FINITE-DIMENSIONAL APPROXIMATION FOR ILL-POSED VECTOR OPTIMIZATION OF CONVEX FUNCTIONALS IN BANACH SPACES

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Abstract. In this paper we present the convergence and convergence rate for regularization solutions in connection with the finite-dimensional approximation for ill-posed vector optimization of convex functionals in reflexive Banach space. Convergence rates of its regularized solutions are obtained on the base of choosing the regularization parameter a priori as well as a posteriori by the modified generalized discrepancy principle. Finally, an application of these results for convex optimization problem with inequality constraints is shown.

Tóm tắt. Trong bài báo này chúng tôi trình bày sự hỗ trợ và tốc độ hỗ trợ của nghiệm hiệu chỉnh trong tập bảng hạn chiếu cho bài toán cực trị đa mục tiêu các biện đã mất lợi trong không gian Banach phân xa. Tốc độ hỗ trợ của nghiệm hiệu chỉnh nhân được dự trên việc chọn tham số hiệu chỉnh trước hoặc sau bằng nguyên lý lệch suy rỗng ổn định cải biến. Cuối cùng là một ứng dụng của các kết quả đạt được cho bài toán cực trị lời với ràng buộc bất đẳng thức.

1. INTRODUCTION

Let $X$ be a real reflexive Banach space preserved a property that $X$ and $X^*$ are strictly convex, and weak convergence and convergence of norms of any sequence in $X$ imply its strong convergence, where $X^*$ denotes the dual space of $X$. For the sake of simplicity, the norms of $X$ and $X^*$ are denoted by the symbol $||.||$. The symbol $\langle x^*, x \rangle$ denotes the value of the linear continuous functional $x^* \in X^*$ at the point $x \in X$. Let $\varphi_j(x)$, $j = 0, 1, ..., N$, be the weakly lower semicontinuous proper convex functionals on $X$ that are assumed to be Gâteaux differentiable with the hemicontinuous derivatives $A_j(x)$ at $x \in X$.

In [6], one of the authors has considered a problem of vector optimization: find an element $u \in X$ such that

$$
\varphi_j(u) = \inf_{x \in X} \varphi_j(x), \quad \forall j = 0, 1, ..., N. 
$$

(1.1)

Set

$$
Q_j = \{ \hat{x} \in X : \varphi_j(\hat{x}) = \inf_{x \in X} \varphi_j(x) \}, \quad j = 0, 1, ..., N, \quad Q = \bigcap_{j=0}^{N} Q_j.
$$

It is well know that $Q_j$ coincides with the set of solutions of the following operator equation

$$
A_j(x) = \theta,
$$

(1.2)

and is a closed convex subset in $X$ (see [11]). We suppose that $Q \neq \emptyset$, and $\theta \notin Q$, where $\theta$ is the zero element of $X$ (or $X^*$).
In [6] it is showed the existence and uniqueness of the solution \( x_h^\alpha \) of the operator equation

\[
\sum_{j=0}^{N} \alpha^j A_j^h(x) + \alpha U(x) = \theta, \quad (1.3)
\]

where \( \alpha > 0 \) is the small parameter of regularization, \( U \) is the normalized duality mapping of \( X \), i.e., \( U : X \to X^* \) satisfies the condition

\[
\langle U(x), x \rangle = \|x\|^2, \quad \|U(x)\| = \|x\|,
\]

\( A_j^h \) are the hemicontinuous monotone approximations for \( A_j \) in the forms

\[
\|A_j(x) - A_j^h(x)\| \leq h g(\|x\|), \quad \forall x \in X,
\]

with level \( h \to 0 \), and \( g(t) \) is a bounded (the image of the bounded set is bounded) nonnegative function, \( t \geq 0 \).

Clairly, the convergence and convergence rates of the sequence \( x_h^\alpha \) to \( u \) depend on the choice of \( \alpha = \alpha(h) \). In [6], one has showed that the parameter \( \alpha \) can be chosen by the modified generalized discrepancy principle, i.e., \( \alpha = \alpha(h) \) is constructed on the basis of the following equation

\[
\rho(\alpha) = h^p \alpha^{-q}, \quad p, q > 0,
\]

where \( \rho(\alpha) = \alpha(a_0 + t(\alpha)) \), the function \( t(\alpha) = \|x_h^\alpha\| \) depends continuously on \( \alpha \geq \alpha_0 > 0 \), \( a_0 \) is some positive constant.

