ON THE STOPAGE RULE IN SOLUTION FOR MONOTONE ILL-POSED PROBLEMS

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Abstract. The purpose of this note is to present an iterative method for solving a regularized equation for nonlinear monotone ill-posed problems in Banach space and to study its stoppage rule so that iteration sequence converges to a solution of initial problem, as the noisy data of the right-hand side converges to its exact.

1. INTRODUCTION

Let $X$ be a real reflexive Banach space and $X^*$ be its dual space. For the sake of simplicity norms of $X$ and $X^*$ will be denoted by one symbol $\| \cdot \|$. Let $A$ be a continuous and monotone operator with domain of definition $D(A) = X$ and range $R(A) \subseteq X^*$. Let $f_0$ be an element of $R(A)$.

Consider the nonlinear ill-posed problem.

$$A(x) = f_0.$$ (1.1)

By ill-posedness we mean that solution of (1.1) do not depend continuously on the data $f_0$. To solve (1.1) we can use variational method of Tikhonov regularization that consists of minimizing the functional

$$\| A(x) - f_\delta \|^2 + \alpha \Omega(x), \text{ over } D(A),$$ (1.2)

where $\Omega(x)$ is a some functional that plays the role of regularization, $\alpha > 0$ is a small parameter and the noisy data $f_\delta$ satisfies the condition

$$\| f_0 - f_\delta \| \le \delta \to 0.$$ 

In [1] and [2] they showed another version of Tikhonov regularization, that is the use of regularized equation

$$A(x) + \alpha U(x) = f_\delta$$ (1.3)

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instead of (1.2), where $U$ is a uniformly monotone operator or dual mapping of $X$. The regularized equation (1.3) is difficultly solved by iteration methods, because iterative parameters must satisfy very complex condition (see [3]). To overcome this difficulty [4] we showed another approach of regularization for (1.1). That is the use of linear and strongly monotone operator $B$ instead of $U$ in (1.3), i.e. the use of equation

$$A(x) + \alpha Bx = f_\delta,$$

(1.4)

with $\overline{D}(B) = X$, $S_0 \subset D(B)$, $S_0$ denotes the set of solutions of (1.1), $B$ is linear and

$$|x|^2 := \langle Bx, x \rangle \geq m_B \|x\|^2, \ x \in D(B), \ m_B > 0.$$  

(1.5)

Without loss of generality, assume that $m_B = 1$.

In [4], we present an iteration method for solving (1.4) with variable parameters of iteration. In this note, we consider another one for (1.4) and the stoppage rule for it, when $\delta \to 0$ and $\alpha$ is chosen such that $\delta/\alpha \to 0$. Our methods is a generation of [5] and [6] in Hilbert spaces for a Banach spaces.

Below the symbols $\to$ and $\rightharpoonup$ denote the strong and the weak convergence for any sequence, respectively.

2. MAIN RESULTS

As in [5], we require an additional condition on $A$:

For each $N > 0 \ \exists C_B(N) > 0$ such that

$$\langle A(x) - A(y), w \rangle \leq C_B(N) \|x - y\| |w|, \ x, y \in D(B),$$

$$|x|, |y| \leq N, \ \forall w \in X.$$  

(21)

Note that this condition was considered in [5] under $D(B) = X = H$, is a Hilbert space, and $B$ is bounded.

For finding $x_{\alpha}^\delta$, the regularized solution of (1.4), we study the iteration method

$$x^{n+1} = x^n - \rho B^{-1}(A_{\alpha}(x^n) - f_\delta), \ A_{\alpha} = A + \alpha B,$$

(2.2)

$\alpha > 0$ and $f_\delta$ are fixed ($x^0 \in D(B)$).

