PHASE STRUCTURE OF BOSE - EINSTEIN CONDENSATE IN ULTRA - COLD BOSE GASES

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Abstract. *The Bose - Einstein condensation of ultra - cold Bose gases is studied by means of the Cornwall - Jackiw - Tomboulis effective potential approach in the improved double - bubble approximation which preserves the Goldstone theorem. The phase structure of Bose - Einstein condensate associating with two different types of phase transition is systematically investigated. Its main feature is that the symmetry which was broken at zero temperature gets restored at higher temperature.*

Keywords: Bose-Einstein condensation, superfluid and restoration phase transition.

I. INTRODUCTION

The experiments on Bose - Einstein condensation (BEC) of atomic vapors ^{87}Rb and ^{23}Na was successfully realized in 1995 [1, 2]. Just after this first achievement a series of experiments [3], [6] dealing with BEC of systems composed of two distinct species of atoms revealed that the multi - component BEC is not a simple extension of the single component BEC. There arise many fascinating phenomena such as the quantum tunneling of spin domain [7], vortex configuration [8], phase segregation [9] and so on. This event marked a very rapid development in the study of quantum gases. The most attractive feature of all experiments realizing BEC in dilute ultra - cold gases is that almost every parameter of the system is controllable, thereby many novel phenomena, in particular, various types of phase transition, could be created experimentally by simply adjusting the model parameters. In parallel with experimental efforts we witness theoretical progresses [10, 21] aiming at describing different observed phenomena of multi - component systems as well as predicting new quantum effects. In this connection, to study the phase structure of Bose - Einstein condensates in ultra - cold gases, both experimental and theoretical, is of special interest. This

paper is devoted to investigating the phase structure of one component Bose - Einstein condensate. To this end, let us begin with the dilute Bose gases given by the Lagrangian:

$$L = \phi^*(-i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m})\phi - \mu\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2, \tag{1}$$

where μ represents the chemical potential, m represents the mass of Bose atom, and the coupling constant λ is expressed through the s - wave scattering length a by

$$\lambda = \frac{4\pi\hbar^2 a}{m}. \tag{2}$$

For repulsive interaction $\lambda > 0$.

The controllable parameters associating with the model (1) involve two model parameters μ, λ and the temperature T.

This article is organized as follows. In section II based on [22,23] we establish the Cornwall - Jackiw - Tomboulis (CJT) effective potential for the system (1) in the improved double - bubble approximation together with the equations of state. The section III deals with the numerical study of various scenarios of phase transition. The conclusion is presented in section IV.

II. EFFECTIVE POTENTIAL IN DOUBLE BUBBLE APPROXIMATION AND EQUATION OF STATE

In the tree approximation the condensate density

$$\rho_0 = \phi_0^2 \tag{3}$$

corresponds to the local minimum of the potential. It fulfills

$$\mu\phi_0 - \frac{\lambda}{2}\phi_0^3 = 0, \tag{4}$$

yielding

$$\frac{\phi_0^2}{2} = \frac{\mu}{\lambda} \tag{5}$$

for $\phi_0 \neq 0$.

Let us now focus on the calculation of the CJT effective potential in the double - bubble approximation [22]. At the first, the field operator is decomposed

$$\phi = \frac{1}{\sqrt{2}}(\phi_0 + \phi_1 + i\phi_2). \tag{6}$$

Inserting Eq. (6) into Eq. (1) we get, among other, the interaction Lagrangian

$$L_{\text{int}} = \frac{\lambda}{2}\phi_0\phi_1(\phi_1^2 + \phi_2^2) + \frac{\lambda}{8}(\phi_1^2 + \phi_2^2)^2, \tag{7}$$

and the inverse propagator in the tree approximation in momentum space

$$\begin{aligned}
D_0^{-1}(k) &= \|A_{ij}\|, \\
A_{11} &= \frac{\vec{k}^2}{2m} - \mu + \frac{3\lambda}{2}\phi_0^2, \\
A_{12} &= -A_{21} = -\omega,
\end{aligned}$$

$$A_{22} = \frac{\vec{k}2}{2m} - \mu + \frac{\lambda}{2}\phi_0^2.$$ (8)

