1 &= C : [5,7] \rightarrow B : [3,5] \\
J_2 &= B : [2,5] \land C : [5,7] \rightarrow A : [2,5]
\end{align*}

Calculate the interval of truth probabilities of the sentence $A$.

**Step 1.** Applying the operator $\mathfrak{R}$, we get

\begin{align*}
A & : [2,7] \\
B & : [2,6] \\
C & : [6,7]
\end{align*}

**Step 2.** Both rules $J_1$ and $J_2$ are satisfied, so applying the operator $\mathfrak{T}$ we obtain

\begin{align*}
A & : [2,5] \\
B & : [3,5] \\
C & : [6,7]
\end{align*}

**Step 3.** Iterate $\mathfrak{R}$, where $B : [2,6]$ is now replaced by $B : [3,5]$, we get

\begin{align*}
A & : [3,5]
\end{align*}

As both $J_1$ and $J_2$ are satisfied after step 1, it is not necessary to repeat $\mathfrak{T}$ after step 3 and we get the final result $A : [3,5]$.

Note that $B^E$ is a type-A problem (see [2]); hence, we can apply the type-A rules to computing the intervals for $A, B, C$ instead of solving linear programming problems.
Example 2. This example is more complex than the above; it illustrates the iteration of the operator $T$. Suppose that we wish to derive the interval of truth probabilities of the sentence $A \land D$ from the knowledge base $B = B^E \cup B'$ where $B^E$ is the set defined as follows

\begin{align*}
B &\rightarrow A : [.9,1] \\
D &\rightarrow B : [.8,.9] \\
A &\rightarrow C : [.6,.8] \\
D &\rightarrow C : [.8,1] \\
C &\rightarrow D : [.2,3]
\end{align*}

and $B'$ is the set of the rules $J_j (j = 1,2,3)$:

\begin{align*}
J_1 &= C : [.1,.5] \rightarrow B \land D : [.7,1] \\
J_2 &= C : [.2,.3] \land (B \land D) : [.7,.9] \rightarrow A \land D : [.4,.7] \\
J_3 &= A \land D : [.6,.9] \rightarrow C : [.2,.3].
\end{align*}

**Step 1** Applying $R$, we have

\begin{align*}
A &\rightarrow [.2,.8] \\
B \land D &\rightarrow [.6,.9] \\
A \land D &\rightarrow [.5,.8] \\
C &\rightarrow [.2,.4] \\
D &\rightarrow [.8,1].
\end{align*}

**Step 2.** Since only $J_1$ is satisfied, we get

\begin{align*}
B \land D &\rightarrow [.7,.9]
\end{align*}

and the interval values of $A, A \land D, C, D$ are not varied.

**Step 3.** Repeating $R$, only the interval value of $A \land D$ is varied

\begin{align*}
A \land D &\rightarrow [.6,.8]
\end{align*}

**Step 4.** Repeating $T$, now $J_3$ and $J_2$ are satisfied, so we have

\begin{align*}
C &\rightarrow [.2,.3] \\
A \land D &\rightarrow [.6,.7].
\end{align*}

**Step 5.** $B^E$ is now changed into $B'$ which consists of:

\begin{align*}
B &\rightarrow A : [.9,1] \\
D &\rightarrow B : [.8,.9] \\
A &\rightarrow C : [.6,.8] \\
D &\rightarrow C : [.8,1] \\
C &\rightarrow [.2,.3].
\end{align*}
\( \mathcal{R} \) is repeated and we have

\[
A \land D : [0.6, 0.7].
\]

By virtue of that all rules in \( \mathcal{R} \) are satisfied in step 4, the interval value for the truth probability of \( A \land D \) is \([0.6, 0.7]\).

References

Abstract

The paper presents a method of logical reasoning in knowledge bases with uncertainty; such a knowledge base is given by a set of "knowledges" of two following forms:

1. \( < S, I > \) where \( S \) is a sentence, and \( I \subseteq [0,1] \) is an interval of the possible values for truth probability of \( S \).
2. \( < S_1, I_1 > \land < S_2, I_2 > \land \cdots \land < S_n, I_n > \rightarrow < S, I > \), where \( S_1, \ldots, S_n, S \) are sentences, and \( I_1, \ldots, I_n, I \) are the corresponding intervals of their truth probabilities.

Let \( \mathcal{B} \) be a such knowledge base, and \( S \) be a goal sentence. The interval of truth probabilities of \( S \) derived from \( \mathcal{B} \) can be found by the proposed method.