DESCRIBING MINIMAL KEYS BY DENSE FAMILIES OF DATABASE RELATIONS

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Abstract. The dense families of database relations were introduced by Järvinen [6]. The aim of this paper is to investigate some new properties of dense families of database relations, and their applications. That is, we characterize minimal keys in terms of dense families. We prove that with a given relation $R$ the equality set $E_R$ is an $R$-dense family whose size is at most $\frac{m(m-1)}{2}$, where $m$ is the number of tuples in $R$. We also prove that the set of all minimal keys of relation $R$ is the transversal hypergraph of the complement of the equality set $E_R$. We give an effective algorithm finding all minimal keys of a given relation $R$. The complexity of this algorithm is also estimated.

Tóm tắt. Họ trụ mặt của quan hệ trong cơ sở dữ liệu được giới thiệu bởi Järvinen [6]. Mục đích của bài báo là nghiên cứu một số tính chất mới của họ trụ mặt của quan hệ và ứng dụng của nó. Đó là, chúng tôi mở tả khóa tối thiểu của quan hệ thông qua họ trụ mặt. Chúng tôi chứng tỏ được rằng với một quan hệ $R$ cho trước, tập bằng nhau $E_R$ là một $R$-trú mặt mà kích thước tối đa của nó là $\frac{m(m-1)}{2}$, ở đây $m$ là số các bộ trong $R$. Chúng tôi cũng chứng tỏ được rằng tập tất cả các khóa tối thiểu của quan hệ $R$ chính là siêu đồ thị transversal của phần bổ của tập bằng nhau $E_R$. Từ đây, chúng tôi đưa ra một thuật toán hiệu quả tìm tất cả các khóa tối thiểu của quan hệ cho trước $R$. Đó phù hợp với thuật toán này cũng được đánh giá.

1. BASIC DEFINITIONS

In this section we present briefly the main concepts of the theory of relational databases which will be needed in sequel. The concepts and facts given in this section can be found in [1,4,7,8,10].

Let $U$ be a nonempty finite set of attributes (e.g. name, age etc). The elements of $U$ will be denoted by $a, b, c, \ldots, x, y, z$, if an ordering on $U$ is needed, by $a_1, \ldots, a_n$. A map $\text{dom}$ associates with each $a \in U$ its domain $\text{dom}(a)$. A relation $R$ over $U$ is a subset of Cartesian product $\prod_{a \in U} \text{dom}(a)$.

We can think of a relation $R$ over $U$ as being a set of tuples: $R = \{h_1, \ldots, h_m\}$,

$$h_i : U \rightarrow \bigcup_{a \in U} \text{dom}(a), \ h_i(a) \in \text{dom}(a), \ i = 1, 2, \ldots, m.$$ 

A functional dependency (FD for short) is a statement of form $X \rightarrow Y$, where $X, Y \subseteq U$. The FD $X \rightarrow Y$ holds in a relation $R = \{h_1, \ldots, h_m\}$ over $U$ if

$$(\forall h_i, h_j \in R) ((\forall a \in X)(h_i(a) = h_j(a)) \Rightarrow (\forall b \in Y)(h_i(b) = h_j(b))).$$
We also say that $R$ satisfies the FD $X \rightarrow Y$.

Let $F_R$ be a family of all FDs that holds in $R$. Then $F = F_R$ satisfies

(F1) \hspace{1cm} X \rightarrow X \in F,
(F2) \hspace{1cm} (X \rightarrow Y \in F, Y \rightarrow Z \in F) \Rightarrow (X \rightarrow Z \in F),
(F3) \hspace{1cm} (X \rightarrow Y \in F, X \subseteq V, W \subseteq Y) \Rightarrow (V \rightarrow W \in F),
(F4) \hspace{1cm} (X \rightarrow Y \in F, V \rightarrow W \in F) \Rightarrow (X \cup V \rightarrow Y \cup W \in F).

A family of FDs satisfying (F1)-(F4) is called an $f$-family over $U$.