In computation the finite-dimensional approximation for (1.3) is the important problem. As usualy, it can be aproximated by the following equation

\[
\sum_{j=0}^{N} \alpha^j A_j^{h_n}(x) + \alpha U^n(x) = \theta, \quad x \in X_n, \quad (1.6)
\]

where \( A_j^{h_n} = P_n^* A_j P_n \), \( U^n = P_n^* U P_n \) and \( P_n : X \to X_n \) the linear projection from \( X \) onto \( X_n, X_n \) is the finite-dimensional subspace of \( X, P_n^* \) is the conjugate of \( P_n \).

Without loss of generality, suppose that \( \|P_n\| = 1 \) (see [11]).

As for (1.3), equation (1.6) has also an unique solution \( x_{\alpha,n}^h \), and for every fixed \( \alpha > 0 \) the sequence \( \{x_{\alpha,n}^h\} \) converges to \( x_{\alpha}^h \), the solution of (1.3), as \( n \to \infty \) (see [11]).

The natural problem is to ask whether the sequence \( \{x_{\alpha,n}^h\} \) converges to \( u \) as \( \alpha, h \to 0 \) and \( n \to \infty \), and how fast it converges, where \( u \) is an element in \( Q \). The purpose of this paper is to answer these questions.

We assume, in addition, that \( U \) satisfies the condition

\[
\langle U(x) - U(y), x - y \rangle \geq m_U \|x - y\|^s, \quad m_U > 0, \quad s \geq 2, \quad \forall x, y \in X.
\]

Set

\[
\gamma_n(x) = \|(I - P_n)x\|, \quad x \in Q,
\]

where \( I \) denotes the identity operator in \( X \).
Hereafter the symbols $\rightrightarrows$ and $\rightarrow$ indicate weak convergence and convergence in norm, respectively, while the notation $a \sim b$ is meant $a = O(b)$ and $b = O(a)$.

2. MAIN RESULT

The convergence of $\{x_{\alpha,n}^h\}$ to $u$ is determined by the following theorem.

**Theorem 1.** If $h/\alpha$ and $\gamma_n(x)/\alpha \to 0$, as $\alpha \to 0$ and $n \to \infty$, then the sequence $x_{\alpha,n}^h$ converges to $u$.

**Proof.** For $x \in Q, x^n = P_n x$, it follows from (1.6) that
\[
\sum_{j=0}^{N} \alpha^j \langle A_j^h(x_{\alpha,n}^h), x_{\alpha,n}^h - x^n \rangle + \alpha \langle U^n(x_{\alpha,n}^h) - U^n(x^n), x_{\alpha,n}^h - x^n \rangle = \alpha \langle U^n(x^n), x^n - x_{\alpha,n}^h \rangle.
\]

Therefore, on the basis of (1.2), (1.7) and the monotonicity of $A_j^h, P_n^* A_j^h P_n$, and $P_n P_n = P_n$ we have
\[
\alpha m_U \| x_{\alpha,n}^h - x^n \|^s \leq \alpha \langle U(x_{\alpha,n}^h) - U(x^n), x_{\alpha,n}^h - x^n \rangle = \alpha \langle U^n(x_{\alpha,n}^h) - U^n(x^n), x_{\alpha,n}^h - x^n \rangle
\]
\[
= \sum_{j=0}^{N} \alpha^j \langle A_j^h(x_{\alpha,n}^h), x^n - x_{\alpha,n}^h \rangle + \alpha \langle U^n(x^n), x^n - x_{\alpha,n}^h \rangle
\]
\[
\leq \sum_{j=0}^{N} \alpha^j \langle A_j^h(x^n), x^n - x_{\alpha,n}^h \rangle + \alpha \langle U^n(x^n), x^n - x_{\alpha,n}^h \rangle
\]
\[
= \sum_{j=0}^{N} \alpha^j \langle A_j^h(x^n) - A_j(x^n) + A_j(x^n) - A_j(x^n), x^n - x_{\alpha,n}^h \rangle + \alpha \langle U(x^n), x^n - x_{\alpha,n}^h \rangle. \quad (2.1)
\]