From Eq. (5) and Eq. (8) it follows that

$$E = \sqrt{\frac{\vec{k}^2}{2m}\left(\frac{\vec{k}^2}{2m} + \lambda\phi_0^2\right)},$$ (9)

which is the Bogoliubov dispersion relation for Bose condensate of ultra - cold Bose gases. For small momenta Eq. (9) reduces to

$$E \sim \sqrt{\frac{\lambda\phi_0^2}{2m}}.k,$$ (10)

associating with the Goldstone boson due to the U(1) breaking.

Next the CJT effective potential is calculated in the double - bubble approximation in which the obtained self - energy is momentum independent. Therefore, we assume the ansatz for the propagator

$$D^{-1}(k) = \left\|B_{ij}\right\|, B_{11} = \frac{\vec{k}^2}{2m} + M_1, B_{22} = \frac{\vec{k}^2}{2m} + M_2, B_{12} = -B_{21} = -\omega.$$

Following closely [23] we arrive at the CJT effective potential $V_\beta(\phi_0, D)$ at finite temperature

$$
\begin{aligned}
V_\beta(\phi_0, D) =&\ -\frac{\mu}{2}\phi_0^2 + \frac{\lambda}{8}\phi_0^4 + +\frac{1}{2}\int_\beta tr\left[\ln D^{-1} + D_0^{-1}D - 1\right] \\
&+ \frac{3\lambda}{8}\left[\int_\beta D_{11}(k)\right]^2 + \frac{3\lambda}{8}\left[\int_\beta D_{22}(k)\right]^2 \\
&+ \frac{\lambda}{4}\left[\int_\beta D_{11}(k)\right]\left[\int_\beta D_{22}(k)\right],
\end{aligned}
$$ (11)

where

$$\int_\beta f(k) = T\sum_{n=-\infty}^{+\infty}\int\frac{d^3k}{(2\pi)^3}f(\omega_n, \vec{k}).$$ (12)

Starting from Eq. (11) we get, respectively,

1. The gap equation

$$\mu - \frac{\lambda}{2}\phi_0^2 - \Sigma_1 = 0.$$ (13)

2. The Schwinger - Dyson equation

$$D^{-1}(k) = D_0^{-1}(k) + \Sigma,$$ (14)

in which

$$\Sigma = \left\|\Sigma_{ij}\right\|, \ \Sigma_{11} = \Sigma_1, \Sigma_{22} = \Sigma_2, \Sigma_{12} = \Sigma_{21} = 0,$$ (15)

and

$$\Sigma_1 = \frac{3\lambda}{2}\int_\beta D_{11}(k) + \frac{\lambda}{2}\int_\beta D_{22}(k),$$

$$\Sigma_2 = \frac{\lambda}{2}\int_\beta D_{11}(k) + \frac{3\lambda}{2}\int_\beta D_{22}(k),$$

$$M_1 = -\mu + \frac{3\lambda}{2}\phi_0^2 + \Sigma_1, M_2 = -\mu + \frac{\lambda}{2}\phi_0^2 + \Sigma_2. \tag{16}$$

Then the explicit form of propagator comes out from combining Eqs. (13) and Eq. (14)

$$D^{-1} = \left\|B_{ij}\right\|, B_{11} = \frac{\vec{k}^2}{2m} - \mu + \frac{3\lambda}{2}\phi_0^2 + \Sigma_1,$$

$$B_{12} = B_{21} = -\omega, B_{22} = \frac{\vec{k}^2}{2m} - \mu + \frac{\lambda}{2}\phi_0^2 + \Sigma_2, \tag{17}$$

which clearly indicates that the Goldstone theorem fails in this approximation. In order to restore it, we follow [24] to add an appropriate correction ΔV to $V_\beta(\phi_0, D)$, namely,

$$\tilde{V}_\beta(\phi_0, D) = V_\beta(\phi_0, D) + \Delta V, \tag{18}$$

with

$$\Delta V = -\frac{\lambda}{4}\left[P_{11}^2 + P_{22}^2 - 2P_{11}P_{22}\right], P_{aa} = \int_\beta D_{aa}(k); a = 1, 2. \tag{19}$$