Clearly, $F_R$ is an $f$-family over $U$. It is known [1] that if $F$ is an arbitrary $f$-family, then there is a relation $R$ over $U$ such that $F_R = F$.

Given a family $F$ of FDs over $U$, there exists a unique minimal $f$-family $F^+$ that contains $F$. It can be seen that $F^+$ contains all FDs which can be derived from $F$ by the rules (F1)-(F4).

A relation scheme $s$ is a pair $(U, F)$, where $U$ is a set of attributes and $F$ is a set of FDs over $U$.

Let $U$ be a nonempty finite set and $\mathcal{P}(U)$ its power set. The mapping $\mathcal{L} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ is called a closure operation on $U$ if it satisfies the following conditions:

1. $X \subseteq \mathcal{L}(X)$,
2. $X \subseteq Y$ implies $\mathcal{L}(X) \subseteq \mathcal{L}(Y)$,
3. $\mathcal{L}(\mathcal{L}(X)) = \mathcal{L}(X)$.

Remark 1. It is clear that, if $F$ is an $f$-family, and we define $\mathcal{L}_F(X)$ as

$$\mathcal{L}_F(X) = \{ a \in U : X \rightarrow \{a\} \in F \}$$

then $\mathcal{L}_F$ is a closure operation over $U$. Conversely, it is known [1,3] that if $\mathcal{L}$ is a closure operation, then there is exactly one $f$-family $F$ over $U$ so that $\mathcal{L} = \mathcal{L}_F$, where

$$F = \{ X \rightarrow Y : X, Y \subseteq U, Y \subseteq \mathcal{L}(X) \}.$$

Thus, there is a one-to-one correspondence between closure operations and $f$-families over $U$.

Let $R$ be a relation over $U$ and $K \subseteq U$. Then $K$ is a key of $R$ if $K \rightarrow U \in F_R$. $K$ is a minimal key of $R$ if $K$ is a key of $R$ and any proper subset of $K$ is not a key of $R$.

Denote $K_R$ the set of all minimal keys of $R$.

2. HYPERGRAPHS AND TRANSVERSALS

Let $U$ be a nonempty finite set and put $\mathcal{P}(U)$ for the family of all subsets of $U$. The family $\mathcal{H} = \{ E_i : E_i \in \mathcal{P}(U), i = 1, 2, \ldots, m \}$ is called a hypergraph over $U$ if $E_i \neq \emptyset$ holds for all $i$ (in [2] it is required that the union of $E_i$s is $U$, in this paper we do not require this).

The elements of $U$ are called vertices, and the sets $E_1, \ldots, E_m$ the edges of the hypergraph $\mathcal{H}$.

A hypergraph $\mathcal{H}$ is called simple if it satisfies $\forall E_i, E_j \in \mathcal{H} : E_i \subseteq E_j \Rightarrow E_i = E_j$. It can be seen that $K_R$ is a simple hypergraph.

Let $\mathcal{H}$ be a hypergraph over $U$. Then $\text{min}(\mathcal{H})$ denotes the set of minimal edges of $\mathcal{H}$ with respect to set inclusion, i.e., $\text{min}(\mathcal{H}) = \{ E_i \in \mathcal{H} : \forall E_j \in \mathcal{H} : E_j \subseteq E_i \}$. It is clear that, $\text{min}(\mathcal{H})$ is a simple hypergraph. Furthermore, $\text{min}(\mathcal{H})$ is uniquely determined by $\mathcal{H}$.
A set $T \subseteq U$ is called a transversal of $\mathcal{H}$ (sometimes it is called hitting set) if it meets all edges of $\mathcal{H}$, i.e., $\forall E \in \mathcal{H} : T \cap E \neq \emptyset$. Denote by $Trs(\mathcal{H})$ the family of all transversals of $\mathcal{H}$. A transversal $T$ of $\mathcal{H}$ is called minimal if no proper subset $T'$ of $T$ is a transversal.

The family of all minimal transversals of $\mathcal{H}$ is called the transversal hypergraph of $\mathcal{H}$, and denoted by $Tr(\mathcal{H})$. Clearly, $Tr(\mathcal{H})$ is a simple hypergraph.

