FURTHER RESULTS ON FUZZY LINGUISTIC LOGIC PROGRAMMING

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Abstract. Fuzzy linguistic logic programming is introduced to represent and reason with linguistically-expressed human knowledge, where the truth of vague sentences is given in linguistic terms, and linguistic hedges can be used to indicate different levels of emphasis. Fuzzy linguistic logic programming has been shown to have fundamental notions and results of a logic programming framework, especially of the declarative semantics, procedural semantics, and fixpoint semantics. The procedural semantics are sound, complete and directly manipulates linguistic terms in order to compute answers to queries. In this paper, we prove some additional results of fuzzy linguistic logic programming, which can be considered as a counterpart of those of traditional definite logic programming. We also show that it has a generalised Pavelka-style completeness. Moreover, the possibility that aggregation operators can occur in rule bodies is also discussed.

Key words. Logic programming, fuzzy logic, hedge algebra, computing with words, completeness.

1. INTRODUCTION

Fuzzy linguistic logic programming (FLLP) \([1]\), developed from fuzzy logic programming \([2]\), is introduced for representing and reasoning with linguistically-expressed human knowledge. FLLP is a many-valued logic programming framework without negation. In FLLP, each

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fact or rule is graded to a certain degree specified by a linguistic truth value, and hedges can be used as unary connectives in rule bodies. Many fundamental notions and results of traditional definite logic programming (TDLP) \cite{8} can have a counterpart in the framework. FLLP can be applied to deductive databases \cite{4}. Other logic programming frameworks developed in a similar approach include multi-adjoint logic programming \cite{5}.

Fuzzy logic in the narrow sense (FLn) \cite{6,7} is a branch of many-valued logic developed for a paradigm of inference under vagueness. Almost all systems of FLn are truth functional. Rational Pavelka logic (RPL) \cite{6} is a simplified version of Pavelka logic \cite{8}. RPL is a system of FLn where truth functions of conjunction and implication are Łukasiewicz t-norm and its residuum. Each evaluation of propositional variables by truth values in [0,1] uniquely extends to an evaluation of all formulae φ using the truth functions. Formula φ is called an \textit{I-tautology} if \( e(φ) = 1 \) for each evaluation e. Several 1-tautology formulae are taken as \textit{axioms}. A \textit{theory} is a set of formulae. Evaluation e is called a \textit{model} of a theory T if \( e(φ) = 1 \) for all φ in T. The \textit{deduction rule} of RPL is modus ponens. A \textit{proof} in a theory T is a sequence \( ϕ_1, \ldots, ϕ_n \) of formulae whose each member is either an axiom of RPL or a member of T or follows from some preceding members of the sequence; \( ϕ_n \) is called a \textit{provable} formula, denoted \( T \vdash ϕ_n \). In the graded approach to syntax, a \textit{graded formula} \((φ, r)\), which is just another notation for the formula \( T \models φ \), states that the truth value of φ is at least r. The deduction rule, called \textit{many-valued modus ponens}, is as follows: if \( T \vdash (φ, r) \) and \( T \vdash (φ → ψ, s) \), then \( T \vdash (ψ, r * s) \), where * is Łukasiewicz t-norm. The \textit{truth degree} of a formula φ over a theory T is defined as \( |φ|_T = \inf\{e(φ)\} | e \text{ is a model of } T \} \), and the \textit{provability degree} of φ is \( |φ|_T = \sup\{r|T \vdash (φ, r)\} \). It is proved that for each theory T and each formula φ, the truth degree and the provability degree of φ coincide. This result is usually referred to as Pavelka-style completeness, one of the most important completeness results in FLn \cite{11,10}.

In addition to the results proved in \cite{11}, this paper will show that FLLP has a counterpart of a number of important results of TDLP, e.g., the model intersection property. Also, the completeness of the procedural semantics of FLLP can be seen as a generalised Pavelkastyle completeness if one considers FLLP as an FLn system and its computation as a proof. Moreover, aggregation operators can occur in rule bodies, enabling us to describe increased fulfillment of user requirements. The remainder of this paper is organised as follows. Section 2 gives an overview of FLLP. Section 3 proves a number of additional results. Section 4 discusses the possibility of using aggregation operators in rule bodies. Section 5 concludes the paper.