It is easily checked that

$$\tilde{V}_\beta(\phi_0, D) = -\frac{\mu}{2}\phi_2^2 + \frac{\lambda}{8}\phi_0^4 + \frac{1}{2}\int_\beta tr[\ln D^{-1} + D_0^{-1}(k)D(k) - 1]$$

$$+ \frac{\lambda}{8}P_{11}^2 + \frac{\lambda}{8}P_{22}^2 + \frac{3\lambda}{4}P_{11}P_{22}. \tag{20}$$

yielding

    1. The gap equation

$$-\mu + \frac{\lambda}{2}\phi_0^2 + \Sigma_2^* = 0. \tag{21}$$

or

$$\frac{\phi_0^2}{2} = \frac{\bar{\mu}}{\lambda}; \ \ \bar{\mu} = \mu - \Sigma_2^*. \tag{22}$$

    2. The SD equation

$$D^{-1}(k) = D_0^{-1}(k) + \Sigma^*, \tag{23}$$

in which

$$\Sigma^* = \|\Sigma_{ij}^*\|, \Sigma_{11}^* = \Sigma_1^* = \frac{\lambda}{2}P_{11} + \frac{3\lambda}{2}P_{22},$$

$$\Sigma_{12}^* = \Sigma_{21}^* = 0, \Sigma_{22}^* = \Sigma_2^* = \frac{3\lambda}{2}P_{11} + \frac{\lambda}{2}P_{22}. \tag{24}$$

Combining Eqs. (22), (23) and (24) provides

$$D^{-1}(k) = \|B_{ij}\|, B_{11} = \frac{\vec{k}^2}{2m} + M_1^*,$$

$$B_{22} = \frac{\vec{k}^2}{2m}, B_{12} = -B_{21} = -\omega, M_1^* = -\mu + \frac{3\lambda}{2}\phi_0^2 + \Sigma_1^*. \tag{25}$$

It is immediately derived from Eq. (25) the dispersion relation

$$E = \sqrt{\frac{\vec{k}^2}{2m}\left(\frac{\vec{k}^2}{2m} + M_1^*\right)},$$

which preserves the Goldstone theorem

$$E \sim \sqrt{\frac{M_1^*}{2m}}.k. \tag{26}$$

as $k \to 0$.

Due to Landau criteria for superfluidity the Bose condensate turns out to be superfluid in broken phase and its sound speed reads

$$C = \sqrt{\frac{M_1^*}{2m}}.$$

Ultimately the one - particle - irreducible effective potential $\tilde{V}_\beta(\phi_0)$ is read off from Eq. (20) with D fulfilling the SD equation Eq. (23),

$$\begin{aligned}
\tilde{V}_\beta(\phi_0) &= -\frac{\mu}{2}\phi_0^2 + \frac{\lambda}{8}\phi_0^4 + \frac{1}{2}\int_\beta tr\ln D^{-1}(k) \\
&\quad + \frac{P_{11}}{2}\left(\frac{3\lambda}{2}\phi_0^2 - \mu - M_1^*\right) + \frac{P_{22}}{2}\left(\frac{\lambda}{2}\phi_0^2 - \mu\right) \\
&\quad + \frac{\lambda}{8}\left(P_{11}^2 + P_{22}^2\right) + \frac{3\lambda}{4}P_{11}P_{22}.
\end{aligned} \tag{27}$$

Based on the effective potential $\tilde{V}_\beta(\phi_0, D)$ given in Eq. (20) various equations of state are easily derived as follows

   - The pressure P is defined by

$$P = -\tilde{V}_\beta(\phi_0, D), \tag{28}$$

taken at minimum. Then the density of condensate reads

$$\rho = -\frac{\partial \tilde{V}_\beta}{\partial \mu} = \frac{\phi_0^2}{2} + \frac{P_{11} + P_{22}}{2}. \tag{29}$$