By the definition of minimal transversal, the following proposition is obvious.

**Proposition 2.1.** Let $\mathcal{H}$ be a hypergraph over $U$. Then

$$Tr(\mathcal{H}) = Tr(min(\mathcal{H})).$$

The following algorithm finds the family of all minimal transversals of a given hypergraph (by induction).

**Algorithm 2.2.** [5]

Input: Let $\mathcal{H} = \{E_1, \ldots, E_m\}$ be a hypergraph over $U$.

Output: $Tr(\mathcal{H})$.

Method:

*Step 0.* We set $L_1 := \{\{a\} : a \in E_1\}$. It is obvious that $L_1 = Tr(\{E_1\})$.

*Step $q+1$ ($q < m$).* Assume that

$$L_q = S_q \cup \{B_1, \ldots, B_{t_q}\},$$

where $B_i \cap E_{q+1} = \emptyset$, $i = 1, \ldots, t_q$, and $S_q = \{A : A \in E_q \land A \cap E_{q+1} \neq \emptyset\}$.

For each $i$ ($i = 1, \ldots, t_q$) constructs the set $\{B_i \cup \{b\} : b \in E_{q+1}\}$. Denote them by $A_{i,1}^q, \ldots, A_{i,t_i}^q$ ($i = 1, \ldots, t_q$). Let

$$L_{q+1} = S_q \cup \{A_{i,p}^q : A \in S_q \land A \not\subset A_{i,p}^q, 1 \leq i \leq t_q, 1 \leq p \leq r_i\}.$$

**Theorem 2.3.** ([5]) For every $q$ ($1 \leq q \leq m$) $L_q = Tr(\{E_1, \ldots, E_q\})$, i.e., $L_m = Tr(\mathcal{H})$.

It can be seen that the determination of $Tr(\mathcal{H})$ based on our algorithm does not depend on the order of $E_1, \ldots, E_m$.

**Remark 2.** Denote $L_q = S_q \cup \{B_1, \ldots, B_{t_q}\}$, and $l_q$ ($1 \leq q \leq m-1$) be the number of elements of $L_q$. Note that, $l_q \geq t_q$. It can be seen that the worst-case time complexity of our algorithm is

$$O(|U|^2 \sum_{q=0}^{m-1} t_q u_q),$$

where $l_0 = t_0 = 1$ and

$$u_q = \begin{cases} l_q - l_q, & \text{if } l_q > l_q, \\ 1, & \text{if } l_q = l_q. \end{cases}$$

Clearly, in each step of our algorithm $L_q$ is a simple hypergraph. It is known that the size of arbitrary simple hypergraph over $U$ cannot be greater than $C_n^{[n/2]}$, where $n = |U|$. $C_n^{[n/2]}$ is asymptotically equal to $2^{n+1/2}/(\pi n)^{1/2}$. From this, the worst-case time complexity of our algorithm cannot be more than exponential in the number of attributes. In cases for which $l_q \leq l_m$ ($q = 1, \ldots, m-1$), it is easy to see that the time complexity of our algorithm is not greater than $O(|U|^2|\mathcal{H}|*|Tr(\mathcal{H})|^2)$. Thus, in these cases this algorithm finds $Tr(\mathcal{H})$ in
polynomial time in \(|U|, |\mathcal{H}|\) and \(|\text{Tr}(\mathcal{H})|\). Obviously, if the number of elements of \(\mathcal{H}\) is small, then this algorithm is very effective. It only requires polynomial time in \(|R|\).

The following proposition is obvious

**Proposition 2.4.** ([5]) The time complexity of finding \(\text{Tr}(\mathcal{H})\) of a given hypergraph \(\mathcal{H}\) is (in general) exponential in the number of elements of \(U\).

Proposition 2.4 is still true for a simple hypergraph.