On the other hand, by using (1.4) and
\[
\| A_j(x^n) - A_j(x) \| \leq K \gamma_n(x),
\]
where $K$ is some positive constant depending only on $x$, it follows from (2.1) that
\[
m_U \| x_{\alpha,n}^h - x^n \|^s \leq \frac{1}{\alpha} \left[(N + 1)(h g(\| x^n \|) + K \gamma_n(x)) \right] \| x^n - x_{\alpha,n}^h \| + \| U(x^n), x^n - x_{\alpha,n}^h \|. \quad (2.2)
\]

Because of $h/\alpha, \gamma_n(x)/\alpha \to 0$ as $\alpha \to 0$, $n \to \infty$ and $s \geq 2$, this inequality gives us the boundedness of the sequence $\{x_{\alpha,n}^h\}$. Then, there exists a subsequence of the sequence $\{x_{\alpha,n}^h\}$ converging weakly to $\hat{x}$ in $X$. Without loss of generality, we assume that $x_{\alpha,n}^h \rightrightarrows \hat{x}$ as $h, h/\alpha \to 0$ and $n \to \infty$. First, we prove that $\hat{x} \in Q_0$. Indeed, by virtue of the monotonicity of $A_j^h = P_n^* A_j^h P_n, U^n = P_n U P_n$ and (1.6) we have
\[
\langle A_j^h(P_n x), P_n x - x_{\alpha,n}^h \rangle \geq \langle A_j^h(x_{\alpha,n}^h), P_n x - x_{\alpha,n}^h \rangle
\]
\[
= \sum_{j=0}^{N} \alpha^j \langle A_j^h(x_{\alpha,n}^h), x_{\alpha,n}^h - P_n x \rangle + \alpha \langle U^n(x_{\alpha,n}^h), x_{\alpha,n}^h - P_n x \rangle
\]
\[
\geq \sum_{j=1}^{N} \alpha^j \langle A_j^h(P_n x), x_{\alpha,n}^h - P_n x \rangle + \alpha \langle U^n(P_n x), x_{\alpha,n}^h - P_n x \rangle, \forall x \in X.
\]
Because of $P_nP_n = P_n$, so the last inequality has form

$$\langle A_h^0(P_n)x, P_n x - x_{a,n}^h \rangle \geq \sum_{j=1}^{N} \alpha^j \langle A_j^h(P_n)x, x_{a,n}^h - P_n x \rangle + \alpha \langle U(P_n)x, x_{a,n}^h - P_n x \rangle, \forall x \in X.$$  

By letting $h, \alpha \to 0$ and $n \to \infty$ in this inequality we obtain

$$\langle A_0(x), x - \hat{x} \rangle \geq 0, \forall x \in X.$$  

Consequently, $\hat{x} \in Q_0$ (see [11]). Now, we shall prove that $\hat{x} \in Q_j, j = 1, 2, \ldots, N$. Indeed, by (1.6) and making use of the monotonicity of $A_j^{h^\alpha}$ and $U^n$, it follows that

$$\langle A_1^h(P_n)x, x_{a,n}^h - P_n x \rangle + \sum_{j=2}^{N} \alpha^{-j} \langle A_j^h(P_n)x, x_{a,n}^h - P_n x \rangle + \alpha^{-1} \langle U^n(P_n)x, x_{a,n}^h - P_n x \rangle$$

$$\leq \frac{1}{\alpha} \left[ h\alpha^{-1}g(\|P_nx\|) + K\gamma_n(x) \right] \|P_n x - x_{a,n}^h\|, \forall x \in Q_0.$$  

After passing $h, \alpha \to 0$ and $n \to \infty$, we obtain

$$\langle A_1(x), \hat{x} - x \rangle \leq 0, \forall x \in Q_0.$$  

Thus, $\hat{x}$ is a local minimizer for $\varphi_1$ on $S_0$ (see [9]). Since $S_0 \cap S_1 \neq \emptyset$, then $\hat{x}$ is also a global minimizer for $\varphi_1$, i.e., $\hat{x} \in S_1$.