2. FUZZY LINGUISTIC LOGIC PROGRAMMING

2.1. Linguistic truth domains and operations

Values of the linguistic variable \textit{Truth}, e.g., \textit{True} and \textit{VeryLittleFalse}, can be characterised by a hedge algebra (HA) \( X = (X, G, H, \leq) \), where X is a term set and \( \leq \) is its \textit{semantic order relation} \cite{11,12}. An \textit{l-limit} HA is a linear HA in which every term has a length of at most \( l + 1 \). A \textit{linguistic truth domain} is a finite and linearly ordered set \( \overline{X} = X \cup \{0, W, 1\} \), where X is the term set of an l-limit HA, and W is the \textit{middle truth value} \cite{11}. Operations are defined on \( \overline{X} = \{v_0, \ldots, v_n\} \) with \( v_0 \leq v_1 \leq \cdots \leq v_n \) as follows: (i) \textit{conjunction}: \( x \land y = \min(x, y) \); (ii) \textit{disjunction}: \( x \lor y = \max(x, y) \); (iii) A non-decreasing \textit{inverse mapping} \( h^- \) for each hedge \( h \); (iv) Łukasiewicz t-norm and its residuum are respectively defined as:

\[
C_L(v_i, v_j) = v_{\max(i+j-n,0)}; \quad L^-_L(v_j, v_i) = v_{\min(n,n+j-i)};
\]
and (v) Gödel t-norm and its residuum are respectively defined as:

\[
C_G(v_i, v_j) = \min(v_i, v_j), \quad \leftarrow^* G (v_j, v_i) = \begin{cases} v_n & \text{if } i \leq j \\ v_j & \text{otherwise.} \end{cases}
\]

Each t-norm and its residuum satisfy the residuation property [5]:

\[
C(b, r) \leq h \iff r \leq \leftarrow^* (h, b)
\]

2.2. Language

The language is a predicate language without function symbols. Connectives can be conjunctions \(\land\) (Gödel) and \(\land_L\) (Łukasiewicz); the disjunction \(\lor\); implications \(\leftarrow\) (Łukasiewicz) and \(\leftarrow_G\) (Gödel); and hedges. For a binary connective \(c\), its truth function is denoted by \(c^*\), and for a hedge connective \(h\), its truth function is its inverse mapping \(h^-\).

A term is either a constant or a variable. An atom is of the form \(p(t_1, ..., t_n)\), where \(p\) is an \(n\)-ary predicate symbol, and \(t_1, ..., t_n\) are terms of corresponding attributes. A body formula is defined inductively as follows: (i) an atom is a body formula; (ii) if \(B_1\) and \(B_2\) are body formulae, then so are \(\land (B_1, B_2)\), \(\lor (B_1, B_2)\), and \(h B_1\), where \(h\) is a hedge connective. A rule is a graded implication \((A \leftarrow B, r)\), where \(A\) is an atom called rule head, \(B\) is a body formula called rule body, and \(r\) is a truth value different from 0; \((A \leftarrow B)\) is called the logical part of the rule. A fact is a graded atom \((A, t)\), where \(A\) is an atom called the logical part of the fact, and \(t\) is a truth value different from 0. All variables are assumed to be universally quantified. A fuzzy linguistic logic program (program, for short) is a finite set of rules and facts. The truth value \(t\) in \((\varphi, t)\) is understood as a lower bound to the exact truth value of \(\varphi\). A program \(P\) can be represented as a partial mapping \(P : \text{Formulae} \to X \setminus \{0\}\), where the domain of \(P\), denoted \(\text{dom}(P)\), is finite and consists only of logical parts, and \(X\) is the linguistic truth domain. For each \((\varphi, t) \in P\), \(P(\varphi) = t\). We refer to the Herbrand base of \(P\) by \(B_P\).