In term of density, the pressure (28) is expressed as

$$P = \frac{\lambda}{2}\rho^2 - \frac{1}{2}\int_\beta tr\ln D^{-1}(k) + \lambda\rho P_{11} - \frac{\lambda}{2}P_{11}^2. \tag{30}$$

The free energy E follows from the Legendre transform $E = \mu\rho - P$ leading to

$$E = \frac{\lambda}{2}\rho^2 + \frac{1}{2}\int_\beta tr D^{-1}(k) + \frac{\lambda}{2}P_{11}^2. \tag{31}$$

Eqs.(30) and (31) are the basic equations of state of BEC which govern all phase transition processes of BEC.

Starting from these equations let us consider respectively the high and low temperature behaviors of P and E. To this end, we first mention that the symmetry U(1), broken at $T = 0$, is restored at high temperature $T = T_c$ if $\phi_0^2 = 0$ as $T \to T_c$.

Using the high temperature expansion of all integrals appearing in Eq. (20) we arrive at

$$T_c = 2\pi \left[\frac{\mu}{2m^{3/2}\lambda \zeta(3/2)} \right]^{2/3}, \tag{32}$$

here $\zeta(z)$ is the zeta function of Riemann.

The behavior of pressure for high T just below the critical temperature is reduced

$$P = \frac{\lambda}{2}\rho + \frac{m^{3/2}\zeta(5/2)}{2\sqrt{2\pi^3}}T^{5/2} + \frac{m^3\lambda[\zeta(3/2)]^2}{16\pi^3}T^3,$$

which is the well -known result of Lee and Yang for Bose gas without invoking the double counting subtraction. The high T behavior of free energy is straightforwardly derived by means of the formula

$$E = -\frac{\partial}{\partial\beta}[\beta P]_\mu, \tag{33}$$

which yields

$$E = -\frac{\lambda}{2}\rho^2 - \frac{3m^{3/2}\lambda\rho\ \zeta(3/2)}{4\sqrt{2\pi^3}}T^{3/2} + \frac{3m^{3/2}\zeta(5/2)}{4\sqrt{2\pi^3}}T^{5/2} + \frac{m^3\lambda[\zeta(3/2)]^2}{8\pi^3}T^3. \tag{34}$$

Next the behaviors of P and E are concerned for $T/\mu \le 1$. At first we note that we have the approximate expression for M_1^* in this regime

$$M_1^* \simeq 2\lambda\rho - \frac{2\sqrt{2}M_1^{3/2}m^{3/2}\lambda}{3\pi^2} - \frac{2\sqrt{2}m^3\lambda\pi^2}{15M_1^{*5/2}},$$

which requires a self - consistent solution for M_1^* as function of density and temperature. The first approximation is chosen as $M_1^* \simeq 2\lambda\rho$ leading to the low temperature behaviors of chemical potential, pressure and free energy:

- Chemical potential

$$\mu \simeq \lambda\rho + \frac{4m^{3/2}\lambda^{5/2}\rho^{3/2}}{3\pi^2} + \frac{m^{3/2}\pi^2}{60\lambda^{3/2}\rho^{5/2}}T^4.$$

- Pressure

$$P = \frac{\lambda\rho^2}{2} + \frac{4m^{3/2}\lambda^{5/2}\rho^{5/2}}{5\pi^2} + \frac{m^{3/2}\pi^2}{36\lambda^{3/2}\rho^{3/2}}T^4.$$

- Free energy

$$E = \mu\rho - P = \frac{\lambda\rho^2}{2} + \frac{8m^{3/2}\lambda^{5/2}\rho^{5/2}}{15\pi^2} + \left(\frac{m^3}{45\rho} - \frac{m^{3/2}\pi^2}{90\lambda^{3/2}\rho^{3/2}} \right)T^4.$$

III. NUMERICAL STUDY

Starting from the analytical expressions above, let us implement the numerical study of various types of phase transitions of Bose - Einstein condensate in ultra - cold Bose gases.

