COMPLETION OF THE CATEGORY OF FINITE-DIMENSIONAL FUZZY SPACES

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Abstract. In this paper we introduce a method to expand the category \( \mathcal{F} \) of all finite-dimensional fuzzy spaces associated with finite-dimensional Chu spaces into a complete system.

Tóm tắt. Bài này tiếp tục nghiên cứu phạm trù các không gian mở hữu hạn chiều đã được đề cập đến trong [7] và [8]. Như đã được chứng minh trong [7], phạm trù \( \mathcal{F} \) các không gian mở hữu hạn chiều liên kết với các không gian Chu hữu hạn chiều là một hệ thống tương đương, tuy nhiên, \( \mathcal{F} \) không đóng đối với phép lấy tích chéo nên nó không là một hệ thống đầy đủ. Trong bài này, chúng tôi đưa ra một phương pháp mở rộng phạm trù \( \mathcal{F} \) thành một hệ thống đầy đủ. Để làm điều đó, chúng tôi xây dựng một phạm trù \( n \)-tập hợp đối ngoài \( \mathcal{F}^* \) chứa \( \mathcal{F} \) như một phạm trù con, trong đó \( \mathcal{F}^* \) là một hệ thống đầy đủ.

1. INTRODUCTION

It is shown in [7] that the category \( \mathcal{F} \) of all finite-dimensional fuzzy spaces associated with finite-dimensional Chu spaces is an equivalent system. Unfortunately, \( \mathcal{F} \) is not closed under the cross product, therefore \( \mathcal{F} \) is not a complete system. In this paper we introduce a method to expand the category \( \mathcal{F} \) into a complete system, that is, we construct a "dual" \( n \)-set category \( \mathcal{F}^* \) containing \( \mathcal{F} \) as a subcategory, where \( \mathcal{F}^* \) is a complete system.

2. FINITE-DIMENSIONAL *-FUZZY SPACES AND THE *-FUZZY FUNCTORS

By \( n \)-set we mean a cartesian product \( X = \prod_{i=1}^{n} X_i \). Let \( S \) denote the \( n \)-set category, when the category \( S^* \) is defined as follows:

1. Objects of \( S^* \) are morphisms in \( S \).
2. If \( \alpha : X = \prod_{i=1}^{n} X_i \rightarrow Y = \prod_{i=1}^{n} Y_i \) and \( \alpha' : X' = \prod_{i=1}^{n} X'_i \rightarrow Y' = \prod_{i=1}^{n} Y'_i \) are two objects of \( S^* \), then a morphism \( \varphi : \alpha \rightarrow \alpha' \) from \( \alpha \) to \( \alpha' \) in \( S^* \) is a map (in the \( n \)-set category) \( \varphi : Y = \prod_{i=1}^{n} Y_i \rightarrow X' = \prod_{i=1}^{n} X'_i \).

Let \( \alpha : X = \prod_{i=1}^{n} X_i \rightarrow Y = \prod_{i=1}^{n} Y_i \), \( \alpha' : X' = \prod_{i=1}^{n} X'_i \rightarrow Y' = \prod_{i=1}^{n} Y'_i \), and \( \alpha'' : X'' = \prod_{i=1}^{n} X''_i \rightarrow Y'' = \prod_{i=1}^{n} Y''_i \) be objects in \( S^* \), \( \varphi : \alpha \rightarrow \alpha' \) and \( \varphi' : \alpha' \rightarrow \alpha'' \) be morphisms of \( S^* \) (i.e., \( \varphi : Y = \prod_{i=1}^{n} Y_i \rightarrow X' = \prod_{i=1}^{n} X'_i \) and \( \varphi' : Y' = \prod_{i=1}^{n} Y'_i \rightarrow X'' = \prod_{i=1}^{n} X''_i \)).

Then composition of \( \varphi \) and \( \varphi' \), denoted by \( \varphi' \varphi \), is given by

\[ \varphi' \varphi = \varphi' \alpha' \varphi : \alpha \rightarrow \alpha'' \].

It is easy to check that with the above definition \( S^* \) is a category.

For a given set \( X = \prod_{i=1}^{n} X_i \), let \( X^* = [0,1]^X \) denote collection of all fuzzy sets of \( X \). For a map \( \alpha : X = \prod_{i=1}^{n} X_i \rightarrow Y = \prod_{i=1}^{n} Y_i \), we define the conjugate \( \alpha^* : Y^* \rightarrow X^* \) of \( \alpha \) by the formula

\[ \alpha^*(a)(x) = a(\alpha(x)) \text{ for } x \in X \text{ and } a \in Y^* \].

