SOME INTUITIONISTIC LINGUISTIC AGGREGATION OPERATORS

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Abstract. Information aggregation is a usual task in human activity. Linguistic aggregation operators are used to aggregate information given in terms of linguistic labels. The use of linguistic labels has been posed due to the nature of the information or the habit of experts when they give assessments. In this paper, the notion of intuitionistic linguistic label is first introduced. This notion may be useful in situations when evaluations of experts are presented as two labels such that the first expresses the degree of membership, and the second expresses the degree of non-membership as in the intuitionistic fuzzy theory [1, 2]. Some intuitionistic linguistic aggregation operators are also proposed.

Keywords. Linguistic aggregation operator, intuitionistic fuzzy set.

1. INTRODUCTION

Group decision making (GDM) has played an important role in daily activities, such as economic, engineering, education, medical, etc. In GDM, one of the problems involves gathering many sources of information, giving the final result via aggregating process. Due to the nature of the information or the habit of experts when they give assessments, information could be given as linguistic labels. Many aggregation operators and linguistic aggregation procedures in GDM problems were presented (see [3] for an overview). In this paper, the novel notion of intuitionistic linguistic label, which inherits ideas of intuitionistic fuzzy set and linguistic label, is first introduced. Then, some linguistic aggregations are presented in intuitionistic linguistic environment.

In this section, a short overview of linguistic aggregation operators and intuitionistic fuzzy sets are presented.
1.1. Linguistic Aggregation Operators

In many problems, the information about quality, comforts, suitability, efficiency, etc., of objects may be given as linguistic labels \([3, 6, 9]\). For example, the comforts of a car can be evaluated using linguistic labels: poor, fair, good, etc. The set of linguistic labels can be constructed depending on the characteristic real word problems. However, it generally contains an odd number of linguistic labels (7 and 9 for example). The set of linguistic labels is theoretically given by \(S = \{s_1, s_2, \ldots, s_n\}\), where the odd number \(n\) is the cardinality of \(S\), \(s_i\) is a possible value of linguistic evaluation in some situations. The set \(S\) is equipped an order relation and a negation operator \([9]\):

\[
s_i \geq s_j \iff i \geq j;
\]

\[
\text{neg} (s_i) = s_j \iff j + i = n + 1.
\]

Linguistic aggregation operators are including \([16]\): linear order based linguistic aggregation operators, extension principle and symbols based linguistic aggregation operators, linguistic 2-tuple based linguistic aggregation operators, linguistic aggregation operators computing with words directly.

In this paper, the linear order based linguistic aggregation operators should be extended to intuitionistic case.

1.2. Linear order based linguistic aggregation operators

Let \(\{a_1, a_2, \ldots, a_m\}\) be a collection of linguistic labels, \(a_i \in S\), and \(\{b_1, b_2, \ldots, b_m\}\) is a permutation of \(\{a_1, a_2, \ldots, a_m\}\) yields \(b_1 \geq b_2 \geq \cdots \geq b_m\). Yager et al. \([18-20]\) introduced some simple linguistic aggregation operators:

- linguistic max operator: \(\max (a_1, a_2, \ldots, a_m) = b_1\);
- linguistic min operator: \(\min (a_1, a_2, \ldots, a_m) = b_m\);

and linguistic median operator: \(\text{med} (a_1, a_2, \ldots, a_m) = \begin{cases} \frac{b_{m+1}}{2} & \text{if } m \text{ is odd}, \\ \frac{b_m + b_{m+1}}{2} & \text{if } m \text{ is even}. \end{cases}\)

Using above operators, many other operators were developed for aggregating linguistic information: ordinal ordered weighted averaging operator (Yager \([14]\)), linguistic weighted disjunction and linguistic weighted conjunction operators (Herrera and Herrera-Viedma \([13]\)), hybrid aggregation operators (Xu \([15]\)), etc.