2.3. Declarative semantics

Let \(P\) be a program, and \(X\) the linguistic truth domain; an Herbrand interpretation \(f\) of \(P\) is a mapping from \(B_P\) to \(X\); \(f\) can be extended to all ground formulae, denoted \(\overline{f}\), as follows: (i) \(\overline{f}(A) = f(A)\), if \(A\) is a ground atom; (ii) \(\overline{f}(c(B_1, B_2)) = c^*(\overline{f}(B_1), \overline{f}(B_2))\) and \(\overline{f}(h B) = h^- (\overline{f}(B))\), where \(B_1, B_2, B\) are ground formulae, \(c\) is a binary connective, and \(h\) is a hedge connective. For non-ground formulae, \(\overline{f}\) is defined as \(\overline{f}(\varphi) = \overline{f}(\forall \varphi) = \inf \{\overline{f}(\varphi \theta) \mid \varphi \theta\text{ is a ground instance of } \varphi\}\), where \(\forall \varphi\) denotes the universal closure of \(\varphi\). An interpretation \(f\) is an Herbrand model of \(P\) if for all \(\varphi \in \text{dom}(P)\), \(\overline{f}(\varphi) \geq P(\varphi)\). A query is an atom used as a question \(A\). A pair \((x; \theta)\), where \(x \in X\), and \(\theta\) is a substitution, is called a correct answer for \(P\) and a query \(A\) if for every model \(f\) of \(P\), we have \(\overline{f}(A \theta) \geq x\).

2.4. Procedural semantics

Admissible rules are defined as follows:

Rule 1. From \(((X A_m Y) ; \theta)\) infer \(((X C(B, r) Y) ; \theta \theta)\) if (1) \(A_m\) is an atom; (2) \(\theta\) is an mgu of \(A_m\) and \(A\); and (3) \((A \leftarrow B, r)\) is a rule in the program.

Rule 2. From \((X A_m Y)\) infer \((X 0 Y)\). This rule is usually used for situations where \(A_m\) does not unify with any rule head or logical part of facts.
Rule 3. From \((XhBY)\) infer \((Xh^{-1}(B)Y)\) if \(B\) is a body formula, \(h\) is a hedge connective.

Rule 4. From \(((X \alpha_0 \gamma)\theta)\) infer \(((X \gamma Y)\theta; \emptyset)\) if (1) \(\alpha_0\) is an atom; (2) \(\theta\) is an mgu of \(\alpha_0\) and \(\gamma\); and (3) \((A, r)\) is a fact in the program.

Rule 5. If there are no more predicate symbols and hedge connectives in the expression, replace all connectives \(\land\)’s, and \(\lor\)’s with \(\land^*\), and \(\lor^*\), respectively, and then evaluate it to obtain a truth value. The substitution remains unchanged.

A pair \((r; \emptyset)\), where \(r\) is a truth value, and \(\emptyset\) is a substitution, is said to be a computed answer for a program \(P\) and a query \(?A\) if there is a sequence \(G_0, \ldots, G_n\) such that (1) every \(G_i\) is a pair consisting of an expression and a substitution; (2) \(G_0 = (A; \emptyset)\) (\(\emptyset\) is the identity (empty) substitution); (3) every \(G_i+1\) is inferred from \(G_i\) by one of the admissible rules; and (4) \(G_n = (r; \emptyset')\) and \(\emptyset' = \emptyset\) restricted to variables of \(A\).

Example 2.1. Assume that we use the linguistic truth domain taken from the 2-limit HA \(X = (X, \{F, T\}, \{V, M, R, L\}, \leq)\), where \(F, T, V, M, R, \) and \(L\) stand for False, True, Very, More, Rather, and Little, respectively, and there is a piece of knowledge as follows: (i) “If a student studies very hard, and his/her university is rather high-ranking, then he/she will be a good employee” is Very More True; (ii) “The university where Ann is studying is high-ranking” is Very True; and (iii) “Ann is studying hard” is More True. Let \(gd\_em, st\_hd,\) and \(hira\_un\) stand for “good employee”, “study hard”, and “high-ranking university”, respectively. The piece of knowledge can be represented by the following program:

\[
(gd\_em(X) \leftarrow_G \land(V st\_hd(X), R hira\_un(X)), VMT) \\
(hira\_un(\text{ann}), VT) \\
(st\_hd(\text{ann}), MT)
\]

Given a query \(?gd\_em(ann)\), we have the following computation (the substitution is \(id\)):

\[
?gd\_em(ann) \\
C_G(\land(V st\_hd(ann), R hira\_un(ann)), VMT) \\
C_G(\land(V^{-}(st\_hd(ann))), R hira\_un(ann)), VMT) \\
C_G(\land(V^{-}(st\_hd(ann))), R^{-}(hira\_un(ann))), VMT) \\
C_G(\land(V^{-}(MT), R^{-}(hira\_un(ann)))), VMT) \\
C_G(\land(V^{-}(MT), R^{-}(VT)), VMT) \\
C_G(\land(V^{-}(MT), R^{-}(VT)), VMT) \\
RT
\]

That is, “Ann will be a good employee” is at least Rather True.