3. DENSE FAMILIES

Let \(\mathcal{D} \subseteq \mathcal{P}(U)\) be a family of subsets of a \(U\). We define a set \(F_\mathcal{D}\) over \(\mathcal{D}\) as follows

\[
F_\mathcal{D} = \{X \rightarrow Y : (\forall A \in \mathcal{D}) X \subseteq A \Rightarrow Y \subseteq A\}.
\]

We have the following proposition.

**Proposition 3.1.** ([6]) If \(\mathcal{D}\) is a family of subsets of a finite set \(U\), then \(F_\mathcal{D}\) is an \(f\)-family over \(U\).

The notion of dense family of a database relation is defined in [6], as follows

Let \(R\) be a relation over \(U\). We say that a family \(\mathcal{D} \subseteq \mathcal{P}(U)\) of attribute sets is \(R\)-dense (or dense in \(R\)) if \(F_R = F_\mathcal{D}\).

The following proposition guarantees the existence of at least one dense family. In the sequel we denote \(L_{F_R}\) simply by \(L_R\).

**Proposition 3.2.** ([6]) The family \(L_R\) is \(R\)-dense.

For any \(A \subseteq U\), we denote by \(\overline{A}\) the complement of \(A\) with respect to the set \(U\), that is, \(\overline{A} = \{a \in U : a \notin A\}\).

**Theorem 3.3.** ([6]) Let \(R\) be a relation over \(U\). If \(\mathcal{D} \subseteq \mathcal{P}(U)\) is \(R\)-dense, then the following conditions hold

1. \(K\) is a key of \(R\) if and only if it contains an element from each set in \(\{\overline{A} : A \in \mathcal{D}, A \neq U\}\).
2. \(K\) is a minimal key of \(R\) if and only if it minimal with respect to the property of containing an element from each set in \(\{\overline{A} : A \in \mathcal{D}, A \neq U\}\).

Note that an element \(a \in U\) belongs to all minimal keys if \(\overline{A} = \{a\}\) for some \(A \in \mathcal{D}\), where \(\mathcal{D}\) is an \(R\)-dense family. Now we investigate some properties of dense families of database relations, and their applications.

Let \(U\) be a nonempty finite set and \(\mathcal{P}(U)\) its power set. For every family \(\mathcal{D} \subseteq \mathcal{P}(U)\), the complement family of \(\mathcal{D}\) is the family \(\overline{\mathcal{D}} = \{\overline{A} : A \in \mathcal{D}\}\) over \(U\).

Let \(R = \{h_1, \ldots, h_m\}\) be a relation over \(U\), and \(E_R\) the equality set of \(R\), i.e.,

\[
E_R = \{E_{ij} : 1 \leq i < j \leq m\},
\]

where \(E_{ij} = \{a \in U : h_i(a) = h_j(a)\}\).

**Proposition 3.4.** The equality set \(E_R\) is \(R\)-dense.

**Proof.** Assume that \(X \rightarrow Y \in F_R\). Let \(E_{ij} \in E_R\) such that \(X \subseteq E_{ij}\). This means that \(h_i(X) = h_j(X)\). From this, and according to the definition of FDs, we have \(h_i(Y) = h_j(Y)\). Thus, \(Y \subseteq E_{ij}\). By the definition of \(F_{E_R}\), that is,
\[ F_{E_{R}} = \{ X \rightarrow Y : (\forall E_{ij} \in E_{R}) \ X \subseteq E_{ij} \Rightarrow Y \subseteq E_{ij} \}, \]

we obtain \( X \rightarrow Y \in F_{E_{R}}. \)

Conversely, let \( X \rightarrow Y \in F_{E_{R}}. \) Suppose that there are \( h_i, h_j \in R \) such that \( h_i(X) = h_j(X), 1 \leq i < j \leq m. \) Which means that \( X \subseteq E_{ij}. \) By \( X \rightarrow Y \in F_{E_{R}}, Y \subseteq E_{ij}. \) Hence, we also obtain \( h_i(Y) = h_j(Y). \) Consequently, \( X \rightarrow Y \in F_{R}. \)

The proposition is proved. \[ \blacksquare \]

It is easy to see that the dense family \( E_{R} \) has at most \( \frac{m(m-1)}{2} \) elements.