As a similarity of weighted median in statistics, Yager \([15, 16, 17]\) defined weighted median of linguistic labels:

Considering a collection of linguistic labels \(\{a_1, a_2, \ldots, a_m\}\), each label \(a_i\) has corresponding weight: \(w_i, w_i \in [0, 1], \sum_{i=1}^{m} w_i = 1\). Such collection is denoted by \(\{(w_1, a_1), (w_2, a_2), \ldots, (w_m, a_m)\}\). Assume that \(\{(u_1, b_1), (u_2, b_2), \ldots, (u_m, b_m)\}\) is the decreasingly ordered collection of \(\{(w_1, a_1), (w_2, a_2), \ldots, (w_m, a_m)\}\), i.e., \(b_j\) is the \(j\)-th largest of \(a_i\), and \(u_j\) is the weight of \(j\)-th largest of \(a_i\). Let \(T_j = \sum_{i=1}^{j} u_j\) be, the linguistic weighted median (\(LWM\)) operator was defined as:

\[
LWM \left((w_1, a_1), (w_2, a_2), \ldots, (w_m, a_m)\right) = b_k,
\]

where \(k\) is the value such that \(T_k\) first crosses 0.5. Yager \([12]\) proved that \(LWM\) operator is idempotent, commutative, and monotonous.
1.3. Intuitionistic Fuzzy Set

The intuitionistic fuzzy set first launched by Atanassov [1] is one of the significant extensions of Zadeh’s fuzzy set [20]. An intuitionistic fuzzy set has two components: a membership function and a non-membership function, it is different from fuzzy set which characterized by only a membership function.

Definition 1.1 ([1]) An intuitionistic fuzzy set $A$ on a universe $X$ is an object of the form $A = \{ (x, \mu_A(x), \nu_A(x)) | x \in X \}$, where $\mu_A(x) \in [0,1]$ is called the “degree of membership of $x$ in $A$”, $\nu_A(x) \in [0,1]$ is called the “degree of non-membership of $x$ in $A$”, and following condition is satisfied

$$\mu_A(x) + \nu_A(x) \leq 1, \ \forall x \in X.$$ 

Some recent developments of the intuitionistic fuzzy set theory with applications could be found in [4, 5, 7, 10, 11].

2. INTUITIONISTIC LINGUISTIC LABELS

The intuitionistic linguistic label defined below can be seen as a linguistic aspect supplement of intuitionistic fuzzy set. It may be helpful when the information is expressed in terms of pair of labels $(s_i, s_j)$, where $s_i$ represents the degree of membership and $s_j$ the degree of non-membership.

Example 2.1. We recall the intuitionistic approach of De and Biswas in medical diagnosis [9], the correspondences between the set of patients and the set of symptoms were be described via an intuitionistic fuzzy relation as in Table 1 (see [5] for intuitionistic fuzzy relation). It is reasonable and meaningful that we allow experts to use linguistic labels instead of numbers. Such situation raised the need of using linguistic in intuitionistic assessments. Using linguistic label set $S$ containing $s_1 = \text{impossibly}$, $s_2 = \text{very unlikely}$, $s_3 = \text{less likely}$, $s_4 = \text{likely}$, $s_5 = \text{more likely}$, $s_6 = \text{very likely}$, and $s_7 = \text{certainly}$, experts’ assessments may be given in Table 1 (membership degree of Paul to the set of all patients who have a temperature is assigned to $s_7 = \text{certainly}$, non-membership degree of Paul to the set of all patients who have a temperature is assigned to $s_1 = \text{impossibly}$).

<table>
<thead>
<tr>
<th>Q</th>
<th>Temperature</th>
<th>Headache</th>
<th>Stomach pain</th>
<th>Cough</th>
<th>Chest pain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paul</td>
<td>(0.8, 0.1)</td>
<td>(0.6, 0.1)</td>
<td>(0.2, 0.8)</td>
<td>(0,6, 0.1)</td>
<td>(0.1, 0.6)</td>
</tr>
<tr>
<td>Jadu</td>
<td>(0, 0.8)</td>
<td>(0.4, 0.4)</td>
<td>(0.6, 0.1)</td>
<td>(0.1, 0.7)</td>
<td>(0.1, 0.8)</td>
</tr>
<tr>
<td>Kundu</td>
<td>(0.8, 0.1)</td>
<td>(0.8, 0.1)</td>
<td>(0, 0.6)</td>
<td>(0.2, 0.7)</td>
<td>(0, 0.5)</td>
</tr>
<tr>
<td>Rohit</td>
<td>(0.6, 0.1)</td>
<td>(0.5, 0.4)</td>
<td>(0.3, 0.4)</td>
<td>(0.7, 0.2)</td>
<td>(0.3, 0.4)</td>
</tr>
</tbody>
</table>

Table 1: Intuitionistic fuzzy relation between patients and symptoms [9]