At the first, we deal with the symmetry restoration of condensate. To this end, in every case of interest the model parameters μ, λ and m are chosen to be close to those of experimental setting. To begin with, let us focus on what was observed in the first experimental realization of BEC: the appearance of BEC at given values of λ and m. Selecting, for example, $\lambda = 10^{-11}eV^{-2}, m = 8.10^{10}eV$ we consider the T dependence of condensate ϕ_0 at several values of chemical potential. It is plotted in Fig. 1 which indicates that the critical temperature T_c depends on μ as given in Eq. (31). On the other hand, Fig.1 also reveals that the symmetry U(1), which was broken at $T = 0$, gets restored at higher value of $T = T_c$. Fig.2, which displays the evolution of effective potential V against ϕ_0 at several values of chemical potential μ, proves that the restoration phase transition is second order.
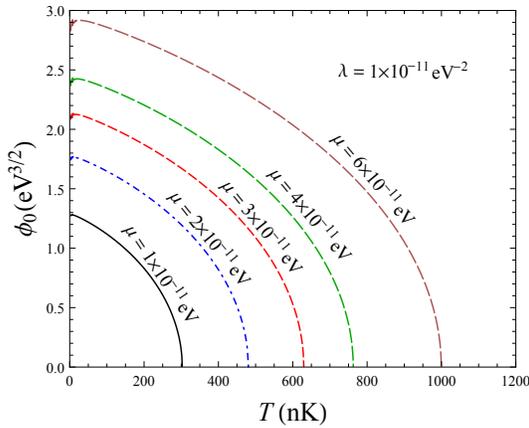

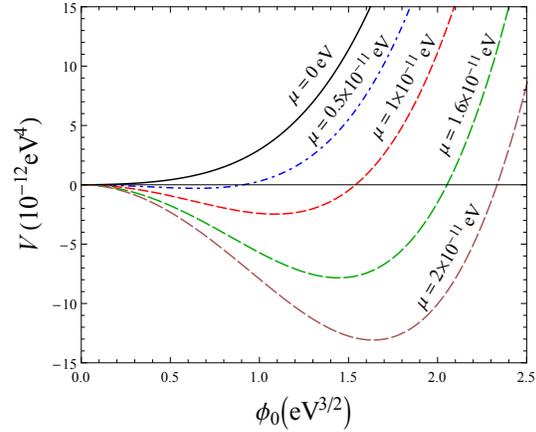**Fig. 1.** The T dependence of order parameter ϕ_0 at $\lambda = 10^{-11}eV^{-2}$ and several values of μ

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Fig. 2. The evolution of effective potential V against ϕ_0 at $\lambda = 10^{-11}eV^{-2}$ and several values of μ

Next, let us deal with another type of phase transition which is generated by changing the coupling constant λ at fixed μ. In Fig.3 we plot the T dependence of ϕ_0 at several values of coupling constant λ for m given above and $\mu = 10^{-11}eV$. It represents another scenario of Bose - Einstein condensation: the appearance of Bose -Einstein condensate depends on the coupling constant. Fig.3 also confirms the restoration of U(1) as T increases from zero and the restoration transition is second order, as is shown in Fig. 4.