Set $\hat{Q}_i = \cap_{k=0}^{i} Q_k$. Then, $\hat{Q}_i$ is also closed convex, and $\hat{Q}_i \neq \emptyset$.

Now, suppose that we have proved $\hat{x} \in \hat{Q}_i$ and we need to show that $\hat{x}$ belongs to $Q_{i+1}$. Again, by virtue of (1.6) for $x \in \hat{Q}_i$, we can write

$$\langle A_{i+1}^{h^n}(x_{a,n}^h), x_{a,n}^h - P_n x \rangle + \sum_{j=i+2}^{N} \alpha^{-j} \langle A_j^{h^n}(x_{a,n}^h), x_{a,n}^h - P_n x \rangle$$

$$+ \alpha^{-1} \langle U^n(x_{a,n}^h), x_{a,n}^h - P_n x \rangle = \sum_{k=0}^{i} \alpha^{-k} \langle A_k^{h^n}(x_{a,n}^h), P_n x - x_{a,n}^h \rangle$$

$$\leq \frac{1}{\alpha} \sum_{k=0}^{i} \alpha^{-k+1} \langle A_k^h(P_n)x - A_k(P_n)x + A_k(P_n)x - A_k(x), P_n x - x_{a,n}^h \rangle$$

$$\leq \frac{1}{\alpha} (i+1) \left( h\alpha^{-1}g(\|P_nx\|) + K\gamma_n(x) \right) \|P_n x - x_{a,n}^h\|.$$  

Therefore,
\[ \langle A_{i+1}^h(P_n x), x_{\alpha,n}^h - P_n x \rangle + \sum_{j=1+2}^N \alpha^{\lambda_j - \lambda_{i+1}} \langle A_j^h(P_n x), x_{\alpha,n}^h - P_n x \rangle + \alpha^{1 - \lambda_{i+1}} \langle U(P_n x), x_{\alpha,n}^h - P_n x \rangle \leq \frac{h g(||P_n x||) + K \gamma_n(x)}{\alpha} (N + 1) ||P_n x - x_{\alpha,n}^h||. \]

By letting \( h, \alpha \to 0 \) and \( n \to \infty \), we have

\[ \langle A_{i+1}(x), \hat{x} - x \rangle \leq 0, \ \forall x \in \bar{Q}_i. \]

As a result, \( \hat{x} \in Q_{i+1} \).

On the other hand, it follows from (2.2) that

\[ \langle U(x), x - \hat{x} \rangle \geq 0, \ \forall x \in Q. \]

Since \( Q_j \) is closed convex, \( Q \) is also closed convex. Replacing \( x \) by \( t \hat{x} + (1 - t)x \), \( t \in (0, 1) \) in the last inequality, and dividing by \( (1 - t) \) and letting \( t \) to 1, we obtain

\[ \langle U(\hat{x}), x - \hat{x} \rangle \geq 0, \ \forall x \in Q. \]

Hence \( ||\hat{x}|| \leq ||x||, \ \forall x \in Q. \) Because of the convexity and the closedness of \( Q \), and the strictly convexity of \( X \) we deduce that \( \hat{x} = u \). So, all sequence \( \{x_{\alpha,n}^h\} \) converges weakly to \( u \). Set \( x^n = u^n = P_n u \) in (2.2) we deduce that the sequence \( \{x_{\alpha,n}^h\} \) converges strongly to \( u \) as \( h \to 0 \) and \( n \to \infty \). The proof is complete.

In the following, we consider the finite-dimensional variant of the generalized discrepancy principle for the choice \( \tilde{\alpha} = \alpha(h, n) \) so that \( x_{\tilde{\alpha},n}^h \) converges to \( u \), as \( h, \alpha \to 0 \) and \( n \to \infty \).