Moreover, in intuitionistic fuzzy set theory, the membership degree and the non-membership degree of $x$ in the set $A$ ($\mu_A(x)$ and $\nu_A(x)$ respectively) must satisfy $\mu_A(x) + \nu_A(x) \leq 1$. This condition can be rewritten as $\mu_A(x) \leq \text{neg} (\nu_A(x))$, where $\text{neg} : [0,1] \rightarrow [0,1]$, $x \mapsto 1 - x$. So, we propose that for $(s_i, s_j)$ the condition $s_j \leq \text{neg} (s_i) = s_{n+1-i}$ should be satisfied. Then, this implies $s_j \leq s_{n-i+1}$ or $i + j \leq n + 1$. 
Definition 2.1. An intuitionistic linguistic label is defined as a pair of linguistic labels \((s_i, s_j) \in S^2\), such results in \(i + j \leq n + 1\), where \(S = \{s_1, s_2, \ldots, s_n\}\) is the linguistic label set, \(s_i, s_j \in S\) respectively define the degree of membership and the degree of non-membership of an object in a set.

The set of all intuitionistic linguistic labels is denoted by \(IS\), i.e.

\[
IS = \{ (s_i, s_j) \in S^2 \mid i + j \leq n + 1 \}.
\]

Example 2.2. If the linguistic label set \(S\), which may be used in medical diagnoses, contains \(s_1 = \text{impossibly}\), \(s_2 = \text{very unlikely}\), \(s_3 = \text{less likely}\), \(s_4 = \text{likely}\), \(s_5 = \text{more likely}\), \(s_6 = \text{very likely}\) and \(s_7 = \text{certainly}\); then, the corresponding intuitionistic linguistic label set of \(IS\) is given below:

\[
\begin{align*}
(s_7, s_1) & \quad (s_6, s_1) & \quad (s_6, s_2) \\
(s_5, s_1) & \quad (s_5, s_2) & \quad (s_5, s_3) \\
(s_4, s_1) & \quad (s_4, s_2) & \quad (s_4, s_3) & \quad (s_4, s_4) \\
(s_3, s_1) & \quad (s_3, s_2) & \quad (s_3, s_3) & \quad (s_3, s_4) & \quad (s_3, s_5) \\
(s_2, s_1) & \quad (s_2, s_2) & \quad (s_2, s_3) & \quad (s_2, s_4) & \quad (s_2, s_5) & \quad (s_2, s_6) \\
(s_1, s_1) & \quad (s_1, s_2) & \quad (s_1, s_3) & \quad (s_1, s_4) & \quad (s_1, s_5) & \quad (s_1, s_6) & \quad (s_1, s_7)
\end{align*}
\]

Table 2: Relation between Patients and Symptoms

<table>
<thead>
<tr>
<th>(Q)</th>
<th>Temperature</th>
<th>Headache</th>
<th>Stomach pain</th>
<th>Cough</th>
<th>Chest pain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paul</td>
<td>((s_7, s_1))</td>
<td>((s_6, s_1))</td>
<td>((s_2, s_5))</td>
<td>((s_6, s_1))</td>
<td>((s_1, s_6))</td>
</tr>
<tr>
<td>Jadu</td>
<td>((s_1, s_7))</td>
<td>((s_4, s_4))</td>
<td>((s_6, s_1))</td>
<td>((s_1, s_6))</td>
<td>((s_1, s_7))</td>
</tr>
<tr>
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<td>((s_5, s_1))</td>
<td>((s_4, s_1))</td>
<td>((s_1, s_7))</td>
<td>((s_2, s_6))</td>
<td>((s_1, s_4))</td>
</tr>
<tr>
<td>Rohit</td>
<td>((s_5, s_1))</td>
<td>((s_5, s_3))</td>
<td>((s_3, s_4))</td>
<td>((s_6, s_1))</td>
<td>((s_2, s_3))</td>
</tr>
</tbody>
</table>

3. ORDER RELATIONS ON \(IS\)

In order to define the linear order based intuitionistic linguistic aggregation operators, it is necessary to define order relations on the \(IS\) set.

Let \(A, B\) be an intuitionistic fuzzy set on \(X\), relation \(A \supset B\) is defined as \([1][2]\):

\[
A \supset B \iff (\forall x \in X) (\mu_A(x) \geq \mu_B(x) \land \nu_A(x) \leq \nu_B(x)).
\]

Order relation on two intuitionistic linguistic labels \((\mu_1, \nu_1), (\mu_2, \nu_2)\) can be defined similarly to “\(\supset\)” relation of intuitionistic fuzzy sets:

\[
(\mu_1, \nu_1) \supset (\mu_2, \nu_2) \iff \mu_1 \geq \mu_2 \land \nu_1 \leq \nu_2,
\]

where \((\mu_1, \nu_1), (\mu_2, \nu_2)\) are intuitionistic linguistic labels.