**Theorem 2.1.** Let $\langle Bx, y \rangle = \langle x, By \rangle$, $x, y \in D(B)$ and $0 < \rho < \frac{2\alpha}{(\alpha + C_B(N_0))^2}$, $N_0 : \|x_{\alpha}^\delta\| \leq N_0/2$. Then

$$x^n \to x_{\alpha}^\delta, \ \text{as} \ n \to +\infty.$$
Proof. From (2.2) it implies that
\[
\lambda_{n+1}^2 := \langle B(x^{n+1} - x_\alpha^\delta), x^{n+1} - x_\alpha^\delta \rangle
\]
\[
= \langle B(x^{n+1} - x^n), x^{n+1} - x^n \rangle + \lambda_n^2 + 2\langle B(x^{n+1} - x^n), x^n - x_\alpha^\delta \rangle
\]
\[
= \lambda_n^2 - 2\rho\langle A_\alpha(x^n) - A_\alpha(x_\alpha^\delta), x^n - x_\alpha^\delta \rangle
\]
\[
+ \rho^2\langle A_\alpha(x^n) - A_\alpha(x_\alpha^\delta), B^{-1}(A_\alpha(x^n) - A_\alpha(x_\alpha^\delta)) \rangle.
\]
Since
\[
\langle A_\alpha(x^n) - A_\alpha(x_\alpha^\delta), x^n - x_\alpha^\delta \rangle \geq \alpha\langle B(x^n - x_\alpha^\delta), x^n - x_\alpha^\delta \rangle = \alpha\lambda_n^2,
\]
\[
\langle A_\alpha(x^n) - A_\alpha(x_\alpha^\delta), B^{-1}(A_\alpha(x^n) - A_\alpha(x_\alpha^\delta)) \rangle =
\]
\[
= \langle A_\alpha(x^n) - A(x_\alpha^\delta) + \alpha B(x^n - x_\alpha^\delta), B^{-1}(A(x^n) - A(x_\alpha^\delta)) + \alpha(x^n - x_\alpha^\delta) \rangle =
\]
\[
= \alpha^2\lambda_n^2 + 2\alpha \langle A(x^n) - A(x_\alpha^\delta), x^n - x_\alpha^\delta \rangle
\]
\[
+ \langle A(x^n) - A(x_\alpha^\delta), B^{-1}(A(x^n) - A(x_\alpha^\delta)) \rangle,
\]
we have
\[
\lambda_{n+1}^2 \leq \lambda_n^2 - 2\rho\alpha\lambda_n^2 + \rho^2 \times
\]
\[
[\alpha^2\lambda_n^2 + 2\alpha \langle A(x^n) - A(x_\alpha^\delta), x^n - x_\alpha^\delta \rangle + \langle A(x^n) - A(x_\alpha^\delta), B^{-1}(A(x^n) - A(x_\alpha^\delta)) \rangle].
\]
Let \( N_0 \) be a number so that \( |x_\alpha^\delta| \leq N_0/2 \) and \( C_0 = C_B(N_0) \). We suppose that the recurrent hypothesis \( |x^n - x_\alpha^\delta| \leq N_0/2 \), then \( |x^n| \leq N_0 \) and
\[
\langle A(x^n) - A(x_\alpha^\delta), B^{-1}(A(x^n) - A(x_\alpha^\delta)) \rangle \leq C_0^2\lambda_n^2.
\]
We have
\[
\lambda_{n+1}^2 \leq \lambda_n^2 - 2\rho\alpha\lambda_n^2 + \rho^2[\alpha^2\lambda_n^2 + 2C_0\alpha\lambda_n^2 + C_0^2\lambda_n^2]
\]
\[
= [1 - 2\rho\alpha + \rho^2(\alpha^2 + 2\alpha C_0 + C_0^2)]\lambda_n^2.
\]
If we choose \( 0 < \rho < 2\alpha/(\alpha + C_0)^2 \), then \( \theta_\delta = 1 - 2\rho\alpha + \rho^2(\alpha + C_0)^2 < 1 \). Therefore, the hypothesis is verified. Consequently, \( \lambda_n \to 0 \). Hence, the convergence of \( x^n \) to \( x_\alpha^\delta \) follows from (1.5) and the definition of \( \lambda_n \).