Note that, the generalized discrepancy principle for parameter choice is presented first in [8] for the linear ill-posed problems. For the nonlinear ill-posed equation involving a monotone operator in Banach space the use of a discrepancy principle to estimate the rate of convergence of the regularized solutions was considered in [5]. In [4] the convergence rates of regularized solutions of ill-posed variational inequalities under arbitrary perturbative operators were investigated when the regularization parameter was chosen arbitrarily such that \( \alpha \sim (\delta + \varepsilon)^p \), \( 0 < p < 1 \). In this paper, we consider the modified generalized discrepancy principle for selecting \( \tilde{\alpha} \) in connection with the finite-dimensional and obtain the rates of convergence for the regularized solutions in this case.

The parameter \( \alpha(h, n) \) can be chosen by

\[ \alpha(a_0 + ||x_{\alpha,n}^h||) = h^p \alpha^{-q}, \ p, q > 0 \] (2.3)

for each \( h > 0 \) and \( n \). It is not difficult to verify that \( \rho_n(\alpha) = \alpha(a_0 + ||x_{\alpha,n}^h||) \) possesses all properties as well as \( \rho(\alpha) \) does, and

\[ \lim_{\alpha \to +\infty} \alpha^q \rho_n(\alpha) = +\infty, \ \lim_{\alpha \to +0} \alpha^q \rho_n(\alpha) = 0. \]

To find \( \alpha \) by (2.3) is very complex. So, we consider the following rule.

**The rule.** Choose \( \tilde{\alpha} = \alpha(h, n) \geq a_0 := (c_1 h + c_2 \gamma_n)^p, \ c_i > 1, i = 1, 2, 0 < p < 1 \) such that the following inequalities

\[ \tilde{\alpha}^{1+q}(a_0 + ||x_{\alpha,n}^h||) \geq d_1 h^p, \]

\[ \tilde{\alpha}^{1+q}(a_0 + ||x_{\alpha,n}^h||) \leq d_2 h^p, \ d_2 \geq d_1 > 1, \]

and...
hold.

In addition, assume that $U$ satisfies the following condition
\[ \| U(x) - U(y) \| \leq C(R) \| x - y \| \nu, \quad 0 < \nu \leq 1, \]
(2.4)
where $C(R), R > 0$, is a positive increasing function on $R = \max \{ \| x \|, \| y \| \}$ (see [10]).

Set
\[ \gamma_n = \max_{x \in Q} \{ \gamma_n(x) \}. \]

**Lemma 1.**
\[ \lim_{h \to 0, n \to \infty} \alpha(h, n) = 0. \]

**Proof.** Obviously, it follows from the rule that
\[ \alpha(h, n) \leq d_2^{1/(1+q)} \left( a_0 + \| x_n^{h, \alpha(h, n)} \| \right)^{-1/(1+q)} R^{p/(1+q)} \]
\[ \leq d_2^{1/(q+1)} a_0^{-1/(1+q)} R^{p/(1+q)}. \]

**Lemma 2.** If $0 < p < 1$ then
\[ \lim_{h \to 0, n \to \infty} \frac{h + \gamma_n}{\alpha(h, n)} = 0. \]

**Proof.** Obviously using the rule we get
\[ \frac{h + \gamma_n}{\alpha(h, n)} \leq \frac{c_1 h + c_2 \gamma_n}{(c_1 h + c_2 \gamma_n)^p} = (c_1 h + c_2 \gamma_n)^{1-p} \to 0 \]
as $h \to 0$ and $n \to \infty$.

Now, let $x_n^{h, \alpha(h, n)}$ be the solution of (1.6) with $\alpha = \alpha(h, n)$. By the argument in the proof of Theorem 1, we obtain the following result.

**Theorem 2.** The sequence $x_n^{h, \alpha(h, n)}$ converges to $u$ as $h \to 0$ and $n \to \infty$.

The next theorem shows the convergence rates of $\{ x_n^{h, \alpha(h, n)} \}$ to $u$ as $h \to 0$ and $n \to \infty$.