It is easily seen that there are intuitionistic linguistic labels which cannot be compared by this relation (for example \((s_1, s_5)\) and \((s_2, s_6)\)). However, when comparing two intuitionistic linguistic labels, first we can compare two membership degrees, then two non-membership degrees, vice versa. Then, we can define two order relations on \(IS\) as following definition.
Definition 3.1. For all of \((\mu_1, \nu_1), (\mu_2, \nu_2)\) on IS, membership based order relation \(\geq_M\) and non-membership based order relation \(\geq_N\) are defined as the following:

\[
(\mu_1, \nu_1) \geq_M (\mu_2, \nu_2) \iff \mu_1 \geq \mu_2 \text{ OR } (\mu_1 = \mu_2 \text{ \& \ } \nu_1 \leq \nu_2);
\]

\[
(\mu_1, \nu_1) \geq_N (\mu_2, \nu_2) \iff \nu_1 < \nu_2 \text{ OR } (\nu_1 = \nu_2 \text{ \& \ } \mu_1 \geq \mu_2).
\]

Theorem 3.1. \(\geq_M\) and \(\geq_N\) are total orders.

Proof. Let’s consider \(\geq_M\). It is easily seen that \(\geq_M\) is reflexive. Now we consider the anti-symmetry, transitivity and totality. Let \((\mu_1, \nu_1), (\mu_2, \nu_2), (\mu_3, \nu_3)\) be arbitrary intuitionistic linguistic labels, we obtain:

\[
\text{Anti-symmetry: } \left\{ \begin{array}{c}
(\mu_1, \nu_1) \geq_M (\mu_2, \nu_2) \\
(\mu_2, \nu_2) \geq_M (\mu_1, \nu_1)
\end{array} \right\} \iff \left\{ \begin{array}{c}
\mu_1 \geq \mu_2 \\
\mu_2 = \mu_1 \\
\nu_1 \leq \nu_2 \\
\nu_2 \leq \nu_1
\end{array} \right\} \iff (\mu_1, \nu_1) = (\mu_2, \nu_2).
\]

\[
\text{Transitivity: } \left\{ \begin{array}{c}
(\mu_1, \nu_1) \geq_M (\mu_2, \nu_2) \\
(\mu_2, \nu_2) \geq_M (\mu_3, \nu_3)
\end{array} \right\} \iff \left\{ \begin{array}{c}
\mu_1 > \mu_2 \\
\mu_2 = \mu_3 \\
\nu_1 \leq \nu_2 \\
\nu_2 \leq \nu_3
\end{array} \right\} \iff (\mu_1, \nu_1) \geq_M (\mu_3, \nu_3).
\]

\[
\Rightarrow \mu_1 > \mu_3 \text{ OR } \left\{ \begin{array}{c}
\mu_1 = \mu_3 \\
\nu_1 \leq \nu_3
\end{array} \right\} \iff (\mu_1, \nu_1) \geq_M (\mu_3, \nu_3).
\]

Totality: If \(\mu_1 > \mu_2\), then \((\mu_1, \nu_1) \geq_M (\mu_2, \nu_2)\). If \(\mu_1 < \mu_2\), then \((\mu_2, \nu_2) \geq_M (\mu_1, \nu_1)\).

If \(\mu_1 = \mu_2\), then there are following cases:

Case 1. If \(\nu_1 \leq \nu_2\), then \((\mu_1, \nu_1) \geq_M (\mu_2, \nu_2)\).

Case 2. If \(\nu_1 > \nu_2\), then \((\mu_2, \nu_2) \geq_M (\mu_1, \nu_1)\).

So, \(\geq_M\) is a total order. Similarly, \(\geq_N\) is also a total order.

In the following, the relationship between \(\geq, \geq_M\) and \(\geq_N\) is explored. For convenience, in each \(A = (s_i, s_j) \in IS, s_i\) and \(s_j\) are respectively denoted by \(\mu_A, \nu_A\).