Theorem 2.1 (Soundness of the procedural semantics) Every computed answer for a program \(P\) and a query \(?A\) is a correct answer for \(P\) and \(?A\).

Theorem 2.2. For every correct answer \((x; \emptyset)\) of a program \(P\) and a ground query \(?A\), there exists a computed answer \((r; \emptyset)\) for \(P\) and \(?A\) such that \(r \geq x\).

Theorem 2.3 (Completeness of the procedural semantics) Let \(P\) be a program, and \(?A\) a query. For every correct answer \((x; \emptyset)\) for \(P\) and \(?A\), there exists a computed answer \((r; \sigma)\) for \(P\) and \(?A\), and a substitution \(\gamma\) such that \(r \geq x\) and \(\emptyset = \sigma\gamma\).

The completeness of the procedural semantics states that given a correct answer for a query, we always have a computed answer which is more general than the correct answer.
2.5. Fixpoint semantics

Let \( P \) be a program. The immediate consequence operator \( T_P \) mapping from interpretations to interpretations is defined as follows: for an interpretation \( f \) and every ground atom \( A \in B_P \), \( T_P(f)(A) = \max\{\sup\{C_i(f(B), r)\} \mid (A \leftarrow_i B, r) \text{ is a ground instance of a rule in } P\} \), \( \sup\{t(A, t) \text{ is a ground instance of a fact in } P\} \}. It is shown in \( \llbracket 1 \rrbracket \) that the least Herbrand model of the program \( P \) is exactly the least fixpoint of \( T_P \) and can be obtained by finitely iterating \( T_P \) from the bottom interpretation \( \bot \), mapping every ground atom into 0.

3. ADDITIONAL RESULTS OF FUZZY LINGUISTIC LOGIC PROGRAMMING

The ordering \( \leq \) in \( \mathcal{X} \) is extended to interpretations pointwise as follows: for any interpretations \( f_1 \) and \( f_2 \) of a program \( P \), \( f_1 \subseteq f_2 \) iff \( f_1(A) \leq f_2(A) \), \( \forall A \in B_P \). Let \( \otimes \) and \( \oplus \) denote the meet (or infimum, greatest lower bound) and join (or supremum, least upper bound) operators, respectively; for all interpretations \( f_1 \) and \( f_2 \) of \( P \) and for all \( A \in B_P \), we have: (i) \( (f_1 \otimes f_2)(A) = f_1(A) \otimes f_2(A) \), and (ii) \( (f_1 \oplus f_2)(A) = f_1(A) \oplus f_2(A) \).

**Proposition 3.1.** Let \( \mathcal{F}_P \) be the set of all interpretations of a program \( P \). Then \( \langle \mathcal{F}_P, \otimes, \oplus \rangle \) is a complete lattice.

**Proof.** We show that for any subset \( F \) of \( \mathcal{F}_P \), \( \bigotimes F \) and \( \bigoplus F \) exist. For all \( A \in B_P \), \( (\bigotimes F)(A) = \otimes \{f(A) \mid f \in F\} \). It suffices that the set of all truth values is a complete lattice for \( \bigotimes \{f(A) \mid f \in F\} \) to exist, and a finite and linearly ordered linguistic truth domain is obviously a complete lattice. The case of \( \bigoplus \) is similar.

**Lemma 3.1.** Let \( f_1 \) and \( f_2 \) be two interpretations of a program \( P \) such that \( f_1 \subseteq f_2 \). For any ground body formula \( B \), we have \( \overline{f_1}(B) \leq \overline{f_2}(B) \).