**Theorem 3.** Assume that the following conditions hold:
(i) $A_0$ is continuously Fréchet differentiable, and satisfies the condition
\[ \| A_0(x) - A_0(u)(x - u) \| \leq \tau \| A_0(x) \|, \quad \forall u \in Q, \]
where $\tau$ is a positive constant, and $x$ belongs to some neighbourhood of $Q$;
(ii) $A_h(X_n)$ are contained in $X_n^*$ for sufficiently large $n$ and small $h$;
(iii) there exists an element $z \in X$ such that $A_0^*(u) z = U(u)$;
(iv) the parameter $\tilde{\alpha} = \alpha(h, n)$ is chosen by the rule.

Then, we have
\[ \| x_n^{h, \alpha(h, n)} - u \| = O((h + \gamma_n)^{\eta_1} + \gamma_n^{\eta_2}), \]
\[ \eta_1 = \min \left\{ \frac{1 - p}{s - 1}, \frac{\mu_1 p}{s(1 + q)} \right\}, \quad \eta_2 = \min \left\{ \frac{1}{s}, \frac{\nu}{s - 1} \right\}. \]
Proof. Replacing \( x^n \) by \( u^n = P_n u \) in (2.2) we obtain
\[
m_U \| x^h_{\tilde{a},n} - u^n \|^s \leq \frac{1}{\alpha} \left[ (N + 1)h g(\| u^n \|) + K \gamma_n \right] \| u^n - x^h_{\tilde{a},n} \|
+ \langle U(u^n) + U(u) - U(u), u^n - x^h_{\tilde{a},n} \rangle. \tag{2.5}
\]
By (2.4) it follows that
\[
\langle U(u^n) - U(u), u^n - x^h_{\tilde{a},n} \rangle \leq C(\tilde{R})\| U(u^n - u) \| \| u^n - x^h_{\tilde{a},n} \| \leq C(\tilde{R})\gamma_n \| u^n - x^h_{\tilde{a},n} \|, \tag{2.6}
\]
where \( \tilde{R} > \| u \| \).

On the other hand, using conditions (i), (ii), (iii) of the theorem we can write
\[
\langle U(u), u^n - x^h_{\tilde{a},n} \rangle = \langle U(u), u^n - u \rangle + \langle z, A_0'(u)(u - x^h_{\tilde{a},n}) \rangle \leq \tilde{R}\gamma_n + \| z \| (\tau + 1)\| A_0(x^h_{\tilde{a},n}) \|
\leq \tilde{R}\gamma_n + \| z \| (\tau + 1)\left[ h g(\| x^h_{\tilde{a},n} \|) + \| A_0(x^h_{\tilde{a},n}) \| \right]
\leq \tilde{R}\gamma_n + \| z \| (\tau + 1)\left[ \sum_{j=1}^N \alpha^j (\| A_j(x^h_{\tilde{a},n}) \| + \| x^h_{\tilde{a},n} \| + h g(\| x^h_{\tilde{a},n} \|) \right]. \tag{2.7}
\]
Combining (2.6) and (2.7) inequality (2.5) has form
\[
m_U \| x^h_{\tilde{a},n} - u^n \|^s \leq \frac{1}{\alpha} \left[ (N + 1)h g(\| u^n \|) + K \gamma_n \right] \| u^n - x^h_{\tilde{a},n} \|
+ C(\tilde{R})\gamma_n \| u^n - x^h_{\tilde{a},n} \|
+ \tilde{R}\gamma_n + \| z \| (\tau + 1)\left[ \sum_{j=1}^N \alpha^j (\| A_j(x^h_{\tilde{a},n}) \| + \| x^h_{\tilde{a},n} \| + h g(\| x^h_{\tilde{a},n} \|) \right]. \tag{2.8}
\]
On the other hand, making use of the rule and the boundedness of \( \{ x^h_{\tilde{a},n} \} \) it implies that
\[
\tilde{\alpha} = \alpha(h, n) \geq (c_1 h + c_2 \gamma_n)^p,
\tilde{\alpha} = \alpha(h, n) \leq C_1 h^{b/(1+q)}, \quad C_1 > 0,
\tilde{\alpha} = \alpha(h, n) \leq 1,
\]
for sufficiently small \( h \) and large \( n \).