Theorem 3.2. For all \(A, B \in IS\), we obtain

\[
A \geq B \iff A \geq_M B \text{ \& \ } B \geq_N A,
\]

where \(\geq\) is defined as \(\geq_M\).
Proof. If \( A \geq B \), then
\[
\begin{align*}
\{ \mu_A \geq \mu_B \} &\Rightarrow \mu_A > \mu_B \quad \text{OR} \quad \{ \mu_A = \mu_B \} \quad \text{OR} \quad \{ \mu_A \geq \mu_B \} \\
\nu_A \leq \nu_B &\Rightarrow \nu_A < \nu_B \quad \text{OR} \quad \{ \nu_A = \nu_B \} \quad \text{OR} \quad \{ \nu_A \geq \nu_B \} \\
\end{align*}
\]

We assume that \( A \geq_M B \) and \( B \geq_N A \). Then,
\[
\begin{align*}
\left\{ \mu_A > \mu_B \right\} &\quad \text{OR} \quad \left\{ \mu_A = \mu_B \right\} \quad \text{OR} \quad \left\{ \mu_A \geq \mu_B \right\} \\
\nu_B > \nu_A &\quad \text{OR} \quad \{ \nu_A = \nu_B \} \quad \text{OR} \quad \{ \nu_A \geq \nu_B \} \\
\end{align*}
\]

\[\iff \{ \mu_A > \mu_B \} \quad \text{OR} \quad \{ \mu_A = \mu_B \} \quad \text{OR} \quad \{ \mu_A \geq \mu_B \} \quad \text{OR} \quad \{ \mu_A = \mu_B \} \quad \text{OR} \quad \{ \nu_A = \nu_B \} \]

\[\iff \{ \mu_A > \mu_B \} \quad \text{OR} \quad \{ \mu_A = \mu_B \} \quad \text{OR} \quad \{ \nu_B \geq \nu_A \} \impliedby \quad A \geq B.\]

4. LINEAR ORDERING BASED INTUITIONISTIC LINGUISTIC AGGREGATION OPERATORS

4.1. Definitions

Definition 4.1. Let \( \{A_1, A_2, \ldots, A_m\} \) be a collection of intuitionistic linguistic labels on \( IS \), and \( \{B_1, B_2, \ldots, B_m\} \) be a permutation of \( \{A_1, A_2, \ldots, A_m\} \), such yields \( B_1 \geq_M B_2 \geq M \cdots \geq_M B_m \).

- Membership based intuitionistic linguistic max and min operators are determined as
  \[
  \max_M (A_1, A_2, \ldots, A_m) = B_1 \text{ and } \min_M (A_1, A_2, \ldots, A_m) = B_m.
  \]

- Membership based intuitionistic linguistic median operator is determined as
  \[
  \text{med}_M (A_1, A_2, \ldots, A_m) = \begin{cases} 
  B_{m+1} & \text{if } m \text{ is odd}, \\
  B_m & \text{if } m \text{ is even}.
  \end{cases}
  \]

- Membership based intuitionistic linguistic weighted median: The collection of \( \{(w_1, A_1) \), \((w_2, A_2)\), \ldots, \((w_m, A_m)\)\} is considered, where \( A_i \) is an intuitionistic linguistic label, and \( w_i \) is its associated weight, \( w_i \in [0, 1], \sum_{i=1}^{m} w_i = 1 \). We assume that \( \{(u_1, B_1) , (u_2, B_2) , \ldots, (u_m, B_m)\} \) is the ordered collection of \( \{(w_1, A_1) , (w_2, A_2), \ldots, (w_m, A_m)\} \), where \( B_j \) is the \( j \)-th largest of the \( A_i \), and \( u_j \) is the weight of the \( j \)-th largest of \( A_i \). Let \( T_i = \sum_{j=1}^{i} u_j \), membership based intuitionistic linguistic weighted median \( (ILW M_M) \) operator was defined as
  \[
  ILW M_M ((w_1, A_1) , (w_2, A_2), \ldots, (w_m, A_m)) = B_k,
  \]
  where \( k \) is the value such that \( T_k \) first crosses 0.5.
Similarly, notions of non-membership based intuitionistic linguistic max, min, median and weighted median (max, min, med, ILWM) are also given:

{B_1, B_2, \ldots, B_m} is a permutation of {A_1, A_2, \ldots, A_m}, such result in \( B_1 \succeq_N B_2 \succeq_N \ldots \succeq_N B_m \).