**Proof.** The lemma is proved by induction on the structure of \( B \). In the base case, where \( B \) is a ground atom, we have \( \overline{f_1}(B) = f_1(B) \leq f_2(B) = \overline{f_2}(B) \). For the inductive case, by case analysis and induction hypothesis, we have \( B = \land(B_1, B_2) \), or \( B = \lor(B_1, B_2) \), or \( B = hB_1 \) such that \( \overline{f_1}(B_1) \leq \overline{f_2}(B_1) \) and \( \overline{f_1}(B_2) \leq \overline{f_2}(B_2) \). By definition, \( \overline{f_1}(B) = \land^\ast(\overline{f_1}(B_1), \overline{f_1}(B_2)) \leq \land^\ast(\overline{f_2}(B_1), \overline{f_2}(B_2)) = \overline{f_2}(B) \), or \( \overline{f_1}(B) = \lor^\ast(\overline{f_1}(B_1), \overline{f_1}(B_2)) \leq \lor^\ast(\overline{f_2}(B_1), \overline{f_2}(B_2)) = \overline{f_2}(B) \), or \( \overline{f_1}(B) = h^\ast(\overline{f_1}(B_1)) \leq h^\ast(\overline{f_2}(B_1)) = \overline{f_2}(B) \), respectively, since truth functions of all connectives in rule bodies are monotone in all arguments. This completes the proof of the lemma.

The following theorem is the counterpart of the model intersection property in TDLP \( \llbracket 3 \rrbracket \).

**Theorem 3.1.** Let \( P \) be a program, and \( F \) a non-empty set of Herbrand models of \( P \). Then \( \otimes F \) is an Herbrand model of \( P \).

**Proof.** Since \( \otimes F \) always exists, we put \( g = \otimes F \). Let \( \varphi \) be any formula in \( \text{dom}(P) \). There are two cases: (i) \( (\varphi, t) \), where \( t \) is a truth value, is a fact in \( P \). For each ground instance \( A \) of \( \varphi \) and each model \( f \in F \) of \( P \), by hypothesis, we have \( f(A) \geq t \). Therefore, \( g(A) = \otimes \{f(A) \mid f \in F\} \geq t \). Then \( \overline{g}(\varphi) = \otimes \{g(A) \mid A \text{ is a ground instance of } \varphi \} \geq t = P(\varphi) \); or (ii) \( (\varphi, t) \) is a rule in \( P \). For each ground instance \( A \leftarrow_i B \) of \( \varphi \) and each model \( f \in F \), by hypothesis, we have \( \overline{f}(A) = \leftarrow_i \sup \{(f(A), \overline{f}(B)) \geq t \}. By the residuation property
Consider the top interpretation \( T \) of a program \( P \) which maps every \( A \in B_P \) to 1. For any \( \varphi \in \text{dom}(P) \), it is easily verified that \( T(\varphi) = 1 \geq P(\varphi) \). Thus, \( T \) is an Herbrand model of \( P \), and the set of all Herbrand models of \( P \) is non-empty.

The following theorem follows immediately from Theorem 3.1 and the definition of the least model.

\[ \textbf{Theorem 3.2.} \text{ Let } P \text{ be a program. Then } M_P = \otimes \{ f | f \text{ is an Herbrand model of } P \} \text{ is the least Herbrand model of } P. \]

Hence, the least Herbrand model of a logic program can be characterised by the greatest lower bound of the set of all its Herbrand models. As in TDLP, \( M_P \) can be regarded as the natural interpretation of \( P \), which gives intuitive description of the meaning of \( P \).

The following proposition shows that if we consider FLLP as a system of FLn and the program \( P \) as a fuzzy theory, then for each body formula \( \varphi \), \( M_P(\varphi) \) is the truth degree of \( \varphi \) over \( P \) in the sense of Pavelka [8, 6].

\[ \textbf{Proposition 3.2.} \text{ Let } P \text{ be a program. For every body formula } \varphi, M_P(\varphi) = \otimes \{ \overline{\text{f}}(\varphi) | f \text{ is an Herbrand model of } P \}. \]

\textbf{Proof.} Let \( f \) be any Herbrand model of \( P \). For each ground instance \( \varphi \theta \) of \( \varphi \), by Lemma 3.1 we have \( M_P(\varphi \theta) \leq \overline{f}(\varphi \theta) \). Thus, \( M_P(\varphi) = \otimes \{ M_P(\varphi \theta) | \varphi \theta \text{ is a ground instance of } \varphi \} \leq \otimes \{ \overline{f}(\varphi \theta) | \varphi \theta \text{ is a ground instance of } \varphi \} = \overline{f}(\varphi) \). Since \( f \) is arbitrary, we have \( M_P(\varphi) \leq \otimes \{ \overline{f}(\varphi) | f \text{ is an Herbrand model of } P \} \), and thus \( M_P(\varphi) = \otimes \{ \overline{f}(\varphi) | f \text{ is an Herbrand model of } P \} \). \]