It is easy to see that

\[ (\beta \alpha)^* = \alpha^* \beta^* \text{ for every } \alpha : X \rightarrow Y \text{ and } \beta : Y \rightarrow Z \].
Now for $\alpha : X = \prod_{i=1}^{n} X_i \rightarrow Y = \prod_{i=1}^{n} Y_i$ we define $F^*(\alpha) = (\prod_{i=1}^{n} X_i, f_\alpha, Y^*)$, where $Y^*$ denotes the collection of all fuzzy sets of $Y = \prod_{i=1}^{n} Y_i$, and $f_\alpha : \prod_{i=1}^{n} X_i \times Y^* \rightarrow [0,1]$ is given by

$$f_\alpha(x_1, x_2, \ldots , x_n, a) = a(\alpha(x_1, x_2, \ldots , x_n))$$

for every $(x_1, x_2, \ldots , x_n, a) \in \prod_{i=1}^{n} X_i \times Y^*$.

The $(n+1)$-dimensional Chu space $F^*(\alpha) = (\prod_{i=1}^{n} X_i, f_\alpha, Y^*)$ is called the $(n+1)$-dimensional $\ast$-fuzzy space associated with the map $\alpha : X = \prod_{i=1}^{n} X_i \rightarrow Y = \prod_{i=1}^{n} Y_i$. The category of all $(n+1)$-dimensional $\ast$-fuzzy spaces associated with maps in the $n$-set category $S$ is called the $(n+1)$-dimensional $\ast$-fuzzy category and denoted by $\mathcal{F}^*$.

3. RESULTS

At first, we will show that the $(n+1)$-dimensional $\ast$-fuzzy category $\mathcal{F}^*$ defined above contains the category $\mathcal{F}$ as a subcategory. In fact, we have the following theorem.

**Theorem 1.** Any $(n+1)$-dimensional fuzzy space is a $(n+1)$-dimensional $\ast$-fuzzy space.

**Proof.** If $F(X) = (\prod_{i=1}^{n} X_i, f_X, X^*)$ then clearly that $F(X) = F^*(1_X)$ is a $(n+1)$-dimensional $\ast$-fuzzy space.

**Theorem 2.** $\mathcal{F}^*$ is a complete system.

**Proof.** Assume that $\langle \phi \rangle = (\prod_{i=1}^{n} \phi_i, \psi) : F^*(\alpha) = (\prod_{i=1}^{n} X_i, f_\alpha, Y^*) \rightarrow F^*(\alpha') = (\prod_{i=1}^{n} X_i', f_{\alpha'}, Y'^*)$ is a $(n+1)$-Chu morphism, where $F^*(\alpha)$ and $F^*(\alpha')$ are $(n+1)$-dimensional $\ast$-fuzzy spaces associated with the maps $\alpha = \prod_{i=1}^{n} \alpha_i : X = \prod_{i=1}^{n} X_i \rightarrow Y = \prod_{i=1}^{n} Y_i$ and $\alpha' = \prod_{i=1}^{n} \alpha'_i : X' = \prod_{i=1}^{n} X'_i \rightarrow Y' = \prod_{i=1}^{n} Y'_i$, respectively. Putting $\beta = \alpha' \circ \phi = \prod_{i=1}^{n} \alpha'_i \circ \phi_i : X = \prod_{i=1}^{n} X_i \rightarrow Y' = \prod_{i=1}^{n} Y'_i$, we get the cross product $\tilde{\mathcal{C}} = (\prod_{i=1}^{n} X_i, f_\alpha \times f_{\alpha'}, Y'^*)$, which is a $(n+1)$-dimensional $\ast$-fuzzy space associated with the map $\beta = \prod_{i=1}^{n} \alpha'_i \circ \phi_i$. In fact, for every $(x_1, \ldots , x_n, b) \in \prod_{i=1}^{n} X_i \times Y'^*$, we have

$$(f_\alpha \times f_{\alpha'})(x_1, \ldots , x_n, b) = f_{\alpha'}(\phi_1(x_1), \ldots , \phi_n(x_n), b) = b(\alpha'_1 \phi_1(x_1), \ldots , \alpha'_n \phi_n(x_n)) = f_{\alpha'}(x_1, \ldots , x_n, b) = f_{\beta}(x_1, \ldots , x_n, b).$$

Thus, the category $\mathcal{F}^*$ is closed under the cross product. Therefore the theorem is proved.