\[
\max_N(A_1, A_2, \ldots, A_m) = B_1 \quad \text{and} \quad \min_N(A_1, A_2, \ldots, A_m) = B_m.
\]

\[
\text{med}_N(A_1, A_2, \ldots, A_m) = \begin{cases} 
B_{\frac{m+1}{2}} & \text{if } m \text{ is odd}, \\
B_{\frac{m}{2}} & \text{if } m \text{ is even}.
\end{cases}
\]

We assume that \( u_j \) is the weight of the \( j \)-th largest of \( A_i \). Let \( T_i = \sum_{j=1}^{i} u_j \).

\[
ILWM_N((w_1, A_1), (w_2, A_2), \ldots, (w_m, A_m)) = B_k,
\]

where \( k \) is the value such that \( T_k \) first crosses 0.5.

**Example 4.1.** Considering \( S = \{s_1, s_2, \ldots, s_9\}, A_1 = (s_1, s_6), A_2 = (s_2, s_7), A_3 = (s_5, s_4), A_4 = (s_7, s_3) \) and \( A_5 = (s_4, s_2) \). We obtain

- \( 4 >_M A_3 >_M A_5 >_M A_2 >_M A_1 \) and \( A_5 >_N A_4 >_N A_3 >_N A_1 >_N A_2 \).
- \( \max_M(A_1, A_2, A_3, A_4, A_5) = A_4 \) and \( \min_M(A_1, A_2, A_3, A_4, A_5) = A_1 \).

\[ \text{max}_N(A_1, A_2, A_3, A_4, A_5) = A_5 \text{ and } \text{min}_N(A_1, A_2, A_3, A_4, A_5) = A_2. \]

- \( \text{med}_M(A_1, A_2, A_3, A_4, A_5) = A_5 \) and \( \text{med}_N(A_1, A_2, A_3, A_4, A_5) = A_3. \)
- Considering \( w_1 = 0.15, w_2 = 0.34, w_3 = 0.25, w_4 = 0.12, w_5 = 0.14. \)

If order relation is \( \geq_M \), we obtain

| \( j \) | \( B_j \) | \( u_j \) | \( T_j \) |
|---|---|---|
| 1 | \( A_4 \) | \( w_4 = 0.12 \) | 0.12 |
| 2 | \( A_3 \) | \( w_3 = 0.25 \) | 0.37 |
| 3 | \( A_5 \) | \( w_5 = 0.14 \) | 0.51 |
| 4 | \( A_2 \) | \( w_2 = 0.34 \) | 0.87 |
| 5 | \( A_1 \) | \( w_1 = 0.15 \) | 0.87 |

So, \( ILWM_M((w_1, A_1), (w_2, A_2), (w_3, A_3), (w_4, A_4), (w_5, A_5)) = A_3. \)

Similarly, if order relation is \( \geq_N \), we obtain

\[ ILWM_N((w_1, A_1), (w_2, A_2), (w_3, A_3), (w_4, A_4), (w_5, A_5)) = A_3. \]

**Remark 4.1.** The application of \( \max_M, \min_M, \text{med}_M, \text{ILWM}_M \) or \( \max_N, \min_N, \text{med}_N, \text{ILWM}_N \) may obtain different results from each other. Subject to more due attention to membership or non-membership degree of assessment, the first or second group of operator is respectively proposed to use.
4.2. Some properties

The following theorem gives an efficient method for calculating with operators $\max_M$ and $\min_M$.

**Theorem 4.1.** Let $\{A_1, A_2, \ldots, A_m\}$ be a collection of intuitionistic linguistic arguments on $\text{IS}$. The following properties yield:

\[
\max_M (A_1, A_2, \ldots, A_m) = \left( \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m}), \min_{i: \mu_{A_i} = \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})} \{\nu_{A_i}\} \right),
\]

\[
\min_M (A_1, A_2, \ldots, A_m) = \left( \min (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m}), \max_{i: \mu_{A_i} = \min (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})} \{\nu_{A_i}\} \right).
\]

**Proof.** It is easily seen that

\[
\left( \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m}), \min_{i: \mu_{A_i} = \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})} \{\nu_{A_i}\} \right) \in \{A_1, A_2, \ldots, A_m\}.
\]

For each $A_j \in \{A_1, A_2, \ldots, A_m\}$, there are two following cases:

If $\mu_{A_j} < \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})$, then

\[
A_j < M \left( \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m}), \min_{i: \mu_{A_i} = \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})} \{\nu_{A_i}\} \right).
\]

If $\mu_{A_j} = \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})$, then $\nu_{A_j} \geq \min_{i: \mu_{A_i} = \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})} \{\nu_{A_i}\}$. So,

\[
A_j \leq \left( \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m}), \min_{i: \mu_{A_i} = \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})} \{\nu_{A_i}\} \right).
\]

Therefore,

\[
\max_M (A_1, A_2, \ldots, A_m) = \left( \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m}), \min_{i: \mu_{A_i} = \max (\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})} \{\nu_{A_i}\} \right).
\]

The rest of proof runs as before.