Remark. In much cases, we can choose the unbounded operator \( B \) such that \( ||x||, ||y|| \leq N \to |x|, |y| \leq N \) in condition (2.1). In fact, for example \( Bx(t) = -d^2x(t)/dt^2 + c_0x(t) \), \( c_0 > 0 \), where \( D(B) \) is the closure in the norm \( W_q^2, 1 < q < 2 \) of all functions from \( C^2[0, 1] \) satisfying the condition \( u(0) = u(1) = 0 \). Then \( B^{-1}v(t) = \int_0^1 g(t, s) v(s) ds \) with
\[
g(t, s) = \begin{cases} u_1(t) u_2(s), & t \leq s, \\ u_2(t) u_1(s), & t \geq s, \end{cases}
\]
where \( u_1, u_2 \) are the nontrivial solutions of \( Bu = 0 \) such that \( u(0) = u(1) = 0 \). The derivatives are understood in generalized. Then, \( |x|^2 = \langle Bx, x \rangle = \int_0^1 c_0 x(t)^2 \, dt = c_0 \|x\|_{L^2[0,1]} \geq c_0 \tilde{c}_p \|x\|_{L^p[0,1]}, 2 < p < +\infty \) with some \( \tilde{c}_p > 0 \), since in these cases \( L^p[0,1] \) is continuously embedded in \( L^2[0,1] \).

For each \( \delta > 0 \), the value \( \alpha = \bar{\alpha} = \bar{\alpha}(\delta) \) is chosen that \( \delta / \bar{\alpha}(\delta) \to 0 \), then \( x^{\bar{\alpha}(\delta)} \to x_0 \), the solution of (1.1), as \( \delta \to 0 \). In order to approximate the solution \( x^{\bar{\alpha}(\delta)} \) of (1.3) with \( \alpha = \bar{\alpha} \), we can use the iterative process (2.2). It is important to indicate how many iterations (depending on \( \delta \)) are performed. Choices of \( n = n(\delta) \) are also called “stopping rules” in the literature.

We have the result

**Theorem 2.2.** If the first integer \( n = n(\delta) \) satisfying the condition \( |x^{n+1} - x^n| \leq a\delta \), where \( a > 0 \), then \( x^n(\delta) \to x_0 \), as \( \delta \to 0 \).

**Proof.** Indeed,

\[
|x^{n+1} - x^n| = \langle B(x^{n+1} - x^n), x^{n+1} - x^n \rangle \\
= \rho^2 \langle A_{\bar{\alpha}}(x^n) - f_\delta, B^{-1}(A_{\bar{\alpha}}(x^n) - f_\delta) \rangle \\
= \rho^2 \langle A_{\bar{\alpha}}(x^n) - f_0 + f_0 - f_\delta, B^{-1}(A_{\bar{\alpha}}(x^n) - f_0 + f_0 - f_\delta) \rangle \\
\geq \rho^2 \{\delta^2 - 2\delta \rho|x^{n+1} - x^n| + (A_{\bar{\alpha}}(x^n) - A_{\bar{\alpha}}(x_{\bar{\alpha}}), B^{-1}(A_{\bar{\alpha}}(x^n) - A_{\bar{\alpha}}(x_{\bar{\alpha}}))\}
\]

where \( x_{\bar{\alpha}} : A(x_{\bar{\alpha}}) + \bar{\alpha}Bx_{\bar{\alpha}} = f_0 \).

Since

\[
\langle A_{\bar{\alpha}}(x^n) - A_{\bar{\alpha}}(x_{\bar{\alpha}}), B^{-1}(A_{\bar{\alpha}}(x^n) - A_{\bar{\alpha}}(x_{\bar{\alpha}}))\bar{\alpha}|x^n - x_{\bar{\alpha}}| + \bar{\alpha}\langle A(x^n) - A(x_{\bar{\alpha}}), x^n - x_{\bar{\alpha}}\rangle \geq \bar{\alpha}^2|x^n - x_{\bar{\alpha}}|
\]

we have

\[
|x^{n+1} - x^n|^2 \geq \bar{\alpha}^2 \rho^2|x^n - x_{\bar{\alpha}}|^2 - 2\rho \delta|x^{n+1} - x^n| - \rho^2 \delta^2.
\]

Therefore,

\[
|x^n - x_{\bar{\alpha}}| \leq \frac{2\delta}{\bar{\alpha}}.
\]

This means that \( x^n \to x_0 \), as \( \delta \to 0 \).

**REFERENCES**


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