The phase diagrams of these restorations in the (T, μ) and (T, λ) planes are presented in Figs.5 and 6, respectively. Their common feature is that there are two distinct regions of the model parameters: the BEC takes place only when $\bar{\mu} > 0$, at $\bar{\mu} = 0$ there occurs the restoration of broken symmetry. In addition, the behavior of phase diagram in Fig.6 is in accordance with Eq. (32).
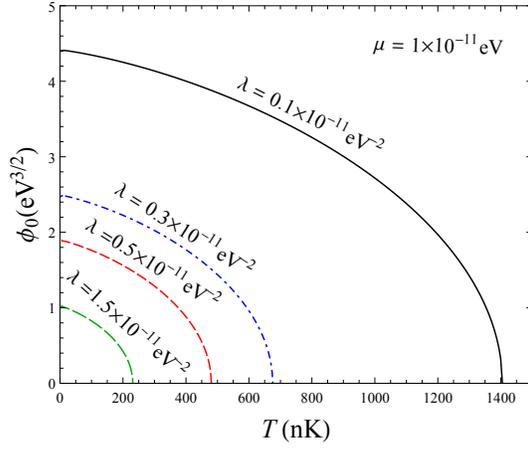
Fig. 3. The T dependence of order parameter ϕ_0 at $\mu = 10^{-11}eV$ and several values of λ
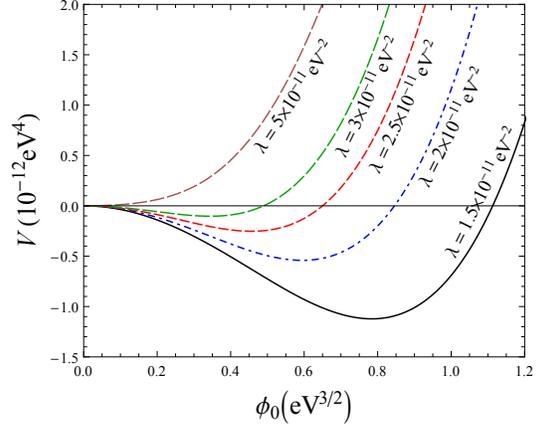

Fig. 4. The evolution of effective potential V against ϕ_0 at $\mu = 10^{-11}eV$ and several values of λ
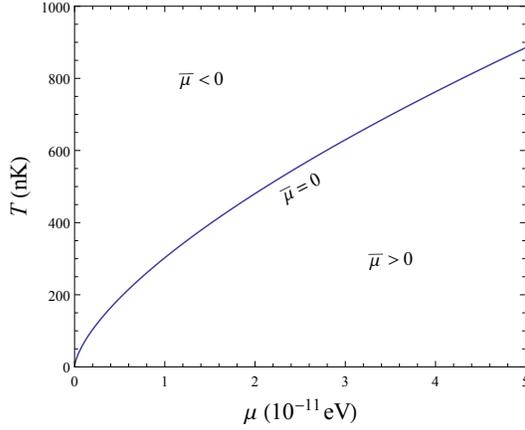
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Fig. 5. Phase diagram of restoration transition in the (T, μ) - plane. The broken phase $(\bar{\mu} > 0)$ and unbroken phase $(\bar{\mu} < 0)$ are separated by the phase diagram $(\bar{\mu} = 0)$
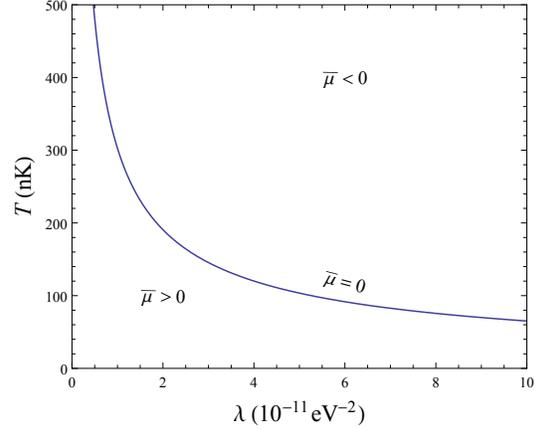
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Fig. 6. Phase diagram of restoration transition in the (T, λ) - plane. The broken phase $(\bar{\mu} > 0)$ and unbroken phase $(\bar{\mu} < 0)$ are separated by the phase diagram $(\bar{\mu} = 0)$

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IV. CONCLUSIONS

In the preceding sections several physical properties of Bose - Einstein condensate were studied by means of the CJT effective potential in the improved double - bubble approximation which preserves the Goldstone theorem. The main results we found are in order:

1. It proved that the condensate is a superfluid in broken phase.
2. Several equations of state were established in the regimes of high and low temperature.

3. The numerical investigation deals with two types of phase transition characterized by adjusting three parameters T, μ and λ.

Here we discovered two difference types of phase structure of one - component Bose - Einstein condensate: the conventional scenario of BEC manifests at given λ; the un - conventional scenario of BEC exhibits when λ is made changed. The last phenomenon could be directly verified in Bose gas experiments at ultra low temperature.

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