**Theorem 4.2.** If $\{A_1, A_2, \ldots, A_m\}$ has the smallest and largest elements, then

\[
\max_M (A_1, A_2, \ldots, A_m) = \max_N (A_1, A_2, \ldots, A_m) = \max (A_1, A_2, \ldots, A_m),
\]

\[
\min_M (A_1, A_2, \ldots, A_m) = \min_N (A_1, A_2, \ldots, A_m) = \min (A_1, A_2, \ldots, A_m).
\]
**Proof.** We have \( (\max(A_1, A_2, \ldots, A_m), \min(\nu_{A_1}, \nu_{A_2}, \ldots, \nu_{A_m})) = \max(A_1, A_2, \ldots, A_m) \)

\[
\begin{align*}
\leq_M \max_M (A_1, A_2, \ldots, A_m) &= \left( \max(\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m}), \min_{i: \mu_{A_i} = \max(\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})} \{\nu_{A_i}\} \right) \\
\Rightarrow \min(\nu_{A_1}, \nu_{A_2}, \ldots, \nu_{A_m}) &\geq \min_{i: \mu_{A_i} = \max(\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})} \{\nu_{A_i}\}.
\end{align*}
\]

On the other hand, \( \min(\nu_{A_1}, \nu_{A_2}, \ldots, \nu_{A_m}) \leq \min_{i: \mu_{A_i} = \max(\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})} \{\nu_{A_i}\} \). Then

\[
\min(\nu_{A_1}, \nu_{A_2}, \ldots, \nu_{A_m}) = \min_{i: \mu_{A_i} = \max(\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_m})} \{\nu_{A_i}\}
\]

\[
\Rightarrow \max_M (A_1, A_2, \ldots, A_m) = \max(A_1, A_2, \ldots, A_m).
\]

The remains are similarly proved.

**Theorem 4.3.** \( ILWM_M \) is idempotent, compensative, commutative and monotonous, i.e.

- \( ILWM_M(A, A, \ldots, A) = A \) for all \( A \in IS \).
- \( \min_M(A_1, A_2, \ldots, A_m) \leq ILWM_M(A_1, A_2, \ldots, A_m) \leq \max_M(A_1, A_2, \ldots, A_m) \) for all \( A_1, A_2, \ldots, A_m \in IS \).
- \( ILWM_M((w_1, A_1), (w_2, A_2), \ldots, (w_m, A_m)) = ILWM_M((w_{\sigma(1)}, A_{\sigma(1)}), (w_{\sigma(2)}, A_{\sigma(2)}), \ldots, (w_{\sigma(m)}, A_{\sigma(m)})) \) for all \( A_1, A_2, \ldots, A_m \in IS, \sigma \) is an arbitrary permutation on the set \( \{1, 2, \ldots, m\} \).
- \( ILWM_M((w_1, A_1), (w_2, A_2), \ldots, (w_m, A_m)) \leq ILWM_M((w_1, C_1), (w_2, C_2), \ldots, (w_m, C_m)) \) if \( A_i \leq C_i \) for all \( i = 1, 2, \ldots, m \).

**Proof.** (1), (2) are straightforward. And (3) is implied from the fact that the \( j \)-th largest of \( \{A_1, A_2, \ldots, A_m\} \) is equal to that of \( \{A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(m)}\} \). It is easily shown that the \( j \)-th largest of \( \{A_1, A_2, \ldots, A_m\} \) is smaller or equal to that of \( \{C_1, C_2, \ldots, C_m\} \); so, (4) is also proved.

**Remark 4.2.** \( \max_N, \min_N, ILWM_M \) also have similar properties.

5. **CONCLUSIONS**

In this paper, the notion of intuitionistic linguistic label is first launched. Some intuitionistic linguistic aggregation operations are also introduced. Besides, some properties of these operators are considered. In future, some new operators should be proposed, and applications in group decision making problems should be presented.

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