On one hand, Theorem 2.3 can be considered as a Pavelka-style completeness in a general sense. On the other hand, we can also have a strict Pavelka-style completeness [8, 6] for ground atoms as follows.

\[ \textbf{Proposition 3.3.} \text{ Let } P \text{ be a program. For every ground atom } A, M_P(A) = \oplus \{ r | (r; id) \text{ is a computed answer for } P \text{ and } ?A \}. \]

\textbf{Proof.} Since \( A \) is ground, every computed answer for \( ?A \) is of the form \( (r; id) \), and the set \( \{ (r; id) \} \) is a computed answer for \( ?A \). For each computed answer \( (r; id) \) for \( ?A \), by Theorem 2.1 \( (r; id) \) is also a correct answer for \( ?A \), thus \( M_P(A) \geq r \). Therefore, \( M_P(A) \geq \oplus \{ r | (r; id) \text{ is a computed answer for } ?A \} \). On the other hand, since \( M_P(A; id) \) is a correct answer for \( ?A \), by Theorem 2.2 there exists a computed answer \( (r' ; id) \) for \( ?A \) such that \( r' \geq M_P(A) \). Hence \( M_P(A) = \oplus \{ r | (r; id) \text{ is a computed answer for } ?A \} \).

Therefore, if one considers FLLP as an FLn system and a computation as a proof, Proposition 3.3 states that \( M_P(A) \) is the provability degree of atom \( A \) in the sense of Pavelka, which, together with Proposition 3.2 establishes a strict Pavelka-style completeness for ground atoms. However, we do not have a similar result for non-ground atoms as shown in the following example.
Example 3.1. Consider a simple program $P$ consisting of two ground facts: $(p(a).VT)$ and $(p(b),T)$. It is easily verified that $M_{P}(p(X)) = \otimes\{M_{P}(p(a)), M_{P}(p(b))\} = T$, where $X$ is a variable. However, since $\exists p(X)$ has a computed answer $(VT; \{X/a\})$, $\oplus\{r|\sigma\}$ is a computed answer for $P$ and $\exists p(X) = VT \neq M_{P}(p(X))$.

The reason is that in logic programming, besides the truth values, one also considers the substitutions, which are an important part of computations.

It can be seen that an arbitrary body formula may occur as a query in our framework, and if so, the same result can be obtained for ground ones.

4. USING AGGREGATION OPERATORS IN RULE BODIES

Recall that in fuzzy logic programming [2], body formulae can be built using aggregation operators, which subsume all kinds of fuzzy conjunctions and disjunctions. Aggregation operators are very useful since they enable us to describe increased fulfillment of user requirements. In this section, we discuss the possibility of extending our rule bodies with aggregation connectives whose truth functions can directly act on linguistic truth values.

In the literature, there are several kinds of aggregation operators which can directly compute with linguistic labels. The most well-known example is the LOWA (Linguistic Ordered Weighted Averaging) operator [13]. The LOWA operator is developed based on the ordered weighted averaging (OWA) operator defined in [14] and the convex combination of linguistic labels defined in [15].

Definition 4.1 (LOWA operator) [13] Let $S = \{s_{1}, \ldots, s_{m}\}$ be a set of linguistic terms to be aggregated, the LOWA operator $\phi(s_{1}, \ldots, s_{m}) = C^m\{w_{k}, t_{k}, k = 1 \ldots m\}$ is defined inductively as follows.

For $m = 2$,

$$C^2\{\{w_{1}, 1 - w_{1}\}, \{t_{1}, t_{2}\}\} = (w_{1} \circ v_{j}) \oplus ((1 - w_{1}) \circ v_{i}) = v_{k}$$

where $t_{1} = v_{j}, t_{2} = v_{i} \in X, j \geq i$, and $k = \min\{n, i + \text{round}(w_{1}.(j - i))\}$, in which $n + 1$ is the cardinality of $X$, $\text{round}(.)$ is the usual round operation.