Consequently, in view of (2.8) it follows that
\[
m_U \| x^h_{\tilde{a},n} - u^n \| \leq \left[ \frac{(N + 1)h g(\| u^n \|) + K \gamma_n}{(c_1 h + c_2 \gamma_n)^p} + C(\tilde{R})\gamma_n \right] \| u^n - x^h_{\tilde{a},n} \|
+ \tilde{R}\gamma_n + C_2(h + \gamma_n)^{\lambda_1 p/(1+q)}
\leq \tilde{C}_1 (h + \gamma_n)^{1-p} + \gamma_n \| u^n - x^h_{\tilde{a},n} \|
+ \tilde{C}_2 \gamma_n + \tilde{C}_3(h + \gamma_n)^{\lambda_1 p/(1+q)}, \quad C_2 \text{ and } \tilde{C}_i, \ i = 1, 2, 3 \text{ are the positive constants.}
\]
Using the implication
\[ a, b, c \geq 0, \quad p_1 > q_1, \quad a^{p_1} \leq ba^{q_1} + c \Rightarrow a^{p_1} = O(b^{p_1/(p_1-q_1)} + c) \]
we obtain
\[ \|x_{\tilde{\alpha},n}^h - u^n\| = O((h + \gamma_n)^{p_1} + \gamma_n^{q_1}). \]
Thus,
\[ \|x_{\tilde{\alpha},n}^h - u\| = O((h + \gamma_n)^{p_1} + \gamma_n^{q_1}), \]
which completes the proof.

**Remarks.** If \( \tilde{\alpha} = \alpha(h, n) \) is chosen a priori such that \( \tilde{\alpha} \sim (h + \gamma_n)^{\eta}, \quad 0 < \eta < 1 \), then inequality (2.8) has the form
\[ m_U\|x_{\tilde{\alpha},n}^h - u^n\| \leq C_1 \left( (h + \gamma_n)^{1-\eta} + \gamma_n^{\nu} \right)\|u^n - x_{\tilde{\alpha},n}^h\| + C_2 \gamma_n + C_3 (h + \gamma_n)^{\lambda_1 \eta}, \]
where \( C_i, i = 1, 2, 3 \) are the positive constants.

Therefore,
\[ \|x_{\tilde{\alpha},n}^h - u^n\| = O((h + \gamma_n)^{\theta_1} + \gamma_n^{\theta_2}), \]
whence,
\[ \|x_{\tilde{\alpha},n}^h - u\| = O((h + \gamma_n)^{\theta_1} + \gamma_n^{\theta_2}), \]
\[ \theta_1 = \min \left\{ \frac{1 - \eta}{s-1}, \frac{\lambda_1 \eta}{s} \right\}, \quad \theta_2 = \min \left\{ \frac{1}{s}, \frac{\nu}{s-1} \right\}. \]

### 3. AN APPLICATION

In this section we consider a constrained optimization problem:
\[ \inf_{x \in X} f_N(x) \]  
subject to
\[ f_j(x) \leq 0, \quad j = 0, \ldots, N-1, \]  
where \( f_0, f_1, \ldots, f_N \) are weakly lower semicontinuous and properly convex functionals on \( X \) that are assumed to be Gâteaux differentiable at \( x \in X \).

Set
\[ Q_j = \{ x \in X : f_j(x) \leq 0 \}, \quad j = 0, \ldots, N-1. \]  
Obviously, \( Q_j \) is the closed convex subset of \( X, \quad j = 0, \ldots, N-1. \)

Define
\[ \varphi_N(x) = f_N(x), \quad \varphi_j(x) = \max\{0, f_j(x)\}, \quad j = 0, \ldots, N-1. \]  
Evidently, \( \varphi_j \) are also convex functionals on \( X \) and
\[ Q_j = \{ \bar{x} \in X : \varphi_j(\bar{x}) = \inf_{x \in X} \varphi_j(x) \}, \quad 0, 1, \ldots, N. \]
So, \( \bar{x} \) is a solution of the problem:
\[ \varphi_j(\bar{x}) = \inf_{x \in X} \varphi_j(x), \quad \forall j = 0, 1, \ldots, N. \]
REFERENCES


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