AN ITERATIVE METHOD FOR SOLUTION OF NONLINEAR OPERATOR EQUATION

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Abstract. In the note, for finding a solution of nonlinear operator equation of Hammerstein's type an iterative process in infinite-dimensional Hilbert space is shown, where a new iteration is constructed basing on two last steps. An example in the theory of nonlinear integral equations is given for illustration.

1. INTRODUCTION

Let $H$ be a real Hilbert space with the norm and scalar product denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively.

Let $F_i$, $i = 1, 2$, be nonlinear monotone operators in $H$, i.e.

$$\langle F_i(x) - F_i(y), x - y \rangle \geq 0, \forall x, y \in D(F_i) \equiv H, \ i = 1, 2.$$ 

The operator equation of Hammerstein's type

$$x + F_2F_1(x) = f_0, \ f_0 \in H \quad (1.1)$$

was considered by several authors (see [1], [2], [4-7], [12-17] and bibliography there). In [10], an iterative process was given for solving (1.1) with the linear property of $F_2$. In [6], the author proposed a method of regularization for the solution of (1.1) in the case, where both the operators $F_i$ are nonlinear and monotone.

In the note, basing on our result in [6] and the idea of iterative regularization proposed by A. Bakyshinski (see [3]), we give a two-step iteration method for solving (1.1) in infinite-dimensional Hilbert space $H$. The result is illustrated by an example in the theory of nonlinear integral equations.

Note that, recently, the problem of approximating a solution of (1.1) is investigated extensively because of its importance in applications (see [8], [9], [11], [16]).
2. MAIN RESULT

Let \( x^1 \) and \( x^2 \) be two arbitrary elements of \( H \). The iteration procedure is defined by

\[
x^{n+2} = \varphi_1^{n+1}(x^{n+1}) + \beta_{n+1} \left[ \varphi_2^n \left( \frac{(x^{n+1} - \varphi_1^n(x^n))}{\beta_n} \right) - \beta_n x^n \right], \quad (2.1)
\]

where

\[
\varphi_i^n(x) = x - \beta_n (F_i(x) + \alpha_n x + a_i f_0), \quad i = 1, 2,
\]

\[
a_1 = 0, \quad a_2 = -1,
\]

and \( \{\alpha_n\} \) and \( \{\beta_n\} \) are two sequences of positive numbers. Later, we see, as in [3], that \( \alpha_n \) plays the role of regularization and \( \beta_n \), the role of iteration parameter.

Theorem. If (1.1) has a solution and there exist the constants \( L_i > 0 \) such that

\[
||F_i(x)|| \leq L_i (1 + ||x||), \quad i = 1, 2, \forall x \in H,
\]

then iteration process (2.1) converges to a solution of (1.1) under the condition

\[
\alpha_n, \beta_n > 0, \quad \alpha_n \to 0, \quad \lim_{n \to \infty} \frac{\beta_n - \beta_{n+1}}{\beta_n \alpha_n^2} = 0, \quad \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty.
\]

Proof. Put

\[
y^n = \frac{x^{n+1} - \varphi_1^n(x^n)}{\beta_n}, \quad n = 1, 2, \ldots \quad (2.3)
\]

Then from (2.1) and (2.2) we have

\[
y^{n+1} = \frac{x^{n+2} - \varphi_1^{n+1}(x^{n+1})}{\beta_{n+1}}
\]

\[
= \varphi_2^n(y^n) - \beta_n x^n
\]

\[
= y^n - \beta_n (F_2(y^n) + x_n + \alpha_n y^n - f_0).
\]

On the other hand, from (2.3) and (2.2) we also obtain

\[
x^{n+1} = \varphi_1^n(x^n) + \beta_n y^n
\]

\[
= x^n - \beta_n (F_1(x^n) - y_n + \alpha_n x^n), \quad n = 1, 2, \ldots
\]
In the Hilbert space $H_1 = H \times H$ with the scalar product denoted by $\langle z_1, z_2 \rangle_1 = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle$, where $z_i = [x_i, y_i]$, $x_i, y_i \in H$, we can write

$$z^{n+1} = z^n - \beta_n \left( \mathcal{F}(z^n) + \alpha_n z^n - f_0 \right), \quad (2.4)$$

$$\mathcal{F}(z^n) = [F_1(x^n), F_2(y^n)] + [-y^n, x^n],$$

where $z = [x, y]$, $f_0 = [\theta, f_0]$, $\theta$ denotes the zero element in $H$. It is easy to verify that in the Hilbert space $H_1$, $\mathcal{F}$ is a monotone operator. However, without any difficulty we can see that $\mathcal{F}$ satisfies the condition

$$||\mathcal{F}(z)|| \leq \sqrt{2} \max L_i (1 + ||z||_1),$$

where $||.||_1$ is the norm of $H_1$ generated by $\langle ., . \rangle_1$.

Applying Theorem 5.1 (p. 144) in [3] to the process (2.4), we can conclude that the sequence $\{z^n\}$ converges in $H_1$ to $z_0 = [x_0, F_1(x_0)]$, one solution of the equation

$$\mathcal{F}(z) = f_0.$$ 

Therefore, the sequence $\{x^n\}$ converges in $H$ to $x_0$, as $n \to \infty$. Theorem is proved.

**Remarks.**

1. The sequence $\beta_n = (1 + n)^{-1/2}$ and $\alpha_n = (1 + n)^{-p}$, $0 < p < 1/2$, satisfy all the conditions in the theorem.

2. If $F_i$ are Lipschitz continuous with a Lipschitz constants $L_i$, then $\mathcal{F}$ also is Lipschitz continuous with Lipschitz constant $\mathcal{L} = 2\sqrt{\max \{1, L_1, L_2\}}$. Applying Theorem 5.2 in [3], we obtain the result that the iteration process (2.1) converges in $H$ to a solution of (1.1), if

$$\lim_{n \to \infty} \beta_n \frac{(1 + \alpha_n^2)}{\alpha_n} < \frac{2}{\mathcal{L}^2}.$$ 

In this case, we can chose the sequence $\beta_n = \theta \alpha_n$,

$$\alpha_n = (1 + n)^{-p}, \quad 0 < p < 1/2, \quad \theta < \frac{2}{(1 + \alpha_0)^2 \mathcal{L}^2}.$$ 

3. **APPLICATION**

Consider the nonlinear integral equation of Hammerstein's type

$$\varphi(t) + \int_0^1 k(t, s) f(\varphi(s)) \, ds = f_0(t), \quad t \in [0, 1], \quad \varphi \in L_2[0, 1], \quad (3.1)$$
where \( f_0(t) \in L_2[0,1] \), \( k(t,s) \geq 0 \) is continuous and \( f(t) \) is a nondecreasing and bounded function satisfying the condition \( |f(t)| \leq a_0 + b_0|t| \), \( t \in \mathbb{R} \). Then,

\[
(F_1 \varphi)(t) = f(\varphi(t)), \quad \varphi(t) \in L_2[0,1],
\]

\[
(F_1 \xi)(t) = \int_0^1 k(t,s)\xi(s)ds, \quad \xi(t) \in L_2[0,1].
\]

Since \( k(t,s) \geq 0 \) and \( f(t) \) is nondecreasing, then \( F_i, i = 1, 2 \), are monotone. The continuity of \( k(t,s) \) implies that \( F_2 \) is bounded. It is not difficult to prove that \( F_i \) satisfy the conditions of the main theorem. Therefore, in order to obtain approximate solution for (3.1) we can apply the process (2.1) with defined above \( F_i \) and \( \alpha_n, \beta_n \) in the remark 1.

**REFERENCES**


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