For $m > 2$,

$$C^m\{w_{k}, t_{k}, k = 1 \ldots m\} = C^2\{\{w_{1}, 1 - w_{1}\}, \{t_{1}, C^{m-1}\{\eta_{h}, t_{h}, h = 2 \ldots m\}\}\}$$

where $W = [w_{1}, \ldots, w_{m}]$ is a weighting vector associated with $S$ such that: (i) $w_{i} \in [0, 1]$, and (ii) $\sum_{i=1}^{m} w_{i} = 1$; $T = [t_{1}, \ldots, t_{m}]$ is a vector such that $t_{i}$ is the $i$th largest element in the collection $s_{1}, \ldots, s_{m}$; $\eta_{h} = w_{h}/\sum_{k=2}^{m} w_{k}, h = 2, \ldots, m$.

A natural question arising is how to obtain the associated weighting vector. Yager [14] proposed an interesting way to compute the weights of the OWA operator using linguistic quantifiers. More precisely, if $Q$ is a relative or proportional quantifier such as "Most", $Q$ can be expressed by a fuzzy subset of $[0, 1]$ such that for each $r \in [0, 1]$, $Q(r)$ indicates the degree to which $r$ portion of objects satisfies the concept denoted by $Q$. Then, the weights can be obtained by:

$$w_{i} = Q(i/n) - Q((i - 1)/n), i = 1, \ldots, n$$
The membership function of such a quantifier $Q$ can be:

$$Q(r) = \begin{cases} 
0 & \text{if } 0 \leq r < a \\
\frac{r-a}{b-a} & \text{if } a \leq r \leq b \\
1 & \text{if } 1 \geq r > b
\end{cases}$$

where $0 \leq a \leq b \leq 1$.

Because of the non-decreasing nature of $Q$, it follows that $w_i \geq 0$. Furthermore, since $Q(1) = 1$ and $Q(0) = 0$, we have $\sum_{i=1}^{n} w_i = 1$. The use of such quantifiers to generate the weighting vector for the LOWA operator essentially implies that the more criteria are satisfied, the better the solution is.

**Example 4.1.** Assume that the truth domain in Example 2.1 is taken, and the quantifier "Most" with $a=0.3$ and $b=0.8$ is used to generate the weighting vector for the LOWA operator. The weighting vectors of dimension 3 and of dimension 2 are $[w_1 = 1/15, w_2 = 2/3, w_3 = 4/15]$ and $[w_1 = 2/5, w_2 = 3/5]$, respectively; we have $\phi(AT,LT,AMT) = \phi(v_{35}, v_{30}, v_{25}) = v_{29} = AAT$.

The LOWA operator has the following properties: (i) it is commutative, i.e., $\phi(s_1, \ldots, s_m) = \phi(\pi(s_1), \ldots, \pi(s_m))$, where $\pi$ is a permutation over the set of arguments; (ii) it is non-decreasing in all arguments, i.e., given $S = [s_1, \ldots, s_m]$ and $T = [t_1, \ldots, t_m]$ being two vectors such that for all $i, s_i \geq t_i$, we have $\phi(S) \geq \phi(T)$; and (iii) it is an or-and operator, i.e., $\min(s_i) \leq \phi(s_1, \ldots, s_m) \leq \max(s_i)$.

Since in all the proofs, we only require that the truth function of a connective in body formulae be non-decreasing in all arguments, allowing body formulae to be built using a LOWA operator (i.e., we can have a rule $A \leftarrow \oplus(B_1, \ldots, B_n)$, where the truth function $\oplus$ is a LOWA operator) does not affect any results of our framework at all.

### 5. CONCLUSION

In this paper, we have presented a number of additional results of FLLP including the counterpart of the model intersection property of TDLP and the characterisation of the least Herbrand model of a logic program by the greatest lower bound of the set of all its Herbrand models. We have also shown that the completeness of the procedural semantics of FLLP can be seen as a generalised Pavelka-style completeness if one considers FLLP as an FLn system and its computation as a proof; in fact, FLLP has a strict Pavelka-style completeness for ground queries, but not for non-ground ones. Moreover, aggregation operators can occur in rule bodies, which enables us to describe increased fulfillment of user requirements. Such results, together with the notions and results presented in [1], show that FLLP can be seen as an elegant and natural generalisation of TDLP for inference under vagueness since (1) it allows one to explicitly represent and reason with partial truth expressed in linguistic terms; (2) it has a counterpart of most fundamental notions and results of TDLP; and (3) it enjoys a generalised Pavelka-style completeness.

### REFERENCES


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