**Theorem 3.** $F^* : S^* \rightarrow \mathcal{F}^*$ is a covariant functor.

**Proof.** For a morphism $\varphi = \prod_{i=1}^{n} \varphi_i : \alpha = \prod_{i=1}^{n} \alpha_i \rightarrow \alpha' = \prod_{i=1}^{n} \alpha'_i$, with $\alpha, \alpha' \in S^*$, we define

$$F^*(\varphi) = (\prod_{i=1}^{n} \varphi_i \alpha_i, \varphi^* \alpha'^*)$$

where $\varphi^*$ and $\alpha'^*$ are conjugated of $\varphi = \prod_{i=1}^{n} \varphi_i$ and $\alpha' = \prod_{i=1}^{n} \alpha'_i$, respectively, that is

$$\varphi^*(a)(y_1, \ldots , y_n) = a(\varphi_1(y_1), \ldots , \varphi_n(y_n))$$

for every $(y_1, \ldots , y_n) \in \prod_{i=1}^{n} Y_i$ and $a \in X'^*$

and

$$\alpha'^*(b)(z'_1, \ldots , z'_n) = b(\alpha'_1(z'_1), \ldots , \alpha'_n(z'_n))$$

for every $(z'_1, \ldots , z'_n) \in \prod_{i=1}^{n} X'_i$ and $b \in Y'^*$.

We claim that $F^*(\varphi) : F^*(\alpha) = (\prod_{i=1}^{n} X_i, f_\alpha, Y^*) \rightarrow F^*(\alpha') = (\prod_{i=1}^{n} X'_i, f_{\alpha'}, Y'^*)$ is a $(n+1)$-dimensional Chu morphism. That is, the following diagram commutes:
\[ \prod_{i=1}^{n} X_i \times Y'^* \xrightarrow{(\varphi_1, 1_{Y'^*})} \prod_{i=1}^{n} X'_i \times Y'^* \]

\[ (1_X, \varphi \cdot \alpha^*) \downarrow \]

\[ \prod_{i=1}^{n} X_i \times Y^* \xrightarrow{f_\alpha} \]

In fact, for every \((x_1, \ldots, x_n) \in \prod_{i=1}^{n} X_i\) and \(b \in Y^*\), we have

\[ f_\alpha(x_1, \ldots, x_n, \varphi \cdot \alpha^*(b)) = \varphi \cdot \alpha^*(b)(\alpha_1(x_1), \ldots, \alpha_n(x_n)) = (\alpha' \varphi)(b)(\alpha_1(x_1), \ldots, \alpha_n(x_n)) = b(\alpha'_1 \varphi_1 \alpha_1(x_1), \ldots, \alpha'_n \varphi_n \alpha_n(x_n)) = f_\alpha(\varphi(x), b) \]

Consequently the above diagram commutes.

Hence \(F^*(\varphi) = (\prod_{i=1}^{n} \varphi_i) \cdot \alpha^* \cdot \alpha^*\) is a \((n+1)\)-Chu morphism.

Now we will show that \(F^*\) preserves the composition. In fact, let

\[ \alpha = \prod_{i=1}^{n} \alpha_i : X = \prod_{i=1}^{n} X_i \rightarrow Y = \prod_{i=1}^{n} Y_i, \quad \alpha' = \prod_{i=1}^{n} \alpha'_i : X' = \prod_{i=1}^{n} X'_i \rightarrow Y' = \prod_{i=1}^{n} Y'_i \]

and

\[ \alpha'' = \prod_{i=1}^{n} \alpha''_i : X'' = \prod_{i=1}^{n} X''_i \rightarrow Y'' = \prod_{i=1}^{n} Y''_i \]

be objects in the category \(S^*\). Let \(\varphi = \prod_{i=1}^{n} \varphi_i : \alpha = \prod_{i=1}^{n} \alpha_i \rightarrow \alpha' = \prod_{i=1}^{n} \alpha'_i\) and \(\varphi' = \prod_{i=1}^{n} \varphi'_i : \alpha'' = \prod_{i=1}^{n} \alpha''_i \rightarrow \alpha''' = \prod_{i=1}^{n} \alpha'''_i\) be morphisms in \(S^*\) (i.e., \(\varphi = \prod_{i=1}^{n} \varphi_i : Y = \prod_{i=1}^{n} Y_i \rightarrow X' = \prod_{i=1}^{n} X'_i\) and \(\varphi' = \prod_{i=1}^{n} \varphi'_i : Y' = \prod_{i=1}^{n} Y'_i \rightarrow X'' = \prod_{i=1}^{n} X''_i\) are maps in the n-set category). By the definition we have \(\varphi' \cdot \varphi = \varphi' \cdot \alpha' \cdot \varphi = \prod_{i=1}^{n} \varphi'_i \cdot \varphi_i \cdot \alpha_i\). Therefore

\[ F^*(\varphi' \cdot \varphi) = (\varphi' \cdot \alpha') \cdot \varphi = \prod_{i=1}^{n} \varphi'_i \cdot \varphi_i \cdot \alpha_i \cdot \varphi \]

Consequently \(F^*\) preserves the composition, and hence \(F^* : S^* \rightarrow \mathcal{F}^*\) is a covariant functor.

The functor \(F^* : S^* \rightarrow \mathcal{F}^*\) is called \((n+1)\)-dimensional \(^*\)-fuzzy functor.

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