**Theorem 3.5.** Let \( R \) be a relation over \( U. \) Then

\[ K_{R} = Tr(min(E_{R})). \]

**Proof.** By the definition of relation \( R, \) we have \( U \not\in E_{R}. \) From this, Proposition 2.1, Proposition 3.4 and Theorem 3.3, the theorem is obvious.

The proof is complete. \[ \blacksquare \]

Let \( R = \{h_1, \ldots, h_m\} \) be a relation over \( U, \) and \( N_{R} \) the nonequality set of \( R, \) i.e.,

\[ N_{R} = \{N_{ij} : 1 \leq i < j \leq m\}, \]

where \( N_{ij} = \{a \in U : h_i(a) \neq h_j(a)\}. \)

Note that, because \( R \) is a relation, \( \emptyset \not\in N_{R} \) and \( U \not\in E_{R}. \) Moreover, \( N_{R} = \overline{E_{R}}. \) From this, and Theorem 3.5, we have the following corollary.

**Corollary 3.6.** Let \( R \) be a relation over \( U. \) Then

\[ K_{R} = Tr(min(N_{R})). \]

From Proposition 3.4 and the definition of dense family, the following proposition is obvious.

**Proposition 3.7.** Let \( R = \{h_1, \ldots, h_m\} \) be a relation over \( U = \{a_1, \ldots, a_n\}. \) Then \( E_{R} \cup \{U\} \)
_is \( R \)-dense.

4. FINDING THE SET OF ALL MINIMAL KEYS OF A RELATION

In this section, we present an effective application of Theorem 3.5, which is the following algorithm finding all minimal keys of a given relation \( R. \) Remember that this problem is inherently exponential in the size of \( R \) [4].

**Algorithm 4.1.**

Input: a relation \( R = \{h_1, \ldots, h_m\} \) over \( U. \)

Output: \( K_{R}. \)

Method:

**Step 1.** Construct the equality set

\[ E_{R} = \{E_{ij} : 1 \leq i < j \leq m\}, \]

where \( E_{ij} = \{a \in U : h_i(a) = h_j(a)\}. \)

**Step 2.** Compute the complement of \( E_{R} \) as follows
\[ E_R = \{ E_{ij} : E_{ij} \in E_R \}. \]

Denote elements of \( E_R \) by \( N_1, \ldots, N_k \).

**Step 3.** From \( E_R \) compute the family \( \min(E_R) = \{ N_i \in E_R : \forall N_j \in E_R : N_j \subset N_i \} \).

**Step 4.** By Algorithm 2.2 we construct the set \( Tr(\min(E_R)) \).

Based on Proposition 2.1, Algorithm 2.2 and Theorem 3.5, we have \( K_R = Tr(\min(E_R)) \). It can be seen that the time complexity of this algorithm is the time complexity of Algorithm 2.2. In many cases this algorithm is very effective (see Remark 2).

It can be seen that, if the number of elements of the equality set \( E_R \) is constant, i.e. \( |E_R| \leq k \) for some constant \( k \), then the time complexity of finding \( K_R \) of a given relation \( R \) is polynomial time [9].

Clearly, if we replace \( E_R \) by \( N_R \), we have another similar effective algorithm finding all minimal keys of a relation.

5. CONCLUSIONS

In this paper we have investigated dense families of database relations and characterized minimal keys in terms of dense families. We prove that the set of all minimal keys of relation \( R \) is the transversal hypergraph of the complement of the equality set \( E_R \). We also give an effective algorithm finding all minimal keys of a given relation \( R \).

Our further research will be devoted to the following problems:

1. Let \( R \) be a relation over \( U \) and \( \mathcal{D} \subseteq \mathcal{P}(U) \). What is a necessary and sufficient condition for family \( \mathcal{D} \) to be \( R \)-dense?
2. Let \( R \) be a relation over \( U \). Can we use dense families to sloving the problem of determining a cover of a relation \( R \)?

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